Solutions 4

(1) (a) Start on the lefthand side. Expand brackets using the distributivity laws to get $aa + ba + ab + bb$. Commutativity tells us that $ba = ab$. Now simplify using standard abbreviations to get $a^2 + 2ab + b^2$.

(b) Start on the lefthand side. Write $(a+b)^3 = (a+b)(a+b)^2$. Use the distributivity laws applied to part (a) above to get $aa^2 + a2ab + ab^2 + ba^2 + b2ab + bb^2$. Using commutativity and abbreviations this is just $a^3 + 2a^2b + ab^2 + ba^2 + 2ab^2 + b^3$. Now use commutativity and abbreviations to get $a^3 + 3a^2b + 3ab^2 + b^3$.

(c) Start on the righthand side. Expand brackets using the distributivity laws to get $aa + ba - ab - bb$. Using commutativity the middle terms cancel, and using abbreviations we get $a^2 - b^2$.

(d) Expand the lefthand side using the distributivity laws to get $a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2$. Now expand the righthand side using the distributivity laws and commutativity to get $a^2c^2 - 2abcd + b^2d^2 + a^2d^2 + 2abcd + b^2c^2$. By commutativity two terms cancel. We have shown that the lefthand side is equal to the righthand side.

(2) If $a = 0$ then the result is not true since $0 \times 1 = 0 \times 2$ and $1 \neq 2$. Assume that $a \neq 0$. Then the result is true because from $ab = ac$ we get that $a^{-1}(ab) = a^{-1}(ac)$. By associativity, $a^{-1}(ab) = (a^{-1}a)b$ and $a^{-1}(ac) = (a^{-1}a)c$. But $a^{-1}a = 1$. Thus $1b = 1c$ and so $b = c$.

(3) (a) 16 two real roots.

(b) 0 repeated root.

(c) -16 no real roots.

(4) (a) Complete the square $x^2 + 10x + 16 = (x+5)^2 - 25 + 16 = (x+5)^2 - 9$. Thus $x = -2, -8$.

(b) Complete the square $x^2 + 4x + 2 = (x+2)^2 - 4 + 2 = (x+2)^2 - 2$. Thus $x = -2 \pm \sqrt{2}$.

(c) Complete the square $2x^2 - x - 7 = 2[x^2 - \frac{1}{2}x - \frac{7}{2}] = 2[(x - \frac{1}{4})^2 - \frac{1}{16} - \frac{7}{2}] = 2[(x - \frac{1}{4})^2 - \frac{57}{16}]$. Thus $x = \frac{1 \pm \sqrt{57}}{4}$.

(5) We are given that $x + y = a$ and $xy = b$. Suppose that $b \neq 0$. Then $x, y \neq 0$. Put $y = \frac{b}{x}$. This leads to the quadratic $x^2 - ax + b = 0$. Solving this yields $x = \frac{1}{2}\left(a + \sqrt{a^2 - 4b}\right)$ and

$y = \frac{1}{2}\left(a - \sqrt{a^2 - 4b}\right)$, where we note that it doesn't matter which value is assigned to x as long as the corresponding value is assigned to y. Suppose that $b = 0$. Then without loss of generality, we may assume that $x = 0$. Then $y = a$.

(6) We have that $2x_1 = -b + \sqrt{D}$ and $2x_2 = -b - \sqrt{D}$. Thus $2x_1 - 2x_2 = \sqrt{D} + \sqrt{D}$. It follows that $x_1 - x_2 = \sqrt{D}$ and so $\Delta = (x_1 - x_2)^2$.