COMPLEX FUNCTIONS of COMPLEX VARIABLE

Revision: a complex number is a number of the form

$z = a + ib$, i = imaginary unit, $i^2 = -1$

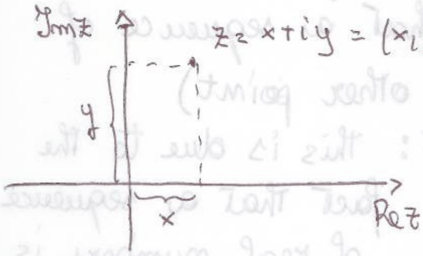
a = real part, b = imaginary part, $a, b \in \mathbb{R}$

REM 1: despite the name the imaginary part is a real number. We denote $a = \mathrm{Re}\, z$, $b = \mathrm{Im}\, z$.

REM 2: given $z_1 = a_1 + i b_1$ and $z_2 = a_2 + i b_2$,

$z_1 = z_2$ if and only if $a_1 = a_2$ and $b_1 = b_2$

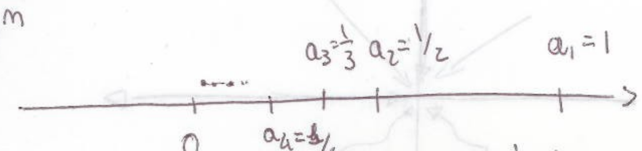
- recall that we can always think of a complex number $z = x + iy$ as of a point in the complex plane:



e.g. $z = 1 - 2i$ can be identified with the point $(1, -2)$.

- In the same way in which we study sequences of real numbers, we can also study sequences of complex numbers: $z_n = a_n + i b_n$; while a sequence of real numbers is a sequence of points on the x-axis

e.g. $a_n = \frac{1}{n}$



and so on ...

a sequence of complex numbers is a sequence of points on the plane. eg. $z_n = \frac{2}{n} + i\left(\frac{1}{n+1}\right)$ can be thought of as the sequence of points $\left(\frac{2}{n}, \frac{1}{n+1}\right) = z_n$

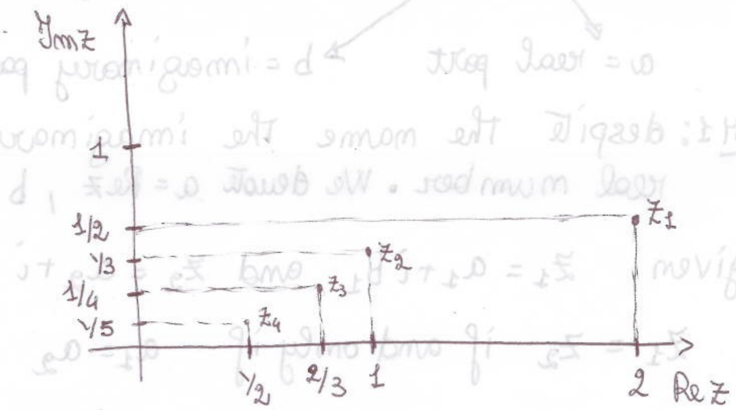
$z_1 = \left(2, \frac{1}{2}\right)$

$z_2 = \left(1, \frac{1}{3}\right)$

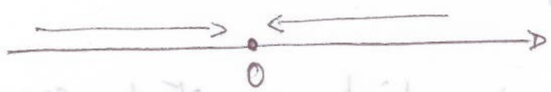
$z_3 = \left(\frac{2}{3}, \frac{1}{4}\right)$

$z_4 = \left(\frac{1}{2}, \frac{1}{5}\right)$

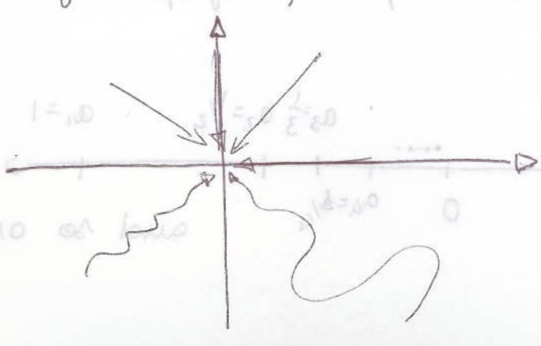
and so on



• The main difference between a sequence of real numbers and a sequence of complex numbers is that a sequence of real numbers can approach zero (or any other point) either from the left or from the right: this is due to the fact that a sequence of real numbers is "constrained to move on the real line".



A sequence of complex numbers has instead the freedom to move in the plain, so it can approach the origin $(0,0)$ from ∞ many directions and following many possible trajectories.



• A sequence of complex numbers $z_n = a_n + i b_n$ <u>tends to zero</u> (or better, to $(0,0)$) if and only if

both $a_n \to 0$ and $b_n \to 0$

• Before getting into the matter of complex valued functions, let me recall another important definition.

<u>Def</u>: let $f(x)$ be a scalar function and $x \in \mathbb{R}$. f is said to be <u>differentiable at the point x</u> if the limit

(*) $\displaystyle\lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta}$ exist and is finite.

In this case we denote $\displaystyle f'(x) = \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta}$.

$f(x)$ is <u>differentiable</u> if it is differentiable $\forall x$.

<u>IMPORTANT REMARK</u>: In (*) we are in fact calculating two limits, $\displaystyle\lim_{\delta \to 0^+} \frac{f(x+\delta) - f(x)}{\delta}$ and $\displaystyle\lim_{\delta \to 0^-} \frac{f(x+\delta) - f(x)}{\delta}$.

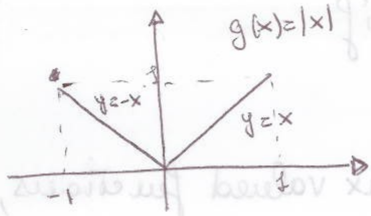
In other words, when we write $\delta \to 0$, we are including two possibilities, $\delta \to 0^+$ and $\delta \to 0^-$. So (*) can be rephrased as follows: the function $f(x)$ is differentiable at x if the limits

$$f'_+(x) = \lim_{\delta \to 0^+} \frac{f(x+\delta) - f(x)}{\delta} \quad \text{and} \quad f'_-(x) = \lim_{\delta \to 0^-} \frac{f(x+\delta) - f(x)}{\delta},$$

exist, are finite and coincide:

$$f'_+(x) = f'_-(x) .$$

Typical example: $g(x) = |x|$ is not differentiable at $x = 0$.



Indeed

$$\lim_{\delta \to 0^+} \frac{g(0+\delta) - g(0)}{\delta} = \frac{\delta - 0}{\delta} = 1$$

$$\lim_{\delta \to 0^-} \frac{g(0+\delta) - g(0)}{\delta} = \frac{-\delta}{\delta} = -1$$

$$g(x) = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$$

if $\delta \to 0^-$ then $\delta < 0$

so $g(0+\delta) = g(\delta) = -\delta$

so $g'_+(0) = 1$, $g'_-(0) = -1$

hence these two limits exist, are finite, but they do not coincide, which causes the dramas of the non differentiability at $x = 0$!!!

- So far we have only considered functions of the type $y = g(x)$, i.e. scalar functions of real variable.

$y \in \mathbb{R}$ $\quad x \in \mathbb{R}$

Now we want to make things more complicated and consider functions

$$w = f(z)$$

$w \in \mathbb{C}$ $\quad z \in \mathbb{C}$

In other words, both the dependent and the independent variable are complex numbers.

EXAMPLE: because $z = x + iy$ we can write

$$f(z) = f(\underbrace{x+iy}_{\text{the input of the function is a complex number}}) = \underbrace{x^2 + i(x-y)}_{\text{The output is still a complex number.}}$$

For this type of functions we want to give a notion of differentiability.

Def: let $h = h_1 + i h_2 = (h_1, h_2)$. A complex valued function $f(z)$, $z \in \mathbb{C}$, is differentiable at the point z if the limit $\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$ $(**)$ exists and is finite.

• Like we did for scalar functions of scalar variable, let us try and understand the limit $(**)$ a bit better.

This time $h \in \mathbb{C}$ so h lives on the complex plane.

$h \to 0$ means that both h_1 and h_2 tend to zero.

Also, h can approach the origin along ∞ many directions, so $(**)$ is not just two limits in one like for functions of one real variable.

Saying that the limit $(**)$ exists and is finite means that infinitely many limits exist, are finite and coincide !!!

These infinitely many limits are the ones calculated "along the infinite trajectories that tend to zero" -

We will give some examples to make this clear.

Before getting to the examples, some more notation:

Ex: $f(z) = z^2 e^z$, where, as usual, $z = x+iy$

Then $f = (x+iy)^2 e^{x+iy} = (x^2-y^2+2ixy)\, e^x (\cos y + i \sin y) =$

$= e^x (x^2-y^2)\cos y + e^x (x^2-y^2)\, i \sin y + 2ixy e^x \cos y$

$- 2xy e^x \sin y =$

$$= \underbrace{e^x (x^2-y^2)\cos y - 2xy e^x \sin y}_{\substack{\text{real part} \\ u(x,y)}} + i \underbrace{\left[e^x (x^2-y^2)\sin y + 2xy e^x \cos y \right)}_{\substack{\text{Imaginary part} \\ v(x,y)}}$$

so $f(z) = f(x+iy) = u(x,y) + i\, v(x,y)$

We can always write a complex-valued function as the real part + i (imaginary part) where the real part and imaginary part are scalar functions of two variables -

Bear this in mind for later on...

Now let's go back to our limits:

Ex: say wether the function $f(z) = z$ is differentiable in complex sense.

We need to check wether the limit

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$ exists and is finite.

Recall $h = h_1 + ih_2$. If $f(z) = z$ then

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0} \frac{z+h-z}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

The limit exists and is finite so $f(z) = z$ is differentiable in complex sense $\forall z$.

Ex: $f(z) = \operatorname{Re} z$. In other words, if $z = x+iy$

then $f(z) = f(x+iy) = \operatorname{Re}(x+iy) = x$

So we will be looking at $f(x+iy) = x$

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0} \frac{x+h_1 - x}{h_1 + ih_2} = \lim_{h\to 0} \frac{h_1}{h_1+ih_2} \quad (*)$$

$h = h_1 + ih_2$ so $z+h = x+h_1 + i(y+h_2)$

Now if we choose $h = (\delta, 0)$, (which means that we let h approach the origin along the x axis)

i.e. $h = \delta + i\cdot 0 = \delta$ then $h \to 0 \iff \delta \to 0$

so from $(*)$ we get $\lim_{h\to 0} \frac{h_1}{h_1+0} = \lim_{h_1\to 0} \frac{h_1}{h_1} = 1$ (1)

If we choose $h = (0, \delta)$, i.e., $h = 0 + i\delta = i\delta$

or, in other words, if we let h approach the origin along the y axis, then from $(*)$

$\lim_{\delta\to 0} \frac{0}{0+i\delta} = 0$ (2) So (1) and (2) do not coincide!

Hence the function $f(z) = \mathrm{Re}(z)$ is not differ. in complex sense! We could also have tried $h = (\delta, \delta)$ so from (*) we get $\lim_{\delta \to 0} \frac{\delta}{\delta + i\delta} = \frac{\delta}{\delta(1+i)} = \frac{1}{1+i}$ a different value!!

REM: if you find two different trajectories $h \to 0$ and say $\tilde{h} \to 0$ such that

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \neq \lim_{\tilde{h} \to 0} \frac{f(z+\tilde{h}) - f(z)}{\tilde{h}}$$

Then you are done, f is not differentiable.

But if

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{\tilde{h} \to 0} \frac{f(z+\tilde{h}) - f(z)}{\tilde{h}}$$

This DOES NOT imply that f is differentiable!

Because there might be another path that tends to zero s.t. the limit along that path differs from the other two!

Def: a function $f(z)$ which is differentiable in complex sense is called HOLOMORPHIC or ANALYTIC

If you are still wondering "what is so special about holomorphic functions?" here is the next thing.

Theorem: let $f(z) = u(x,y) + i\,v(x,y)$. $f(z)$ is holomorphic if and only if the following relations hold:

$$(C\text{-}R)\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$ called the CAUCHY - RIEMANN relations

Before proving this Theorem, let's see a consequence of it.

Corollary: if $f(z) = u(x,y) + i\,v(x,y)$ is holomorphic then $\nabla^2 u = 0$ and $\nabla^2 v = 0$. Also, harmonic funct. are the only functions that can be the real or imaginary part of an analytic funct.

Proof of corollary: if $f(z)$ is analytic then (C-R) hold and $u_x = v_y \Rightarrow u_{xx} = v_{xy}$

but also $u_y = -v_x \Rightarrow u_{yy} = -v_{xy}$,

hence $\nabla^2 u = u_{xx} + u_{yy} = v_{xy} - v_{xy} = 0$.

Analogous proof for v.

proof of theorem: we will prove only one implication, namely if $f(z)$ is holomorphic then (CR) hold. So, if $f(z)$ is analytic then $\exists$ finite the limit

$$(L)\ \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}.$$ Moreover, such limit does not depend on the trajectory along which h tends to zero. Because $f(z) = u(x,y) + i\,v(x,y)$ then

$$(L) = \lim_{h\to 0} \frac{u(x+h_1, y+h_2) + i v(x+h_1, y+h_2) - u(x,y) - i\,v(x,y)}{h}$$

$$\boxed{\begin{aligned} f(z) = f(x+iy) = u(x,y) + i\,v(x,y). \quad \text{so } z+h = (x+h_1) + i(y+h_2) \\ \text{and } f(z+h) = u(x+h_1, y+h_2) + i\,v(x+h_1, y+h_2) \end{aligned}}$$

so $(L) = \lim_{h \to 0} \dfrac{u(x+h_1, y+h_2) - u(x,y) + i\left(v(x+h_1, y+h_2) - v(x,y)\right)}{h}$ (E)

Now we calculate the limit (E) when h approaches 0 along the x-axis, i.e. $h = (\delta, 0)$. So from (E):

$$\lim_{\delta \to 0}\left(\frac{u(x+\delta, y) - u(x,y)}{\delta} + i\,\frac{v(x+\delta, y) - v(x,y)}{\delta}\right) = u_x + i v_x \quad (3)$$

Now choose $h = (0, \delta)$. So from (E):

$$\lim_{\delta \to 0}\left(\frac{u(x, y+\delta) - u(x,y)}{i\delta} + i\,\frac{v(x, y+\delta) - v(x,y)}{i\delta}\right) =$$

$$= \frac{u_y}{i} + v_y = -i u_y + v_y \quad (4)$$

Now because f is analytic we must have that the limit (3) is equal to (4). So

$u_x + i v_x = -i u_y + v_y$ This is True

if and only if $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

(by equating the real parts and the imaginary parts).

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The proof of the theorem also says that if f is holomorphic then its derivative is given by

$f'(z) = u_x + i v_x$ or equivalently by

$f'(z) = v_y - i u_y$.

EX: we have shown, using the definition, that
$f(z) = \mathrm{Re}\, z$ is not holomorphic. Let's check using C-R:
$f(z) = f(x+iy) = x$ so $u(x,y) = x$ and $v(x,y) = 0$.
Hence $u_x = 1$ and $v_y = 0$! C-R do not hold
therefore $f(z)$ is not analytic.

Let's see how we can use the corollary:

EX: Given $u(x,y) = x$, determine wether u is the real or imaginary part of an analytic function. If yes, find the most general analytic function f that has u as real part.

$u_x = 1$ and $u_{xx} = 0$, $u_{yy} = 0$ so $\nabla^2 u = 0$
hence u is the real (or imaginary) part of an analytic function. If I want to determine f that has u as real part, I can use C-R, which is gonna give me v and hence f:

$$u_x = 1 = v_y \Rightarrow \int v_y = \int 1 \Rightarrow v = y + h(x)$$

$$u_y = 0 = -v_x \Rightarrow \int v_x = 0 \Rightarrow v = g(y)$$

$$\Rightarrow y + h(x) = g(y) \Rightarrow g(y) = y \text{ and } h(x) = c$$

$$\Rightarrow v(x,y) = y + c$$ where c is a generic constant

$\Rightarrow$ The most general function having u as real part is $f(x+iy) = x + i(y+c)$.

EX: given $u(x,y) = x^2 - y^2$, Test wether u is the real part of an analytic function. If yes, find the most general analytic function having u as real part.

$\partial_x u = 2x \quad \partial_{xx} u = 2 \quad \partial_{yy} u = -2 \Rightarrow \nabla^2 u = 0$

so u is the real (or imaginary) part of an analytic func.

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Rightarrow \begin{cases} v_y = 2x & (1) \\ v_x = 2y & (2) \end{cases}$$

So from (1) we have $\frac{\partial v}{\partial y} = 2x \Rightarrow \int \frac{\partial v}{\partial y} \cdot dy = \int 2x \, dy$

$\Rightarrow v(x,y) = 2xy + h(x)$

From (2): $\int \frac{\partial v}{\partial x} dx = \int 2y \, dx \Rightarrow v(x,y) = 2yx + g(y)$

$\Rightarrow \cancel{2xy} + h(x) = \cancel{2yx} + g(y) \Rightarrow h(x) = g(y)$

$\Rightarrow h = g =$ constant $= c$

$\Rightarrow v(x,y) = 2xy + c$ and the general analytic f that we were seeking is $f = x^2 - y^2 + i(2xy + c)$.

Let us now observe another funny thing about holomorphic functions. What we have just done says that

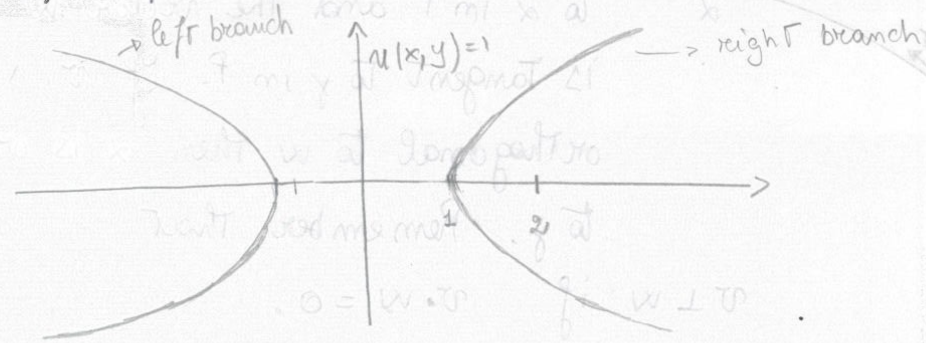
$$f = \underbrace{x^2 - y^2}_{u(x,y)} + i\underbrace{(2xy)}_{v(x,y)} \quad \text{is analytic.}$$

We will see that the family of curves

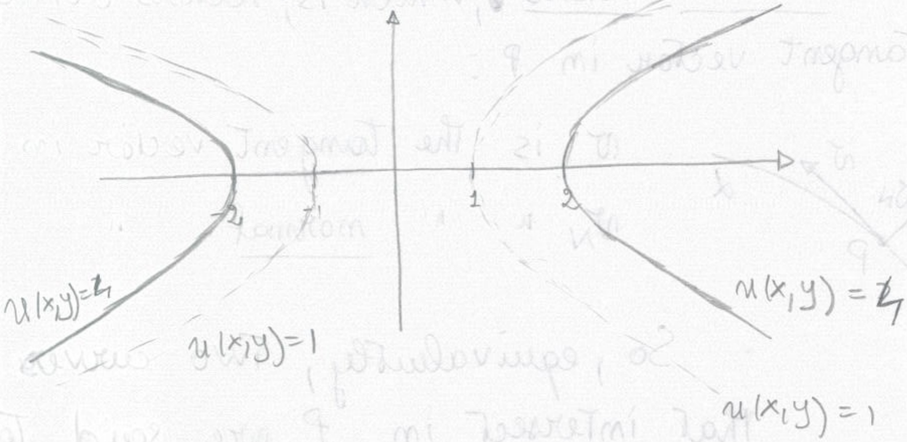
$u(x,y) = c$ and $v(x,y) = k$

with c and k real constants, are families of orthogonal curves.

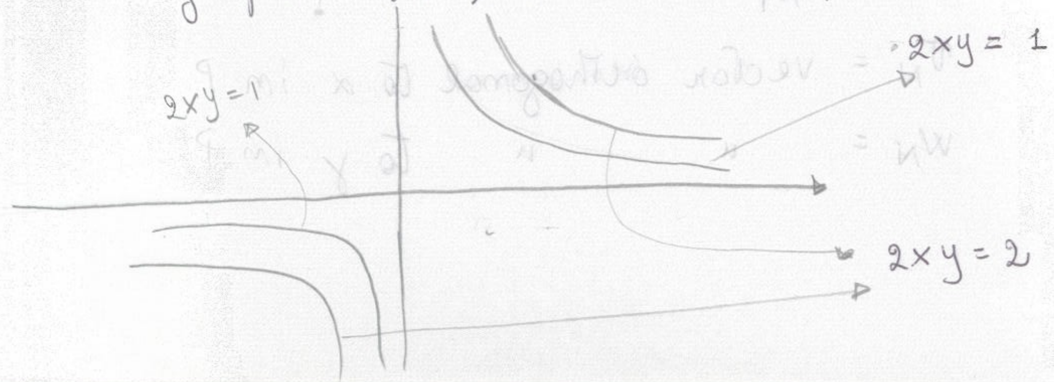
What do I mean by this? Let's get to that step by step.
First of all what do I mean by family of curves?
Take for example $c=1$. Then the equation
$u(x,y)=1$, which is $x^2-y^2=1$, describes an hyperbola



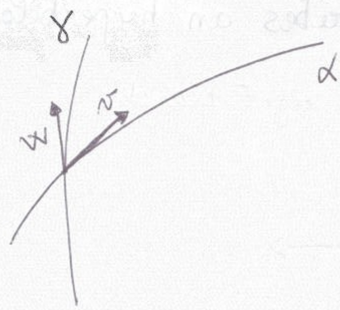
If we take $c=4$ we get another hyperbola, $u(x,y)=4$ i.e.
$x^2-y^2=4$



So, as c varies, $u(x,y)=c$ describes a family of curves.
Same thing for $v(x,y)=k$.



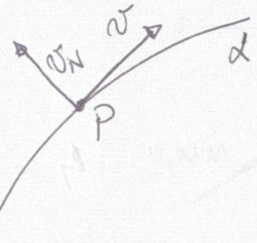
Now next step: what does it mean for two curves to be orthogonal? Take two curves, α and γ. Suppose they intersect at the point P of coordinates (x,y).



Consider the vector v that is tangent to α in P and the vector w that is tangent to γ in P. If v is orthogonal to w then α is orthogonal to γ. Remember that

$$v \perp w \quad \text{if} \quad v \cdot w = 0.$$

But, instead of considering tangent vectors, one can consider <u>normal</u> <u>vectors</u>, which is, vectors orthogonal to the tangent vector in P:



v is the <u>tangent</u> vector in P

v_N " " <u>normal</u> "

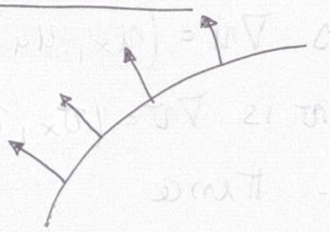
So, equivalently, two curves that intersect in P are said to be orthogonal in P if v_N is orthogonal to w_N, where

v_N = vector orthogonal to α in P

w_N = " " to γ in P.

How do we find the vectors that are orthogonal to a curve?

Def: The vector field $\underline{V}_N(x,y)$ s.t. $\underline{V}_N$ is orthogonal to a curve γ at each point of γ, is called the NORMAL FIELD:

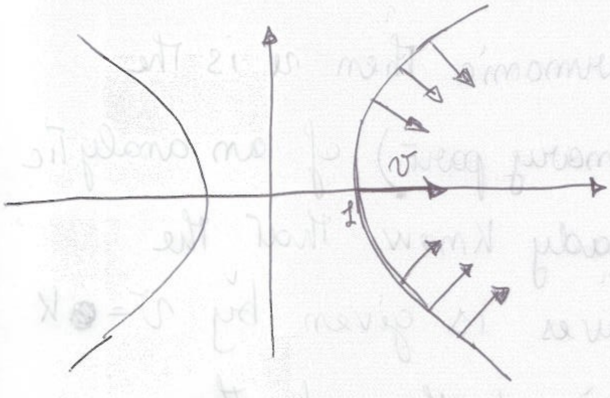


so $\underline{V}_N$ associates to every point (x,y) of the curve a vector, $\underline{V}_N(x,y)$, which is normal to the curve in (x,y).

FACT: given a curve $f(x,y) = c$, the gradient $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ is the normal field of the curve $f = c$.

EX: Take again the curve $u(x,y) = x^2 - y^2 = 1$

then $\nabla u = (2x, -2y) = 2x\,\underline{i} - 2y\,\underline{j}$



at the point $(1,0)$

$(\nabla u)(1,0) = (2,0) = v$

an v is clearly orthog. to the curve in the point $(1,0)$

You can experiment by yourself with other points along the curve.

Thm : let $f = u + iv$ be an holomorphic function. Then the family of curves $u = c$ is orthogonal to the family $v = k$, $\forall k$ and $\forall c$. Meaning that $\forall k$ and $\forall c$ the curve $u = c$ is orthogonal to the curve $v = k$.

proof: the normal field of u is $\nabla u = (u_x, u_y)$, whereas the normal field of v is $\nabla v = (v_x, v_y)$ - f is holomorphic $\Rightarrow$ (CR) hold - Hence

$$\nabla u \cdot \nabla v = (u_x, u_y) \cdot (v_x, v_y) =$$

$$= u_x v_x + u_y v_y = (+v_y) v_x - v_x v_y = 0 .$$

The normal vectors are orthogonal hence the curves are orthogonal.

EX: given the function $u = x^3 - 3xy^2$, find the family of curves $v = k$ which is orthogonal to the family of curves $u = c$.

If u is harmonic then u is the real (or imaginary part) of an analytic function. Then we already know that the family of orthogonal curves is given by $v = k$ where v can be determined through the (C-R) relations.

$u_x = 3x^2 - 3y^2 \qquad u_{xx} = 6x$

$\Rightarrow \nabla^2 u = 0.$

$u_y = -6xy \qquad u_{yy} = -6x$

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Rightarrow \begin{cases} v_y = 3x^2 - 3y^2 & (1) \\ v_x = 6xy & (2) \end{cases}$$

From (1): $\int \frac{\partial v}{\partial y} dy = 3x^2 y - y^3 \Rightarrow v = 3x^2 y - y^3 + h(x)$

From (2): $v = 3x^2 y + g(y)$

$\Rightarrow \cancel{3x^2 y} - y^3 + h(x) = \cancel{3x^2 y} + g(y)$

$\Rightarrow g(y) = -y^3$ and $h(x) = \text{const}$

$\Rightarrow v(x,y) = 3x^2 y - y^3 + \text{const}$

the curves $v = c$ are then

$3x^2 y - y^3 = \tilde{c}$, $\tilde{c}$ generic constants.

REM: if u is NOT HARMONIC then there is another procedure to find the family of orthogonal curves to $u = c$.

REM: The words ANALYTIC, HOLOMORPHIC and DIFFERENTIABLE IN COMPLEX SENSE are all equivalent.