Civil Engineering 2 Mathematics Autumn 2011

Exercise Sheet 2

1. Solve the following linear system by using the Gaussian elimination method. In particular, determine the values of $t \in \mathbb{R}$ such that the system admits a solution and the values for which it does not have a solution.

$$\begin{cases} x_1 + x_2 + tx_3 = 1 \\ x_1 + x_3 = 0 \\ x_1 + x_2 + t^3 x_3 = 3 \\ x_1 + x_2 + x_3 = 0 \end{cases}$$

2. Find an LU factorization of the following matrices:

$$A = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & 0 & -1 \end{vmatrix}, \quad B = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 3 & -1 & 0 \end{vmatrix} \quad \text{and} \quad E = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 1 \end{vmatrix},$$

More specifically, for A and B determine L and U with L lower triangular having ones on the main diagonal and U upper triangular. For E find any LU factorization. Using the LU factorization, solve the systems $Ax = b$, $b = (1,0,0,3)^T$, $Bx = d$, $d = (5,1,0)^T$ and $Ex = v$, $v = (1,1,1)^T$, $x \in \mathbb{R}^3$.

3. Find eigenvalues and eigenvectors of the following matrices

$$\begin{vmatrix} 2 & -2 & -1 \\ -2 & 5 & 2 \\ -1 & 2 & 2 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix}.$$

Determine wether they are diagonalizable. If yes, find the diagonalizing matrix C and the diagonal matrix Δ s.t. $C^{-1}AC = \Delta$. Also, call A the first matrix. Calculate A^{34}. Find the general solution of the systems of ODEs $y' = Ay$ where $y = (y_1, y_2, y_3) \in \mathbb{R}^3$, i.e. solve

$$\begin{cases} y_1' = 2y_1 - 2y_2 - y_3 \\ y_2' = -2y_1 + 5y_2 + 2y_3 \\ y_3' = -y_1 + 2y_2 + 2y_3 \ . \end{cases}$$

4. Find the matrix A s.t. the following system of ODEs

$$\begin{cases} y_1' = y_3 \\ y_2' = 3y_1 + 7y_2 - 9y_3 \\ y_3' = 2y_2 - y_3 \end{cases}$$

can be expressed in the form $y' = Ay,\ y \in \mathbb{R}^3$. Then find the general solution.

5. Find the matrix A s.t. the following system of ODEs

$$\begin{cases} y_1' = ay_1 + y_2 \\ y_2' = ay_2 + y_3 \\ y_3' = ay_3 \end{cases} \qquad a \in \mathbb{R},$$

can be expressed in the form $y' = Ay,\ y \in \mathbb{R}^3$. Then find the solution with $y(0) = (1, 0, 0)^T$. If $a < 0$, what is the behaviour of $y(t)$ as $t \to \infty$? What if $a = 0$?

6. Let M be an $n \times n$ block diagonal matrix, i.e. $M = \begin{vmatrix} M_1 & \mathbf{0} \\ \mathbf{0} & M_2 \end{vmatrix}$, with M_1 and M_2 $m \times m$ and $l \times l$ matrices, $l + m = n$, and $\mathbf{0}$ denote $0-$matrices of appropriate size. Then $e^M = \begin{vmatrix} e^{M_1} & \mathbf{0} \\ \mathbf{0} & e^{M_2} \end{vmatrix}$. Using this result, find the general solution of the system

$$\begin{cases} \dot{x} = x \\ \dot{y} = \omega z \\ \dot{z} = -\omega y. \end{cases} \qquad \omega > 0,\ (x, y, z) \in \mathbb{R}^3$$

Also, find the solution for $x(0) = 1, y(0) = 1, z(\pi/\omega) = -1$.

Answers

1. For $t = -2$ the solution is unique, for $t = 1$ no solutions.

2. For A: $L = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{vmatrix}$, $U = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & -1 \end{vmatrix}$, $x = (1, 0, -3, -3)^T$.
For B: $L = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix}$, $U = \begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{vmatrix}$, $x = (-4/3, -4, 7/3)^T$.
For E: LU factorization is ...well, stare at E for a couple of seconds! $x = (1, 1/3, 2/3)^T$.

3. First matrix: $\lambda = 7$ with eigenvector $v_1 = (-1, 2, 1)$ and $\lambda = 1$ with algebraic multiplicity 2 and eigenvectors $v_2 = (2, 1, -0), v_3 = (1, 0, 1)$. $C = \begin{vmatrix} -1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$ and $\Delta = \text{diag}\{7, 1, 1\}$. Solution of system
$y_1 = \frac{1}{6}[(e^{7t} + 5e^t)c_1 + 2(e^t - e^{7t})c_2 + (e^t - e^{7t})c_3], y_2 = \frac{1}{6}[2(e^t - e^{7t})c_1 + 2(e^t + 2e^{7t})c_2 + 2(-e^t + e^{7t})c_3], y_3 = \frac{1}{6}[(e^t - e^{7t})c_1 + 2(-e^t + e^{7t})c_2 + (5e^t + e^{7t})c_3]$.

Second matrix: $\lambda = 2$ with eigenvector $(0, 0, 1)$ and $\lambda = 1$ with algebraic multiplicity 2 and eigenvector $(1, 0, 0)$. Non diagonalizable.

4. $y_1 = c_1 e^t + c_2 e^{2t} + c_3 e^{3t}$, $y_2 = c_1 e^t + 3c_2 e^{2t} + 6c_3 e^{3t}$, $y_3 = c_1 e^t + 2c_2 e^{2t} + 3c_3 e^{3t}$.

5. $y_1 = e^{at}, y_2 = y_3 = 0$.

6. General solution: $x(t) = e^t c_1,\ y(t) = c_2 \cos \omega t + c_3 \sin \omega t$,
$z(t) = -c_2 \sin \omega t + c_3 \cos \omega t$. Solution with given initial conditions is $x(t) = e^t,\ y(t) = \cos \omega t + \sin \omega t$, $z(t) = -\sin \omega t + \cos \omega t$.