Solutions to Homework 2

1. Roots are $\pm(2+5i)$ [4 marks]

[You must write complex no's in the above form].

Check $(2+5i)^2 = -21+20i$ [1 mark]

Calculations Let $(x+iy)^2 = -21+20i$. Three equations arise ① $x^2-y^2=-21$ ② $2xy=20$ ③ $x^2+y^2=29$.

①+③ gives $x^2=4$ and so $x=\pm 2$. From ② $y=\frac{10}{x}$.

This gives us two solutions to x and y and so the complex no's above.

2. Additional roots are: $1, 2, -i, 1+i, 1-i$ [$2\frac{1}{2}$ marks]

Required factorization: $(x-1)(x-2)(x^2+1)(x^2-2x+2)$ [$2\frac{1}{2}$ marks]

irreducible, real quadratics (discriminants < 0)

(Method in notes).

3. Determinant $= 2$. [1 mark]

Adjugate $= \begin{pmatrix} -24 & 36 & 10 \\ 20 & -30 & -8 \\ -5 & 8 & 2 \end{pmatrix}$ [3 marks]

Inverse $= \frac{1}{2}\begin{pmatrix} -24 & 36 & 10 \\ 20 & -30 & -8 \\ -5 & 8 & 2 \end{pmatrix}$ [1 mark]

4. The correct logic in the solution is essential.

Since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ commutes with all 2×2 real matrices, it must commute in particular with both $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Thus

- $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$
  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$
  These are equal and $b = 0$ and $c = 0$.

- $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix}$
  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}$
  These are equal and so $c = 0$, and $a = d$

It follows that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = aI$.

Observe that, conversely, aI commutes with all 2×2 matrices.

Finally, observe that $\begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix} = a^2$. We are given that $a^2 = 1$. We have proved that the required two matrices are $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

[1 mark] for the matrices

[4 marks] for perfect, 100%, 24 carat logic