# Solutions 6

(1) (a) $6 + 4i$.
(b) $5 + 14i$.
(c) $28 + 96i$.
(d) $\frac{11}{17} + \frac{10}{17}i$.
(e) $\frac{3}{2} - \frac{5}{2}i$.
(f) $\frac{-31}{200} + \frac{367}{200}i$.

(2) (a) $\pm\frac{1}{\sqrt{2}}(1 - i)$.
(b) $\pm(\sqrt{2} + \sqrt{3}i)$.
(c) $\pm(6 - 7i)$.

(3) (a) $\frac{1}{2}(-1 \pm \sqrt{3}i)$.
(b) $\frac{1}{4}(3 \pm \sqrt{7}i)$.
(c) The roots are $1 + 2i$ and $1 + i$.

(4) Let $a+bi$ be a Gaussian integer. Squaring it, we get $(a^2 - b^2) + 2abi$. Observe now that $(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2$. Thus

$$\left(a^2 - b^2, 2ab, a^2 + b^2\right)$$

is a Pythagorean triple.

(5) By De Moivre's theorem

$$(\cos x + i \sin x)^5 = \cos 5x + i \sin 5x.$$

Expand the LHS using the binomial theorem to get

$$(\cos x)^5 + 5(\cos x)^4(i \sin x) + 10(\cos x)^3(i \sin x)^2 + 10(\cos x)^2(i \sin x)^3 \\ +5(\cos x)(i \sin x)^4 + (i \sin x)^5.$$

Simplifying and equating real and complex parts we get

$$\cos 5x = \cos^5 x - 10 \cos^3 \sin^2 x + 5 \cos x \sin^4 x$$

and

$$\sin 5x = 5 \cos^4 x \sin x - 10 \cos^2 x \sin^3 x + \sin^5 x.$$

(6) (a) We have that $e^{ix} = \cos x + i \sin x$ and $e^{-ix} = \cos(-x) + i \sin(-x) = \cos x - i \sin x$. Thus $e^{ix} - e^{-ix} = 2i \sin x$. It follows that $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$
(b) Using the calculations of (a) above, we have that $e^{ix} + e^{-ix} = 2 \cos x$ and so we get the result.

We have that $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$. Taking fourth powers of both sides we get that

$$\cos^4 x = \frac{1}{16}\left(e^{4ix} + 4e^{2ix} + 6 + 4e^{-2ix} + e^{-4ix}\right).$$

This simplifies to

$$\frac{1}{8}\left(\cos 4x + 4\cos 2x + 3\right),$$

as required.

(7) Put $z = i^i$. We interpret $i^i$ to mean $\exp(i\ln(i))$. Now

$$\exp(i(\frac{\pi}{2} + 2\pi k)) = i$$

where $k$ is any integer. It follows that

$$\ln(i) = i(\frac{\pi}{2} + 2\pi k).$$

Thus

$$\ln(z) = -(\frac{\pi}{2} + 2\pi k).$$

Hence

$$i^i = \exp(-\left(\tfrac{\pi}{2} + 2\pi k\right)).$$

All of these values are real.