Vectors

This is a version of parts of Sections 9.1, 9,2, 9.3.

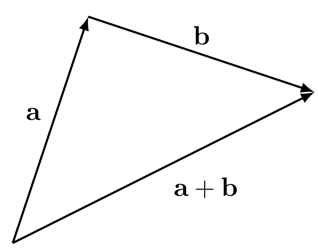
Vectors geometrically

Definition Two directed line segments which are parallel, have the same length and point in the same direction are said to represent the same *vector*.

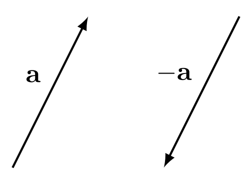
Vectors are denoted by bold letters $\mathbf{a}, \mathbf{b}, \ldots$. If P and Q are points then the directed line segment from P to Q is written $\overrightarrow{PQ}$. The *zero vector* $\mathbf{0}$ is represented by any degenerate directed line segment $\overrightarrow{PP}$ which is just a point. It has zero length and, exceptionally, no uniquely defined direction. *Vectors are arrows that can be moved parallel to themselves in space.*

Vector arithmetic

Let $\mathbf{a}$ and $\mathbf{b}$ be vectors. Their *sum* is defined as follows: slide the vectors parallel to themselves so that the terminal point of $\mathbf{a}$ touches the initial point of $\mathbf{b}$. The directed line segment from the initial point of $\mathbf{a}$ to the terminal point of $\mathbf{b}$ represents the vector $\mathbf{a} + \mathbf{b}$.



If $\mathbf{a}$ is a vector, then $-\mathbf{a}$ is defined to be the vector with the same length as $\mathbf{a}$ but pointing in the opposite direction.



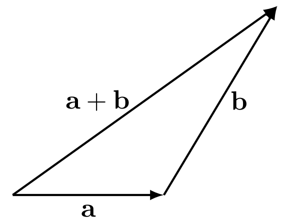
Theorem [Properties of vector addition]

(1) $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$.
(2) $\mathbf{0} + \mathbf{a} = \mathbf{a} = \mathbf{a} + \mathbf{0}$.
(3) $\mathbf{a} + (-\mathbf{a}) = \mathbf{0} = (-\mathbf{a}) + \mathbf{a}$.
(4) $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.

As usual, subtraction is defined in terms of addition

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}).$$

Let $\mathbf{a}$ be a vector. Denote by $\|\mathbf{a}\|$ the *length* of the vector. If $\|\mathbf{a}\| = 1$ then $\mathbf{a}$ is called a *unit vector*. We always have $\|\mathbf{a}\| \geq 0$ with $\|\mathbf{a}\| = 0$ if and only if $\mathbf{a} = \mathbf{0}$. By results on triangles



the *triangle inequality* $\|\mathbf{a}+\mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$ holds.

We now define *multiplication of a vector by a scalar*. Let λ be a scalar and $\mathbf{a}$ a vector. If $\lambda = 0$ define $\lambda\mathbf{a} = \mathbf{0}$. If $\lambda > 0$ define $\lambda\mathbf{a}$ to have the same direction as $\mathbf{a}$ and length $\lambda \|\mathbf{a}\|$. If $\lambda < 0$ define $\lambda\mathbf{a}$ to have the opposite direction to $\mathbf{a}$ and length $(-\lambda) \|\mathbf{a}\|$. Observe that in all cases

$$\|\lambda\mathbf{a}\| = |\lambda| \, \|\mathbf{a}\| \, .$$

If $\mathbf{a}$ is non-zero, define

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{\|\mathbf{a}\|},$$

a unit vector in the same direction as $\mathbf{a}$. The process of constructing $\hat{\mathbf{a}}$ from $\mathbf{a}$ is called *normalization*. Vectors that differ by a scalar multiple are said to be *parallel*.

Theorem [Properties of scalar multiplication] Let λ and μ be scalars.

(1) $0\mathbf{a} = \mathbf{0}$.
(2) $1\mathbf{a} = \mathbf{a}$.
(3) $(-1)\mathbf{a} = -\mathbf{a}$.
(4) $(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$.
(5) $\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$.
(6) $\lambda(\mu\mathbf{a}) = (\lambda\mu)\mathbf{a}$.

Inner products

We now introduce a notion that will enable us to measure angles. It is based on the idea of the perpendicular projection of a line segment onto another line. Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors. If $\mathbf{a}, \mathbf{b} \neq \mathbf{0}$ denote the angle between them by θ and define

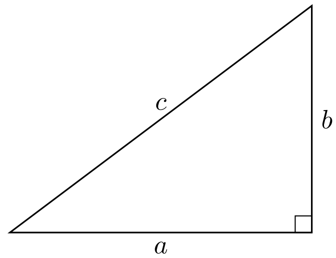
$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta.$$

If either $\mathbf{a}$ or $\mathbf{b}$ is zero, define $\mathbf{a} \cdot \mathbf{b} = 0$. We call $\mathbf{a} \cdot \mathbf{b}$ the *inner product* of $\mathbf{a}$ and $\mathbf{b}$. It is important to remember that it is a scalar and not a vector. The inner product $\mathbf{a} \cdot \mathbf{a}$ is abbreviated $\mathbf{a}^2$. We say that non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are *orthogonal* if the angle between them is a right angle. The key property of the inner product is that for non-zero $\mathbf{a}$ and $\mathbf{b}$ we have that $\mathbf{a} \cdot \mathbf{b} = 0$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

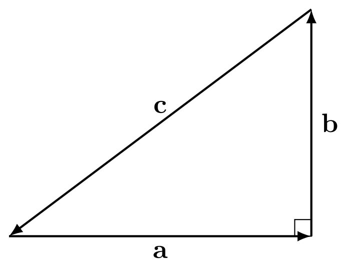
Theorem [Properties of the inner product]

(1) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.
(2) $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$.
(3) $\lambda(\mathbf{a} \cdot \mathbf{b}) = (\lambda\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\lambda\mathbf{b})$ for any scalar λ.
(4) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$.

Example We prove Pythagoras' theorem, that $a^2 + b^2 = c^2$ in the triangle below, using vectors in just a few lines.



Choose vectors as shown in the diagram below.



Then $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$. Thus $\mathbf{a} + \mathbf{b} = -\mathbf{c}$. Now

$$(\mathbf{a} + \mathbf{b})^2 = (-\mathbf{c}) \cdot (-\mathbf{c}) = \|\mathbf{c}\|^2$$

and

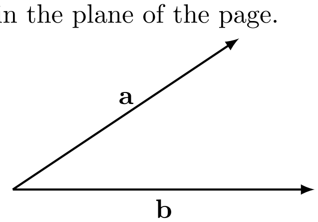
$$(\mathbf{a} + \mathbf{b})^2 = \|\mathbf{a}\|^2 + 2\mathbf{a} \cdot \mathbf{b} + \|\mathbf{b}\|^2 .$$

This is equal to $\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2$ because $\mathbf{a} \cdot \mathbf{b} = 0$. It follows that

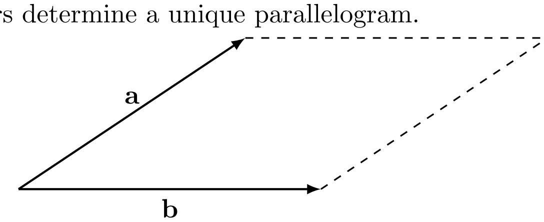
$$\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 = \|\mathbf{c}\|^2 .$$

Vector products

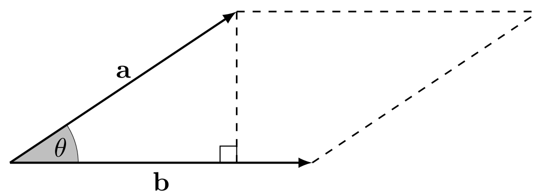
We define another binary operation on the set of vectors. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors. A unique vector $\mathbf{a} \times \mathbf{b}$ is defined in terms of these two which contains information about the area enclosed by $\mathbf{a}$ and $\mathbf{b}$ and about the orientation in space of the plane determined by $\mathbf{a}$ and $\mathbf{b}$. If either $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$, define $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$. If $\theta = 0$ define $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. There is no loss of generality in assuming that $\mathbf{a}$ and $\mathbf{b}$ lie in the plane of the page.



These two vectors determine a unique parallelogram.



With reference to the diagram below



the area enclosed is $\|\mathbf{a}\|\,\|\mathbf{b}\|\sin\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. For the unit vector $\mathbf{n}$, we need a vector orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. There are only two choices: either we choose the vector pointing out of the page or we choose the vector pointing into the page. We choose the vector pointing into the page. With all this in mind, define

$$\mathbf{a} \times \mathbf{b} = (\|\mathbf{a}\|\,\|\mathbf{b}\|\sin\theta)\,\mathbf{n}.$$

We call $\mathbf{a} \times \mathbf{b}$ the *vector product* of $\mathbf{a}$ and $\mathbf{b}$. The key property of the vector product is that for non-zero vectors $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are parallel.

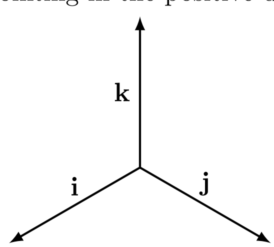
Theorem [Properties of the vector product]

(1) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.
(2) $\lambda(\mathbf{a} \times \mathbf{b}) = (\lambda\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\lambda\mathbf{b})$ for any scalar λ.
(3) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.

The set of vectors equipped with the operations defined in this section is called *three-dimensional Euclidean space*. It is the space of the *Elements* in modern dress.

Vectors algebraically

Set up a *cartesian coordinate system* consisting of x-, y- and z-axes. We orient the system so that in rotating the x-axis clockwise to the y-axis, we are looking in the direction of the positive z-axis. Let $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ be unit vectors parallel to the x-, y- and z-axes, respectively, pointing in the positive directions.



Every vector $\mathbf{a}$ can now be uniquely written in the form

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

for some scalars a_1, a_2, a_3. This is achieved by *orthogonal projection* of the vector $\mathbf{a}$ (moved so that it starts at the origin) onto each of the three coordinate axes. The numbers a_i are called the *components* of $\mathbf{a}$ in each of the three directions.

Theorem

(1) If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then $\mathbf{a} = \mathbf{b}$ if and only if $a_i = b_i$ for $1 \le i \le 3$. That is, corresponding components are equal.
(2) $\mathbf{0} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$.

(3) If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k}.$$

(4) If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ then $\lambda\mathbf{a} = \lambda a_1\mathbf{i} + \lambda a_2\mathbf{j} + \lambda a_3\mathbf{k}$ for any scalar λ.

The vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ can equally well be represented by the column vector

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

In this notation, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ can be represented by the column vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

respectively.

Theorem [Inner products] *Let* $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ *and* $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$. *Then*

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ then $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

Theorem [Vector products] *Let* $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ *and* $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$. *Then*

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

It is important to note that this 'determinant' can only be expanded along the first row.

Determinants

We now by define a third product that is simply a combination of inner and vector products. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three vectors. Then $\mathbf{b} \times \mathbf{c}$ is a vector and so $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is a scalar. Define $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. This is called the *scalar triple product.*

Theorem [Scalar triple products and determinants] *Let*

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},\ \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \text{ and } \mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}.$$

Then

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Thus the properties of scalar triple products are the same as the properties of 3×3 *determinants.*

The connection between determinants and scalar triple products will enable us to describe the geometric meaning of determinants. Start with 1×1 matrices. The determinant of the matrix (a) is just a. The *length* of a is $|a|$, the absolute value of the determinant of (a).

Theorem [2×2 determinants] *Let* $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ *and* $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ *be a pair of plane vectors. Then the area of the parallelogram determined by these vectors is the absolute value of the determinant*

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

Theorem [3×3 determinants] *Let*

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},\ \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \text{ and } \mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$$

be three vectors. Then the volume of the parallelepiped determined by these three vectors is the absolute value of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

or its transpose.

We refer to the diagram below.