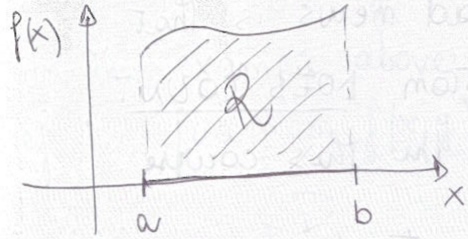
DOUBLE INTEGRALS

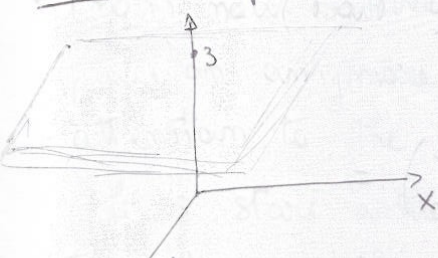
Let's start with a reminder of a couple of things that you know very well.

Given a function $f(x)$ of one variable, calculating the integral $\int_a^b f(x)\,dx$ means calculating the area of the region R under the graph of $f(x)$.



The interval $[a,b]$ is the segment of the line $y=0$, with $a \le x \le b$. (In other words the interval $[a,b]$ lays on the x axis, which is the line $y=0$.) As we have already mentioned, the graph of a function of two variables is a surface

EX the function $f(x,y)=3$ is represented by a plane parallel to the x-y plane which intersects the z axis at the point $(0,0,3)$.



REM : when we integrate a function of two variables, we can integrate either with respect to x, or with resp. to y or w.r. to both.

Integrating w.r. to both gives rise to double integrals.

EX: $f(x,y) = x^2 y$ - We want to calculate

$$\int_0^1 \int_0^2 (x^2 y)\, dx\, dy$$

Rem 1: I can integrate first in dx and then in dy or the other way around. The bad news is that in general the order of integration DOES COUNT. We will not get into this problem in this course.

Rem 2: When we integrate first w.r. to dx we write

$$\int_0^1 dy \int_0^2 (x^2 y)\, dx \quad \text{or} \quad \int \left(\int x^2 y\, dx \right) dy$$

when we integrate first w.r. to dy we write

$$\int_0 dx \int (x^2 y)\, dy \quad \text{or} \quad \int \left(\int x^2 y\, dy \right) dx$$

(In the above I have not written the extrema because they need to be rearranged depending on the order of integration, we'll get to that).

So, say I want to calculate

$$\int_0^1 dy \int_0^2 (x^2 y)\, dx = \int_0^1 dy\, y \left(\frac{x^3}{3} \Big|_0^2 \right) =$$

y can be treated like a constant while I do the first integration in dx

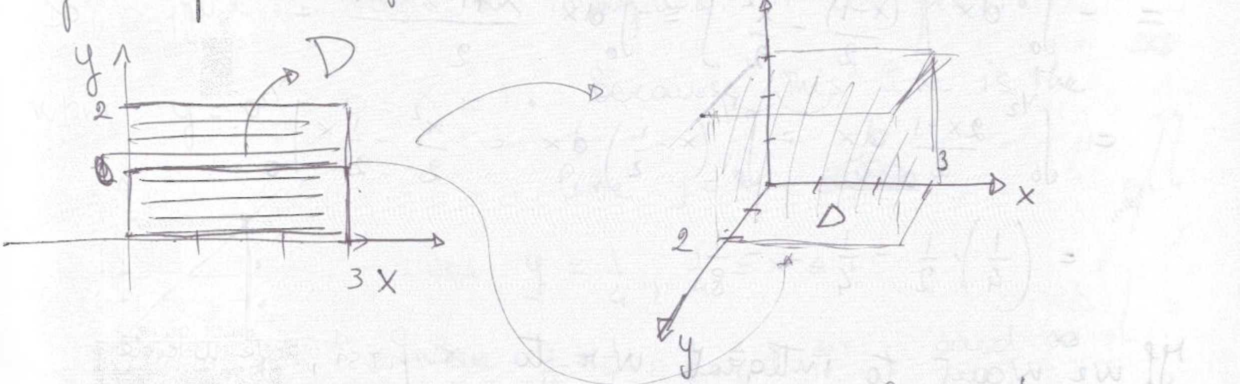
$$= \int_0^1 \frac{8}{3} y\, dy = \frac{8}{3} \frac{y^2}{2} \Big|_0^1 = \frac{8}{6} = \frac{4}{3}.$$

EX: Calculate $\int_0^2 dy \int_2^3 3\, dx =$

$$= \int_0^2 dy \left(3x \Big|_2^3 \right) = \int_0^2 dy\, 3 = 3y \Big|_0^2 = 6$$

So far, everything is easy but... what am I doing?

In the example above we were calculating the integral of the function $f(x,y) = 3$ over the domain $2 \le x \le 3, 0 \le y \le 2$



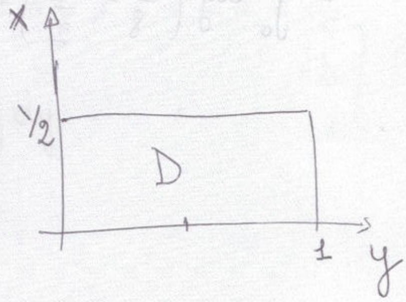
which amounts to calculating the volume between the surface $f(x,y) = 3$ and the domain D.

- From now on we will not sketch the graph of the function anymore, but we'll always pay loads of attention to the domain D.

Let's start with simple examples and work our way towards something more complicated.

EX: Calculate

$\iint_D (x-y)\, dx\, dy$ over the domain



D is the region of points with $0 \le x \le \frac{1}{2}$ and $0 \le y \le 1$ so, if we want to integrate first w.r to y,

$$\iint_D (x-y)\,dx\,dy = \int_0^{1/2} dx \int_0^1 (x-y)\,dy = -\int_0^{\frac{1}{2}} dx \; \frac{(x-y)^2}{2}\Bigg|_{y=0}^{y=1}$$

because $0 \le x \le \frac{1}{2}$ because $0 \le y \le 1$

$$= -\int_0^{1/2} dx \left[\frac{(x-1)^2}{2} - \frac{x^2}{2}\right] = -\int_0^{1/2} dx \; \frac{\cancel{x^2}+1-2x-\cancel{x^2}}{2} =$$

$$= \int_0^{1/2} \frac{2x-1}{2}\,dx = \int_0^{1/2}\left(x-\frac{1}{2}\right)dx = \frac{x^2}{2} - \frac{1}{2}x\Bigg|_0^{1/2} =$$

$$= \left(\frac{1}{4}\right)\cdot\frac{1}{2} - \frac{1}{4} = -\frac{1}{8}$$

If we want to integrate w.r. to x first, we write

$$\iint_D (x-y)\,dx\,dy = \int_0^1 dy \int_0^{1/2} (x-y)\,dx =$$

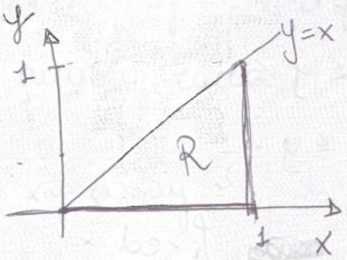
$$= \int_0^1 dy \; \frac{(x-y)^2}{2}\Bigg|_{x=0}^{x=\frac{1}{2}} = \int_0^1 dy \left[\frac{\left(\frac{1}{2}-y\right)^2}{2} - \frac{y^2}{2}\right] =$$

$$= \int_0^1 dy \; \frac{1}{2}\left[\frac{1}{4} + \frac{y^2}{2} - y - \frac{y^2}{2}\right] =$$

$$= \int_0^1 dy \left(\frac{1}{8} - \frac{y}{2}\right) = \left(\frac{1}{8}y - \frac{y^2}{4}\right)\Bigg|_0^1 = \frac{1}{8} - \frac{1}{4} = -\frac{1}{8}$$

Other example, more complicated

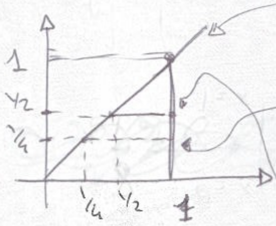
EX Calculate $\iint_R xy \, dx\,dy$ on the region R



This time it is more complicated because "expressing R in formulas" is more complicated.

So, suppose I want to integrate in dx first.

when $y=0$ $\quad 0 \le x \le 1$. Because this line is the line $y=x$,



when $y = \frac{1}{4}$, $\frac{1}{4} \le x \le 1$

when $y = \frac{1}{2}$, $\frac{1}{2} \le x \le 1$, and so on and so forth. In general, once you fix a certain value of y, you have $y \le x \le 1$, and this is true for any $0 \le y \le 1$. So

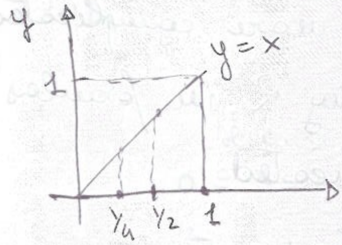
$$\iint_R xy \, dx\,dy = \int_0^1 dy \int_y^1 (xy)\, dx =$$

$$= \int_0^1 dy \left(y \int_y^1 x\, dx \right) = \int_0^1 dy \; y \cdot \left[\frac{x^2}{2} \right]_{x=y}^{x=1} =$$

$$= \int_0^1 dy \; y \cdot \left(\frac{1}{2} - \frac{y^2}{2} \right) = \int_0^1 \left(\frac{y}{2} - \frac{y^3}{2} \right) dy$$

$$= \left(\frac{y^2}{4} - \frac{y^4}{8} \right) \Bigg|_0^1 = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$$

If I want to integrate in dy first, then I do the reasoning the other way around: I fix x and look at the range of y. Meaning



when $x = \frac{1}{4}$ then $0 \le y \le \frac{1}{4}$

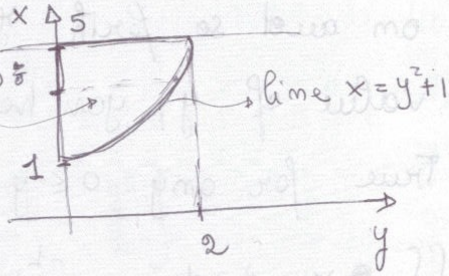
" $x = \frac{1}{2}$ then $0 \le y \le \frac{1}{2}$

In general, for fixed x, we have $0 \le y \le x$; as for x, $0 \le x \le 1$. So

$$\iint_R xy \, dx dy = \int_0^1 dx \int_0^x dy \,(xy) =$$

$$= \int_0^1 dx \; x \cdot \left(\frac{y^2}{2} \Big|_{y=0}^{y=x} \right) = \int_0^1 dx \; \frac{x^3}{2} = \frac{x^4}{8} \Big|_{x=0}^{x=1} = \frac{1}{8}$$

EX: Calculate $\iint_D xy \, dx dy$ over D



Calculate it by first integrating w.r. to dy;

for fixed x we have $0 \le y \le \sqrt{x-1}$

and x ranges in $1 \le x \le 5$. So

$$\int_1^5 dx \int_0^{\sqrt{x-1}} dy \,(xy) = \int_1^5 dx \; x \cdot \left(\frac{y^2}{2} \Big|_{y=0}^{y=\sqrt{x-1}} \right) =$$

$$= \int_1^5 dx \; x \cdot \frac{(x-1)}{2} = \int_1^5 dx \left(\frac{x^2}{2} - \frac{x}{2} \right) = \frac{x^3}{6} - \frac{x^2}{4} \Big|_{x=1}^{x=5} =$$

$$= \frac{125}{6} - \frac{25}{4} - \frac{1}{6} + \frac{1}{4} = \frac{176}{12}$$

Now swap the order of integration and integrate w.r. to ~~y~~ x first. So

for every fixed y we have $y^2+1 \leq x \leq 5$

and y ranges in $0 \leq y \leq 2$.

$$\iint_D xy \, dx\, dy = \int_0^2 dy \int_{y^2+1}^{5} (xy)\, dx =$$

$$= \int_0^2 dy \; y \left(\frac{x^2}{2} \Big|_{x=y^2+1}^{x=5} \right) = \int_0^2 dy \; y \cdot \left(\frac{25}{2} - \frac{(y^2+1)^2}{2} \right) =$$

$$= \int_0^2 dy \left(\frac{25}{2} y - \frac{y^5}{2} - \frac{y}{2} - y^3 \right) =$$

$$= \left(\frac{25}{2}\frac{y^2}{2} - \frac{y^6}{12} - \frac{y^2}{4} - \frac{y^4}{4} \right) \Big|_{y=0}^{y=2}$$

$$= \frac{25}{\cancel{2}} \cdot \frac{\cancel{4}}{\cancel{2}} - \frac{64}{12} - 1 - \frac{16}{4} = \frac{176}{12}.$$

The moral of the story is that changing the order of integration requires a good understanding of the region of integration.

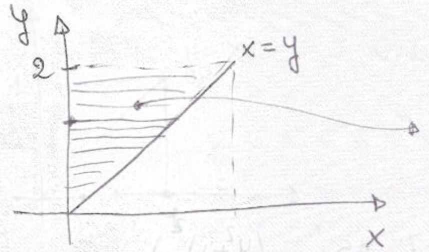
EX: If I ask you to calculate

$$\int_0^2 \left(\int_0^y 3\, dx \right) dy$$, you can easily do that but...

can you also sketch the region of integration?

$$\int_0^2 \left(\int_0^y 3\, dx \right) dy$$

we are integrating in dx first. So this means that for fixed y, $0 \le x \le y$ and $0 \le y \le 2$

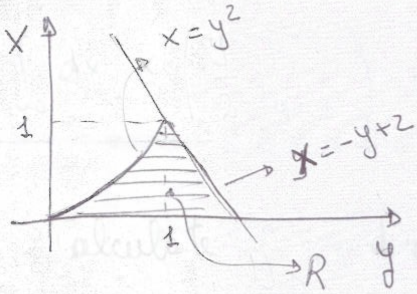


This is R!

EX $\displaystyle\int_0^1 dx \int_{\sqrt{x}}^{2-x} 9\, dy$ for fixed x, we have $\sqrt{x} \le y \le 2-x$ and $0 \le x \le 1$. So

$y = \sqrt{x} \Rightarrow y^2 = x$

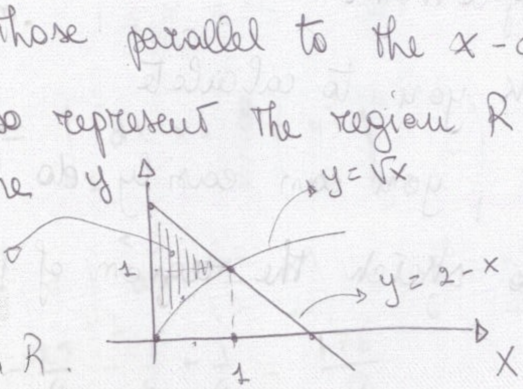
$y = 2 - x \Rightarrow x = -y + 2$

(*) 

representation in the (y, x) plane

Rem: we said that when we integrate for example in dx first, we think of y fixed. This is quite natural because if we are integrating in dx then we can think y as a constant and the lines $y = \text{const}$ are those parallel to the x-axis.

Rem: you can also represent the region R of (*) in the (x, y) plane



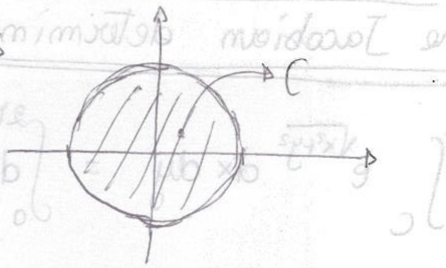
This is the region R.

• Change of variable in double integrals

Polar coordinates

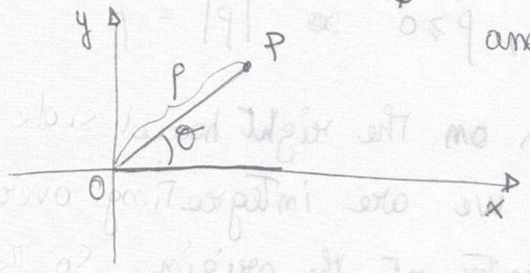
Ex: Calculate $\iint_C e^{\sqrt{x^2+y^2}}\, dx\, dy$ over the region C which is the ball of radius 2 centered at the origin.

It is clear that in cases like this one it is easier to use polar coordinates:



$$\begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta . \end{cases} \quad (*)$$

In this system of coordinates a point in the plane is individuated by ρ = distance from the origin



and θ = angle between the x axis and the segment OP. (In other words, ρ is the length of the segment OP).

So (x,y) are the old coordinates and (ρ,θ) are the new ones.

Def: The matrix $J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix}$ is called the

JACOBIAN MATRIX of the transformation (*)

The determinant of J, det J, is called the JACOBIAN DETERMINANT.

Going back to our example, notice that from (*), we have

$x^2+y^2 = \rho^2\cos^2\theta + \rho^2\sin^2\theta = \rho^2$.

When we change coordinates in double integrals we need to multiply the integrand by the absolute value of the Jacobian determinant:

$$\iint_C e^{\sqrt{x^2+y^2}}\,dx\,dy = \int_0^{2\pi} d\theta \int_0^2 e^{\rho}\,\rho\,d\rho\,d\theta \qquad (**)$$

$$\det J = \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix} = \rho\cos^2\theta + \rho\sin^2\theta = \rho$$

because ρ is a distance, $\rho \geq 0$ so $|\rho| = \rho$.

The extrema of integration on the right hand side of (**) come from the fact that we are integrating over C = ball of radius 2 an center at the origin. So the points inside C are described by $0 \leq \theta \leq 2\pi$, $0 \leq \rho \leq 2$.

Now we are only left with the computation:

$$\int_0^{2\pi} d\theta \int_0^2 e^{\rho}\rho\,d\rho\,d\theta = \int_0^{2\pi} d\theta \left[\rho e^{\rho}\Big|_0^2 - \int_0^2 e^{\rho}\,d\rho \right] =$$

$$= \int_0^{2\pi} d\theta \,(2e^2 - e^2 + 1) = (e^2+1)\cdot 2\pi$$

Where does the Jacobian determinant come from?

When you had integrals of functions of one variable, say x, and you wanted to change variable, say $z = 2x$, you did

$$\int_0^1 x \, dx = \int_0^2 \frac{z}{2} \cdot \frac{dz}{2}$$

because $dz = 2\,dx$

so ① there is a factor $\frac{1}{2}$ swooping in because $dx = \frac{1}{2} dz$

② you have to adjust the extrema of integration: if $0 \le x \le 1$ then $0 \le 2x \le 2$ so $0 \le z \le 2$

In two variables (think of (*)) going from "$dx\,dy$" to "$d\rho\, d\theta$" involves a factor, which is the determinant of the Jacobian matrix.

EX: Use the transformation $u = x - y$, $v = x + y$ to evaluate $\iint_R (x^2 - y^2)\, dx\, dy$ over the region R enclosed by the lines $x = 0$, $y = 0$, $x = 2$, $y = 2$.

Solving an exercise of this type involves 4 steps

STEP 2 sketch the region of integration both in the (x,y) plane and in the (u,v) plane.

STEP 1: Write x and y as functions of u and v and hence calculate $|\det J|$.

STEP 3: change the extremes of integration (which means, change the region of integration)

STEP 4: calculation.

Let's solve the exercise by following these steps.

S1: $u = x - y \Rightarrow x = u + y \Rightarrow \dots \quad x = \frac{u+v}{2}$

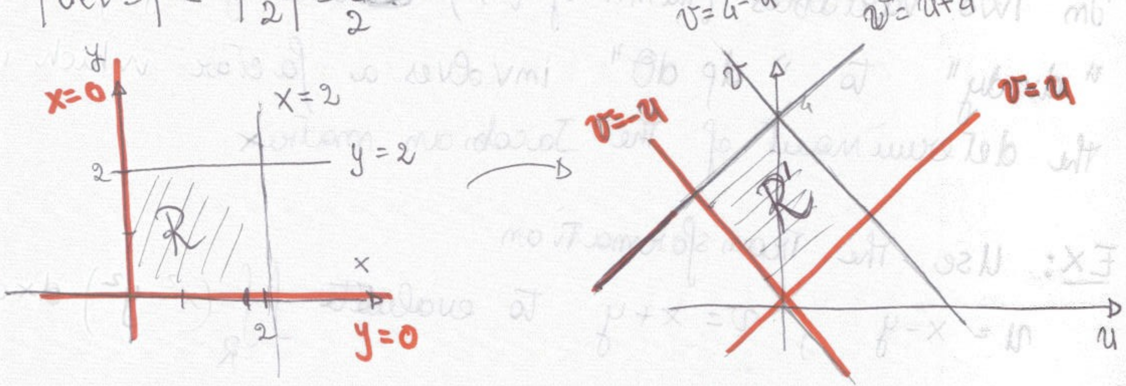
$v = x + y \Rightarrow y = v - x \Rightarrow \dots \quad y = \frac{v-u}{2}$

Now $\frac{\partial x}{\partial u} = \frac{1}{2} \quad \frac{\partial x}{\partial v} = \frac{1}{2} \quad \frac{\partial y}{\partial u} = -\frac{1}{2} \quad \frac{\partial y}{\partial v} = \frac{1}{2}$ so

$$\det J = \det \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

so $|\det J| = \left|\frac{1}{2}\right| = \frac{1}{2}$

S2:



The line $x = 0$ goes, in the (u, v) plane, into the line $\frac{u+v}{2} = 0$

$y = 0$ goes into $\frac{v-u}{2} = 0$

$x = 2$ " $\frac{u+v}{2} = 2$

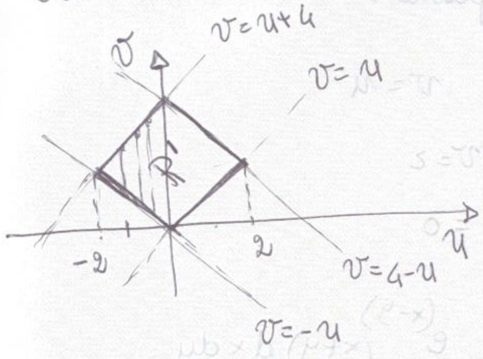
$y = 2$ " $\frac{v-u}{2} = 2$

So $\frac{u+v}{2} = 0 \Rightarrow v = -u$, $\frac{v-u}{2} = 0 \Rightarrow v = u$,

$\frac{u+v}{2} = 2 \Rightarrow v = 4 - u$ and

$\frac{v-u}{2} = 2 \Rightarrow v = 4 + u$

Step 3: In the (x,y)-coordinates the extremes of integration are $\int_0^2 dy \int_0^2 dx\,(x^2-y^2)$; in the (u,v)-coordinates, they become



$$\iint_R (x^2-y^2)\,dx\,dy =$$

$$= \iint_{R'} \underline{uv}\ \underbrace{\tfrac{1}{2}}_{\text{Jacobian determinant}}\ du\,dv =$$

because $u\cdot v = (x-y)\cdot(x+y) = x^2-y^2$

$$= \int_{-2}^{0} du \int_{-u}^{u+4} u\,v\,\frac{1}{2}\,dv + \int_0^2 du \int_u^{4-u} u\,v\,\frac{1}{2}\,dv =$$

$$= \int_{-2}^{0} du\ u\cdot\left(\frac{v^2}{4}\Bigg|_{v=-u}^{v=u+4}\right) + \int_0^2 du\ u\cdot\left(\frac{v^2}{4}\Bigg|_{v=u}^{v=4-u}\right) =$$

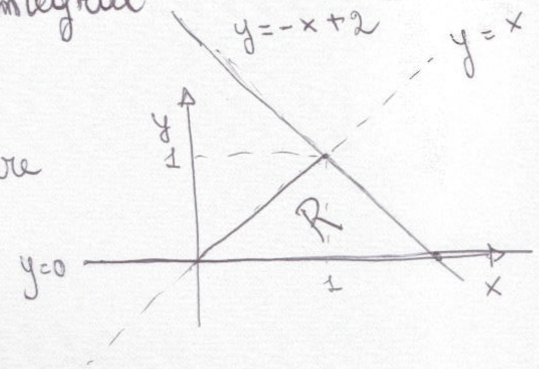
$$= \int_{-2}^{0} du\ u\cdot\left(\frac{(u+4)^2}{4} - \frac{u^2}{4}\right) + \int_0^2 du\ u\cdot\left[\frac{(4-u)^2}{4} - \frac{u^2}{4}\right] = \ldots$$

and now STEP 4: you can do your maths!! -

EX: Use the change of coordinates

$$\begin{cases} u = x-y \\ v = x+y \end{cases}$$ to evaluate the integral

$$\iint_R e^{(x-y)}(x+y)\,dx\,dy$$ where

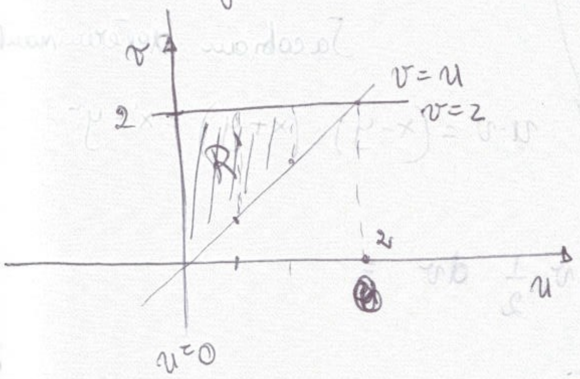


Like before we have $x = \frac{u+v}{2}$ and $y = \frac{v-u}{2}$ so that $|\det J| = \frac{1}{2}$. Now let's sketch the region R', i.e. the region of integration in the (u,v) plane.

The line $y=0$ goes into the line $v=u$

The line $y=-x+2$: $y+x=2 \Rightarrow v=2$

" $y=x$: $u+\cancel{v} = \cancel{v}-u \Rightarrow u=0$



so $\iint_R e^{(x-y)}(x+y)\,dx\,dy =$

$$= \iint_{R'} e^{u}\, v\, \frac{1}{2}\, du\, dv$$

$$= \int_0^2 du \int_u^2 e^{u}\, v\, \frac{1}{2}\, dv$$

now you can do your calculations.