Solutions to Homework 1

1. To prove that $\sqrt{5}$ is irrational, we adopt the same methodology as the proofs that $\sqrt{2}$ is irrational and $\sqrt{3}$ is irrational (exercises!).

We will need the following lemma:

<u>Lemma</u>

If a is a natural number and $5|a^2$ then $5|a$.
(Proof later). [1]

Now for the main course.

1. Assume that $\sqrt{5}$ is rational.
2. Then we can write $\sqrt{5} = \frac{a}{b}$, where this is a fraction in its lowest terms.
(<u>this is a crucial assumption</u>). [1]
3. Square both sides of the equation and rearrange to get $5b^2 = a^2$.
4. We see that $5|a^2$ and so $5|a$ by the Lemma. We can therefore write $a = 5x$ for some x. [1]
5. Substitute this expression for a into (3) to get $b^2 = 5a^2$.
[1]
6. We see that $5|b^2$ and so $5|b$ by Lemma.
7. By (4) and (6), we have shown that $5|a$ and $5|b$. But this contradicts (2).
[1]

It follows that (1) is false and so $\sqrt{5}$ is, in fact, irrational. ∎

Proof of Lemma

We can write $a = 5q + r$ where $0 \leq r < 5$.
Thus $a^2 = 5(5q^2 + 2qr) + r^2$.

If $5 | a$ then $r = 0$ and so $5 | a^2$.

If $5 \nmid a$ then $r = 1, 2, 3$ or 4. In each case, $5 \nmid a^2$. [1]

Thus if $5 | a^2$ then $5 | a$, as required. ■

[Many people assume that if $m | n^2$ then $m | n$. This is false in general.
For example, $4 | 100$ but $4 \nmid 10$.
Thus the claim $5 | a^2 \Rightarrow 5 | a$ needs proving and cannot be assumed.]

2. Observe that if we write $n = 10q + r$ where $0 \le r \le 9$ then r is the units digit. Thus

$$n^5 = (10q + r)^5 .$$

We could use the binomial theorem to expand this or we could simply observe directly that

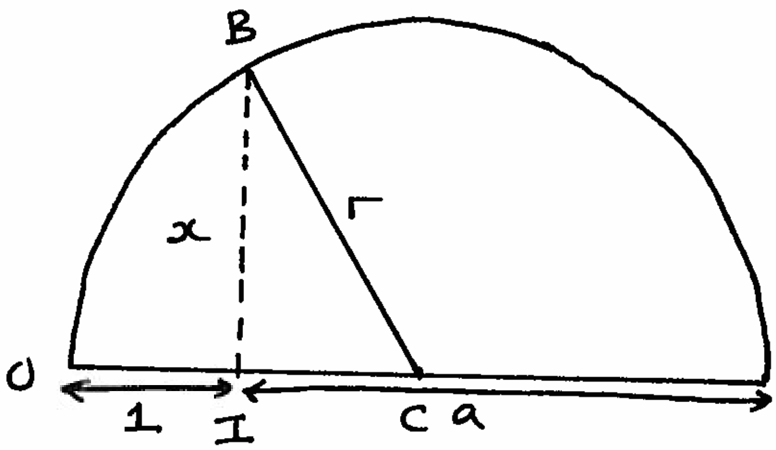
$$n^5 = 10(\text{stuff}) + r^5 .$$

It follows that the units digit of n^5 is equal to the units digit of r^5. 3

We therefore need only verify the claim for $0 \le r \le 9$. The following table shows this. 1

r	0	1	2	3	4	5	6	7	8	9
r^5	0	1	32	243	1024	3125	7776	16807	32768	59049

3.



[I have drawn the diagram in one of the two possible configurations.]

Let the radius of the circle be r.

Then $r = \frac{1}{2}(1+a)$ $(*)$

By Pythagoras' theorem

$$x^2 + IC^2 = r^2$$

But $IC = OC - 1 = r - 1$.

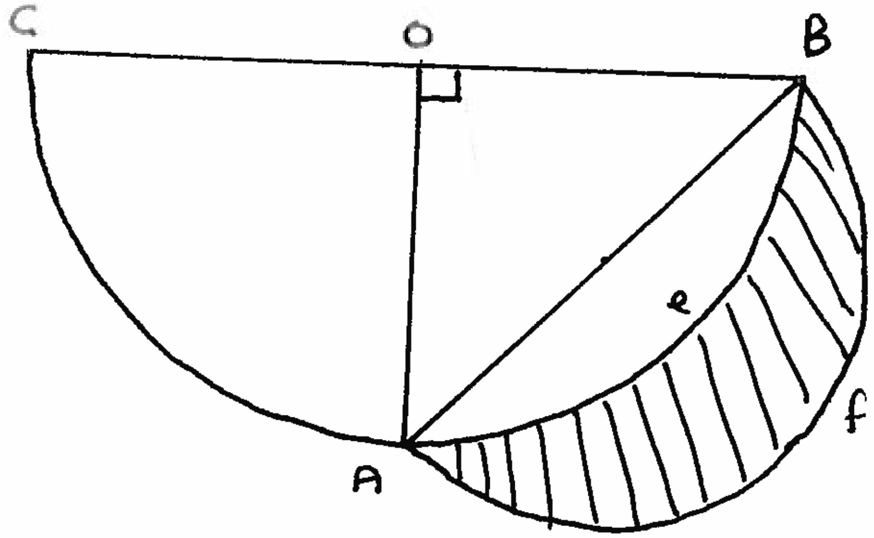
Thus

$$\begin{aligned} x^2 &= r^2 - (r-1)^2 \\ &= r^2 - (r^2 - 2r + 1) \\ &= 2r - 1 = a. \quad \text{by } (*). \end{aligned}$$

It follows that $\underline{\underline{x = \sqrt{a}}}$ (choosing +ve root).

[5]

4.



Let $OB (= OC = OA) = r$.

Area of quarter circle $OAeB = \frac{1}{4}\pi r^2$ (1)

Length of AB by Pythagoras' theorem $= r\sqrt{2}$.

Thus area of semi-circle on AB =

$$\frac{1}{2}\pi\left(\frac{r\sqrt{2}}{2}\right)^2 = \frac{1}{4}\pi r^2. \quad (2)$$

Area of lune = area $OAfB$ − area $OAeB$

= area of $\triangle OAB$ + area of semi-circle on AB (2)

− area of quarter circle $OAeB$ (1)

= area of $\triangle OAB$ since (1) = (2). $\blacksquare$

[5]