Civil Engineering 2 Mathematics Autumn 2011

Exercise Sheet 1

1. Show that x^2 is a solution of the homogeneous differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{4}{x^2}y = 0.$$

Hence find the general solution to the inhomogeneous equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{4}{x^2}y = 1,$$

by seeking a solution of the form $y = x^2 \int v(x)$.

2. Show that $\cos x$ is a solution of

$$\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + \left(\sec^2 x\right)\ y = 0,$$

and hence find the general solution to the equation

$$\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + \left(\sec^2 x\right)\ y = \frac{1}{\cos x}, \quad x \in \left[0, \frac{\pi}{2}\right).$$

Find also the particular solution that satisfies the boundary conditions

$$y\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{16}\log^2 3, \quad y'\left(\frac{\pi}{6}\right) = -\frac{1}{16}\log^2 3.$$

3. Find the value of λ such that $y = x^\lambda$ is a solution of

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - \left(\frac{6}{x^2} + \frac{2}{x}\right)y = 0.$$

4. Let $t > 0$. Show that the substitution $x = t^2$ transforms the ode

$$4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0 \qquad (*)$$

into the equation

$$\frac{d^2y}{dt^2} + y = 0.$$

Hence find the general solution of equation (*).

5. Find the general solution of

$$\frac{d^2y}{dx^2} - (\coth x)\frac{dy}{dx} + (4\sinh^2 x)y = 0$$

by making the substitution $t = \cosh x$ to transform it to a simpler form.

6. Use the substitution $x = \sin t$ to determine the general solution (valid for $-1 \leqslant x \leqslant 1$) of

$$(1-x^2)\frac{d^2y}{dx^2} + \left(2(1-x^2)^{1/2} - x\right)\frac{dy}{dx} + y = \sin^{-1}(x).$$

(Here $\sin^{-1}(x)$ does not denote $1/\sin(x)$ but the inverse function.)

7. Find the general solution of

$$4y''(x) + 4y'(x) + y = x^2.$$

(Hint: first determine a solution of $4y_1'' + 4y_1' + y = 0$ for example the one with initial conditions $y_1(0) = 1, y_1'(0) = -1/2$.)

Answers

1. $y = (1/4)x^2 \ln x + Ax^2 + B/x^2$.

2. GS: $y = \frac{\cos x}{8}\log^2\left(\frac{1+\sin x}{1-\sin x}\right) + C\cos x \log\left(\frac{1+\sin x}{1-\sin x}\right) + D\cos x$. PS: from generic solution when $C = -(\log 3)/2$, $D = (\log^2 3)/2$.

3. $\lambda = -2$.

4. $y = A\cos\left(\sqrt{x}\right) + B\sin\left(\sqrt{x}\right)$.

5. $y = A\cos\left(2\cosh x\right) + B\sin\left(2\cosh x\right)$.

6. $y = A\exp(-\sin^{-1}(x)) + B\sin^{-1}(x)\exp(-\sin^{-1}(x)) - 2 + \sin^{-1}(x)$.

7. $y = (x^2 - 8x + 24) + Cxe^{-\frac{1}{2}x} + De^{-\frac{1}{2}x}$