Solutions 7

(1) (a) The quotient is $2x^2 - 3x$ and the remainder is 1.
(b) The quotient is $x^2 + 2x - 3$ and the remainder is -7.
(c) The quotient is $x^2 - 3x + 8$ and the remainder is $-27x + 7$.

(2) (a) We are given that 4 is a root and so we know that $x - 4$ is a factor. Dividing out we get $3x^2 - 8x + 4$. This is a quadratic and so we can find its roots by means of completing the square. We get 2 and $\frac{2}{3}$. Thus the roots are $4, 2, \frac{2}{3}$.
(b) We are given that -1 and -2 are roots and so $(x + 1)(x + 2)$ is a factor. Dividing out we get $x^2 - x + 1$. The roots of this quadratic are $\frac{1}{2}(1 \pm i\sqrt{3})$. Thus the roots are $-1, -2, \frac{1}{2}(1 \pm i\sqrt{3})$.

(3) The required cubic is $(x - 2)(x + 3)(x - 4) = x^3 - 3x^2 - 10x + 24$.

(4) The required quartic is $(x - i)(x + i)(x - 1 - i)(x - 1 + i) = x^4 - 2x^3 + 3x^2 - 2x + 2$.

(5) By assumption $x^3 + ax^2 + bx + c = (x - x_1)(x - x_2)(x - x_3)$. Multiplying out the RHS we get

$$x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_1x_3 + x_2x_3)x - x_1x_2x_3.$$

Now equate with the coefficients of the LHS to get

$$a = -(x_1 + x_2 + x_3),\ b = x_1x_2 + x_1x_3 + x_2x_3,\ c = -x_1x_2x_3.$$

(6) The polynomial in question has real coefficients and so the complex roots come in complex conjugate pairs. It follows therefore that $3 - i\sqrt{2}$ is also a root. Thus $(x - 3 - i\sqrt{2})(x - 3 + i\sqrt{2}) = x^2 - 6x + 11$ is a factor. Dividing out we get $x^2 + 7x + 6$. This factorizes as $(x + 1)(x + 6)$ and so its roots are -1 and -6. Thus the roots are $-1, -6, 3 + i\sqrt{2}, 3 - i\sqrt{2}$.

(7) The polynomial in question has real roots and so $1 + i\sqrt{5}$ is another root. Thus $(x - 1 - i\sqrt{5})(x - 1 + i\sqrt{5})$ is a factor. Dividing out by $x^2 - 2x + 6$ we get $x^2 - 2$. This factorizes as $(x - \sqrt{2})(x + \sqrt{2})$. Thus the roots are $1 + i\sqrt{5}, 1 - i\sqrt{5}, \sqrt{2}, -\sqrt{2}$.

(8) (a) -1 is a root and so $x + 1$ is a factor. We can write the polynomial as the product $(x + 1)(x^2 + 1)$. The roots are therefore $-1, i, -i$.
(b) -2 is a root and so $x + 2$ is a factor. We can therefore write the polynomial as $(x + 2)(x^2 - 3x + 3)$. The roots are therefore $-2, \frac{1}{2}(3 + i\sqrt{3}), \frac{1}{2}(3 - i\sqrt{3})$.
(c) 1 is a root and so we get a first factorization of our polynomial as $(x - 1)(x^3 + 5x + 6)$. -1 is a root of $x^3 + 5x + 6$. We may therefore factorize $x^3 + 5x + 6 = (x + 1)(x^2 - x + 6)$. The quadratic has the roots $\frac{1}{2}(1 \pm i\sqrt{23})$. The roots are therefore $1, -1, \frac{1}{2}(1 + i\sqrt{23}), \frac{1}{2}(1 - i\sqrt{23})$.

(9) (a) Show that 1 is a root and then divide by $x-1$ to get the required factorization $(x-1)(x^2+x+1)$. Observe that x^2+x+1 has complex roots and so cannot be factorized further in terms of real polynomials.
(b) This is a difference of two squares and so a first factorization is $(x^2+1)(x^2-1)$ and thus the required factorization is $(x-1)(x+1)(x^2+1)$. Observe that x^2+1 has complex roots and so cannot be factorized further in terms of real polynomials.
(c) Put $y=x^2$ and Solve $y^2+1=0$. The solutions are $\pm i$. Thus $x^2=i$ or $x^2=-i$. Taking square roots, yields $x=\frac{1}{\sqrt{2}}(1+i), \frac{-1}{\sqrt{2}}(1+i), \frac{1}{\sqrt{2}}(-1+i), \frac{1}{\sqrt{2}}(1-i)$. Now we collect together complex conjugate pairs, to get $(x-\frac{1}{\sqrt{2}}(1+i))(x-\frac{1}{\sqrt{2}}(1-i)) = x^2-\sqrt{2}x+1$, and $x^2+\sqrt{2}x+1$. Thus $x^4+1=(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$. [A student made a nice observation that leads to a much quicker solution to this question. Observe that $x^4+1=(x^2+1)^2-2x^2$. How does this help?]

(10) $1, i, -1, -i$.

(11) Let $\omega = \frac{1}{2}\left(1+i\sqrt{3}\right)$. Then the roots are $1, \omega, \omega^2, \omega^3, \omega^4, \omega^5$.

(12) Let $\omega = \frac{1}{\sqrt{2}}(1+i)$. Then the roots are $1, \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6, \omega^7$.

(13) This question shows the sorts of insights that are needed to calculate explicit radical expressions for nth roots.

(14) (a) The cube roots are
- $2(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})=2i$.
- $2(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6})=-\sqrt{3}-i$.
- $2(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6})=\sqrt{3}-i$.

(b) The fourth roots are
- $\sqrt[4]{2}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=i\sqrt[4]{2}$.
- $\sqrt[4]{2}\left(\cos\pi+i\sin\pi\right)=-\sqrt[4]{2}$.
- $\sqrt[4]{2}\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)=-i\sqrt[4]{2}$.
- $\sqrt[4]{2}\left(\cos 2\pi+i\sin 2\pi\right)=\sqrt[4]{2}$.

(c) Observe that $1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$. The sixth roots are
- $\sqrt[12]{2}\left(\cos\frac{\pi}{24}+i\sin\frac{\pi}{24}\right)$.
- $\sqrt[12]{2}\left(\cos\frac{9\pi}{24}+i\sin\frac{9\pi}{24}\right)$.
- $\sqrt[12]{2}\left(\cos\frac{17\pi}{24}+i\sin\frac{17\pi}{24}\right)$.
- $\sqrt[12]{2}\left(\cos\frac{25\pi}{24}+i\sin\frac{25\pi}{24}\right)$.
- $\sqrt[12]{2}\left(\cos\frac{33\pi}{24}+i\sin\frac{33\pi}{24}\right)$.
- $\sqrt[12]{2}\left(\cos\frac{41\pi}{24}+i\sin\frac{41\pi}{24}\right)$.