1. Let $y = x^2 \int v$. Then $y' = 2x \int v + x^2 v$, $y'' = 2 \int v + 4xv + x^2 v'$ and $y'' + \frac{1}{x} y' - \frac{4}{x^2} y = 1 \Leftrightarrow x^2 v' + 5xv = 1 \Leftrightarrow v' + \frac{5}{x} v = \frac{1}{x^2}$. Solve equation for v via integrating factor method. $A(x) = \int \frac{5}{x} = 5 \log x = \log x^5$ hence integrating factor is $\exp(\log x^5) = x^5$. Multiplying both sides of the equation by x^5 we obtain $(x^5 v)' = x^3 \Rightarrow x^5 v = x^4/4 + C \Rightarrow v = \frac{1}{4x} + \frac{C}{x^5}$. Hence $\int v = (1/4) \ln x + C/x^4 + D$
$\Rightarrow \underline{y = x^2 \int v = (1/4) x^2 \ln x + C/x^2 + Dx^2}$.

2. Let $y_1 = \cos x$. Then $y_1'' + \tan x \, y_1' + \sec^2 x \, y_1 = -\cos x - \tan x \sin x + 1/(\cos x) = 0$ as required.
Let $y = \cos x \int v$. Then $y' = -\sin x \int v + (\cos x) v$ and $y'' = -\cos x \int v - 2(\sin x) v + (\cos x) v'$. So $y'' + \tan x \, y' + \sec^2 x \, y = 1/(\cos x)$ if and only if $v' - \tan x \, v = 1/(\cos^2 x)$. Now solve equation for v by integrating factor method. Integrating factor is $\exp(A(x))$ where $A(x) = -\int \tan x dx = \log(\cos x)$ (we don't need to put the absolute value as $0 \le x < \pi/2$). So $\exp(A(x)) = \cos x$. Multiply both sides of the equation for v by the integrating factor and obtain $(\cos x \, v)' = 1/(\cos x) \Rightarrow v = 1/(\cos x) \int 1/(\cos x) + C/(\cos x)$. We now need to calculate $\int v$. First, integrating by parts, we have

$$\int v = \frac{1}{2} \left(\int \frac{1}{\cos x} dx \right)^2 + \int \frac{C}{\cos x} dx + D.$$

To calculate $\int 1/(\cos x) dx$:$\int 1/(\cos x) dx = \int \cos x/(\cos x)^2 = \int \cos x/(1 - \sin^2 x) =$ [substitute $z = \sin x$] $\int dz/(1 - z^2)$. So in the end

$$\int \frac{dx}{\cos x} = \frac{1}{2} \log \left(\frac{1 + \sin x}{1 - \sin x} \right).$$

The general solution is

$$y = \frac{\cos x}{8} \log^2 \left(\frac{1 + \sin x}{1 - \sin x} \right) + C \cos x \log \left(\frac{1 + \sin x}{1 - \sin x} \right) + D \cos x.$$

Use initial condition to find $C = -(\log 3)/2$, $D = (\log^2 3)/2$.

3. Let $y_1 = x^\lambda$. Then $y_1'' - y_1' - (6/x^2 + 2/x) y_1 = \lambda(\lambda - 1) x^{\lambda - 2} - \lambda x^{\lambda - 1} - 6x^{\lambda - 2} - 2x^{\lambda - 1} = 0$.
Equate coefficients of $x^{\lambda - 2} \Rightarrow \lambda^2 - \lambda - 6 = 0 \Rightarrow \lambda = 3$ or $\lambda = -2$.
Coefficients of $x^{\lambda - 1} \Rightarrow \lambda + 2 = 0 \Rightarrow \lambda = -2$. So $\underline{\lambda = -2}$ is the only possible value.

4. If $x = t^2$ then $dy/dt = (dx/dt)(dy/dx) = 2t dy/dx = 2x^{1/2} dy/dx$.
Then $d^2 y/dt^2 = 2x^{1/2} \frac{d}{dx} \left(2x^{1/2} \frac{dy}{dx} \right) = 4x d^2 y/dx^2 + 2 dy/dx$.
Therefore the ode becomes $d^2 y/dt^2 + y = 0$, as required.
General solution is $y = A \cos t + B \sin t \Rightarrow$ in terms of x, $\underline{y = A \cos(\sqrt{x}) + B \sin(\sqrt{x})}$.

5. If $t = \cosh x$, then $dy/dt = (dx/dt)\,(dy/dx) = (1/\sinh x)(dy/dx)$.
Then $d^2y/dt^2 = (1/\sinh x)\frac{d}{dx}((1/\sinh x)\frac{dy}{dx}) = (1/\sinh^2 x)d^2y/dx^2 - (\cosh x/\sinh^3 x)dy/dx$

$\Rightarrow (\sinh^2 x)d^2y/dt^2 = d^2y/dx^2 - (\coth x)\,dy/dx$.
Therefore the ode becomes $\left(\sinh^2 x\right) d^2y/dt^2 + \left(4\sinh^2 x\right) y = 0 \Rightarrow d^2y/dt^2 + 4y = 0$
$\Rightarrow y = A\cos 2t + B\sin 2t$. In terms of x, the solution is thus $\underline{y = A\cos(2\cosh x) + B\sin(2\cosh x)}$.

6. Let $x = \sin t$. Then $dy/dt = (dx/dt)(dy/dx) = (\cos t)\,dy/dx = (1-x^2)^{1/2}dy/dx$.
So, $d^2y/dt^2 = (1-x^2)^{1/2}\frac{d}{dx}((1-x^2)^{1/2}dy/dx) = (1-x^2)d^2y/dx^2 - xdy/dx$.
Thus: $(1-x^2)d^2y/dx^2 - xdy/dx + 2(1-x^2)^{1/2}dy/dx = d^2y/dt^2 + 2dy/dt$.
The ode therefore becomes $d^2y/dt^2 + 2dy/dt + y = \sin^{-1}(x) = t$.
We have a constant coefficient 2nd order ode. Solve in the normal way by seeking homogeneous (y_H) and particular (y_P) solutions. For y_H look for solution using associated auxiliary polynomial $\lambda^2 + 2\lambda + 1 = 0$
$\Rightarrow (\lambda+1)^2 = 0 \Rightarrow \lambda = -1 \Rightarrow y_H = (A + Bt)\exp(-t)$.
So you can take $y_1 = e^{-t}$ and then look for a general solution of the non homogeneous equation in the form $y = y_1 \int v$. v solves $dv/dt = te^t$ hence $v = e^t(t-1) + C$ and $\int v = e^t(t-2) + Ct + D$.
So the general solution in terms of t is $y = (D + Ct)\exp(-t) + t - 2$.
Writing back in terms of x we have $\underline{y = (D + C\sin^{-1}(x))\exp(-\sin^{-1}(x)) + \sin^{-1}(x) - 2.}$

7. Following the hint, start from the homogeneous equation $4y_1'' + 4y_1' + y = 0$, which has constant coefficients. Associated auxiliary polynomial has only one root, $\lambda = -1/2$, so general solution of homogeneous equation is $y_1 = Ae^{-x/2} + Bxe^{-x/2}$. Imposing $y(0) = 1$ gives $A = 1$, and $y_1'(0) = -1/2$ gives $B = 0$. So take $y_1(x) = e^{-x/2}$. Let $y = e^{-x/2}\int v$. In order for y to be a solution of the non-homogeneous equation, v has to satisfy the equation $v' = \frac{1}{4}x^2e^{x/2}$ so by simply integrating both sides we get $v = \int \frac{x^2}{4}e^{x/2} + C$. Integrating by parts gives $v = e^{x/2}(\frac{x^2}{2} - 2x + 4) + C$. Then $y = y_1 \int v = (x^2 - 8x + 24) + Cxe^{-x/2} + De^{-x/2}$.