Solutions 9

(1) (a) This is a consistent system of equations with infinitely many solutions. The solution set is
$$\left\{ \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix} : \lambda \in \mathbb{R} \right\}.$$
(b) This is an inconsistent system that has no solutions.
(c) This is a consistent system with the unique solution
$$\begin{pmatrix} -\frac{3}{5} \\ \frac{14}{5} \\ -\frac{7}{5} \end{pmatrix}.$$
(d) This is a consistent system with the unique solution
$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$
(e) This is a consistent system with a unique solution
$$\begin{pmatrix} -6 \\ 5 \\ -1 \end{pmatrix}.$$
(f) This is a consistent system with infinitely many solutions The solution set is
$$\left\{ \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} : \lambda \in \mathbb{R} \right\}.$$

(2) (a) 5.
(b) 0.
(c) 5.
(d) 2.
(e) -1200.
(f) 33.
(g) 4.
(h) 0.

(3) We have that $(1-x)(3-x)-8=0$. Thus $x^2-4x-5=0$. Hence $(x+1)(x-5)=0$. It follows that $x=-1$ or $x=5$.

(4) x.

(5) (a) $\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$.

(b) $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$.

(c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$.

(d) $\begin{pmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ 1 & -1 & -1 \\ -\frac{2}{5} & \frac{3}{5} & \frac{4}{5} \end{pmatrix}$.

(e) $\begin{pmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$.

(f) $\begin{pmatrix} \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \\ \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \end{pmatrix}$.

(6) (a) By definition, $AA^{-1} = I$. Thus by properties of determinants

$$\det(A)\det(A^{-1}) = 1.$$

It follows that $\det(A) \neq 0$ if and only if $\det(A^{-1}) \neq 0$, and that in this case $\det(A^{-1}) = \det(A)^{-1}$.

(b) By properties of deterninants, $\det(A) = \det(A^T)$. Thus $\det(A) \neq 0$ if and only if $\det(A^T) \neq 0$. Now $AA^{-1} = I = A^{-1}A$ and so $(A^{-1})^T A^T = I = A^T(A^{-1})^T$ from properties of the transpose. But these equations say that the inverse of A^T is $(A^{-1})^T$ and we have proved the result.

(7) (a)

$$C^{-1} = \begin{pmatrix} 3 & -2 & -2 \\ -1 & 1 & 1 \\ 2 & -1 & -2 \end{pmatrix}.$$

(b)

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

(8) (a) The characteristic polynomial is $x^2 - 5x + 4$. The eigenvalues are 1 and 4.

(b) The characteristic polynomial is $-x^3 + 6x^2 - 3x - 10$. The eigenvalues are 2, -1 and 5.