Civil Engineering 2 Mathematics Autumn 2011

Solutions 2

1.

$$\begin{array}{c} i \\ ii \\ iii \\ iv \end{array} \left\{ \begin{array}{l} x_1 + x_2 + t x_3 = 1 \\ x_1 + x_3 = 0 \\ x_1 + x_2 + t^3 x_3 = 3 \\ x_1 + x_2 + x_3 = 0 \end{array} \right. \Rightarrow \begin{array}{c} ii \\ ii - i \\ i - iii \\ i - iv \end{array} \left\{ \begin{array}{l} x_1 + x_3 = 0 \\ -x_2 + (1-t)x_3 = -1 \\ (t - t^3)x_3 = -2 \\ (t-1)x_3 = 1 \end{array} \right.$$

So if $t \neq 1$, working bottom to top, we have

$$x_3 = 1/(t-1),\ x_3 = -2/[t(1-t^2)],\ x_2 = 0,\ x_1 = -1/(t-1).$$

In order for the system to have a solution the two expressions for x_3 have to match $1/(t-1) = -2/[t(1-t^2)] \Rightarrow t^2 + t - 2 = 0 \Rightarrow t = -2$ and $t = 1$, but we exclude $t = 1$. If $t = 1$ the system does not have a solution (look at the formula for x_3 or substitute $t = 1$ into the system and try and solve it).

2. For A: write

$$\left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{array} \right| \left| \begin{array}{cccc} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{array} \right| = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & 0 & -1 \end{array} \right|.$$

First row of L by first to last column of U gives
$u_{11} = 1, u_{12} = 0, u_{13} = 0, u_{14} = 0.$
Second row of L by first to last column of U gives
$l_{21} = 0, u_{22} = 1, u_{23} = 1, u_{24} = -1.$
Third row of L by first to last column of U gives
$l_{31} = 0, l_{32} = 1, 1 + u_{33} = -1$ so $u_{33} = -2$, $-1 + u_{34} = 1$ so $u_{34} = 2$.
Fourth row of L by first to last column of U gives
$l_{41} = 0, l_{42} = 2, l_{43} = 1, u_{44} = -1$. To solve the system: $Ax = b \Leftrightarrow LUx = b$.
Let $Ux = y$ and solve $Ly = b$, which is $\left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{array} \right| \left| \begin{array}{c} y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \right| = \left| \begin{array}{c} 1 \\ 0 \\ 0 \\ 3 \end{array} \right|$, so working top to bottom $y_1 = 1, y_2 = 0, y_3 = 0, y_4 = 3$. Now solve $Ux = y$, which is $\left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & -1 \end{array} \right| \left| \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \right| = \left| \begin{array}{c} 1 \\ 0 \\ 0 \\ 3 \end{array} \right|$. Working bottom to top we get
$x_4 = -3, -2x_3 - 6 = 0 \Rightarrow x_3 = -3, x_2 = 0, x_1 = 1.$
For B: write

$$\left| \begin{array}{ccc} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{array} \right| \left| \begin{array}{ccc} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{array} \right| = \left| \begin{array}{ccc} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 3 & -1 & 0 \end{array} \right|$$

We work as before and find $u_{11} = 1, u_{12} = -1, u_{13} = 1$,
$l_{21} = 1, -1 + u_{22} = 0, 1 + u_{23} = 1$,
$l_{31} = 3, -3 + l_{32} = -1, 3 + u_{33} = 0$.
$Bx = d \Leftrightarrow LUx = d$. Let $Ux = y$, so $LUx = d \Leftrightarrow Ly = d$. Solve $Ly = d$: $\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} \begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix} = \begin{vmatrix} 5 \\ 1 \\ 0 \end{vmatrix}$ working top to bottom we have $y_1 = 5, 5 + y_2 = 1, 15 - 8 + y_3 = 0$, hence $y = (5, -4, -7)$. Now solve $Ux = y$: $\begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 5 \\ -4 \\ -7 \end{vmatrix}$ working bottom to top we have $-3x_3 = -7, x_2 = -4, x_1 + 4 + 7/3 = 5$ hence the result.
The last is way too easy...E is already lower diagonal, so an LU factorization is just $E = E\,I$, with I the identity matrix. Because it is lower triangular you can easily solve the system.

3. For the first matrix, call it A. $\det(A - \lambda I) = 0 \Leftrightarrow \lambda^3 - 9\lambda^2 + 15\lambda - 7 = 0 \Leftrightarrow (\lambda - 7)(\lambda - 1)^2 = 0$. So the eigenvalues are $\lambda = 7$ and $\lambda = 1$.
$\lambda = 7$: $Av = \lambda v \Rightarrow (A - 7I)v = 0$

$$\begin{cases} -5v_1 - 2v_2 - v_3 = 0 \\ -12v_2 + 24v_3 = 0 \\ 0v_2 + 0v_3 = 0 \end{cases} \Rightarrow \begin{cases} v_1 = -v_3 \\ v_2 = 2v_3 \\ v_3 \in \mathbb{R} \setminus \{0\} \end{cases}.$$

So if we choose $v_3 = 1$ we get the eigenvector $v = (-1, 2, 1)$.
$\lambda = 1 : (A - I)x = 0 \Rightarrow x_1 = 2x_2 + x_3$, where x_2 and x_3 can be any real number (but they cannot be both zero.) Choose $x_2 = 0, x_3 = 1$ and get the eigenvector $w = (1, 0, 1)$, Choose $x_3 = 0, x_2 = 1$ and get the eigenvector $u = (2, 1, 0)$. The matrix $C = \begin{vmatrix} -1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$ is invertible ($\det C = -6 \neq 0$), so it is the diagonalizing matrix. Check that $C^{-1}AC = \begin{vmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \Delta$.
To calculate A^{34}: $A = C\Delta C^{-1}$ so $A = C\Delta^{34}C^{-1}$.
To solve the system: $e^{At} = Ce^{\Delta t}C^{-1} = C \begin{vmatrix} e^{7t} & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{vmatrix} C^{-1}$
$= \frac{1}{6} \begin{vmatrix} e^{7t} + 5e^t & 2(e^t - e^{7t}) & e^t - e^{7t} \\ 2(e^t - e^{7t}) & 2(e^t + 2e^{7t}) & 2(-e^t + e^{7t}) \\ e^t - e^{7t} & 2(-e^t + e^{7t}) & 5e^t + e^{7t} \end{vmatrix}$. The general solution is $y(t) = e^{At}c$ where $c = (c_1, c_2, c_3)$ is a vector of generic constants.

For the second matrix, let me call it B. Characteristic polynomial: $(1 - \lambda)^2(2 - \lambda) = 0 \Rightarrow \lambda = 1, 2$ and 1 has algebraic multiplicity two. $(B - 2I)x = 0 \Rightarrow$

$\begin{vmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$. Working bottom to top $0x_3 = 0, x_2 = 0, x_1 = 0$ so we choose $x_3 = 1$ and we get $v_1 = (0, 0, 1)$. For $\lambda = 1$: $(B - I)x = 0 \Rightarrow \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix} \Rightarrow x_3 = 0, x_2 = 0, 0x_1 = 0$. So we choose $x_1 = 1$ and we get $v_2 = (1, 0, 0)$. So we get only 2 eigenvectors, hence B is not diagonalizable.

4. The matrix A is

$$A = \begin{vmatrix} 0 & 0 & 1 \\ 3 & 7 & -9 \\ 0 & 2 & -1 \end{vmatrix}$$

Characteristic polynomial is $\det(A - \lambda I) = -\lambda^3 - 11\lambda + 6\lambda^2 + 6$. One root is easy to see and it is $\lambda = 1$. Dividing the characteristic polynomial by $(\lambda - 1)$ gives $-\lambda^2 + 5\lambda - 6$, the roots of which are $\lambda = 2$ and $\lambda = 3$. The corresponding eigenvectors are $(1, 1, 1), (1, 3, 2)$ and $(1, 6, 3)$, so the matrix A is diagonalizable and the diagonalizing matrix is $C = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 6 \\ 1 & 2 & 3 \end{vmatrix}$.

We know that $A = C\Delta C^{-1}$ where $\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$. The general solution is then given by $y = e^{At}D$, $D = (d_1, d_2, d_3) \in \mathbb{R}^3$ vector of generic constants. $y = e^{At}D = e^{(C\Delta C^{-1})t}D = Ce^{\Delta t}C^{-1}D$. Because D is arbitrary, $C^{-1}D$ is still a vector of arbitrary constants, which we keep calling D (with abuse of notation), hence the general solution is

$$y = \begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix} = C \begin{vmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{vmatrix} \begin{vmatrix} d_1 \\ d_2 \\ d_3 \end{vmatrix} = \begin{vmatrix} d_1e^t + d_2e^{2t} + d_3e^{3t} \\ d_1e^t + 3d_2e^{2t} + 6d_3e^{3t} \\ d_1e^t + 2d_2e^{2t} + 3d_3e^{3t} \end{vmatrix}$$

5. $A = \begin{vmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{vmatrix} = aI + N_3$ with N_3 defined in handout. Solution is $y(t) = e^{At}y(0)$ so we need to calculate e^{At}. aI and N_3 commute so $e^{At} = e^{atT}e^{N_3t}$. $e^{atI} = e^{at}I$ and $e^{N_3t} = \sum_{k=0}^{\infty} \frac{(N_3t)^k}{k!} = I + N_3t + N_3t^2/2 = \begin{vmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{vmatrix}$.

Putting everything together $y(t) = e^{at} \begin{vmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} e^{at} \\ 0 \\ 0 \end{vmatrix}$.

If $a < 0$ then $y(t)$ approaches the origin as $t \to +\infty$, if $a = 0$ then $y(t)$ remains in $(1, 0, 0)$ $\forall t \geq 0$.

6. Letting $X = (x, y, z)$, the system can be rewritten as $\dot{X} = AX$ where $A = \left| \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & -\omega & 0 \end{array} \right| = \left| \begin{array}{cc} 1 & \mathbf{0} \\ \mathbf{0} & \omega J \end{array} \right|$ and J has been defined in your lecture notes.
The general solution is $X(t) = e^{At}C$ with $C = (c_1, c_2, c_3)$ generic vector of $\mathbb{R}^3$.
$At = \left| \begin{array}{cc} t & \mathbf{0} \\ \mathbf{0} & \omega J \end{array} \right|$ so
$\left| \begin{array}{c} x(t) \\ y(t) \\ z(t) \end{array} \right| = e^{At}C = \left| \begin{array}{cc} e^t & \mathbf{0} \\ \mathbf{0} & e^{\omega t J} \end{array} \right| C = \left| \begin{array}{ccc} e^t & 0 & 0 \\ 0 & \cos \omega t & \sin \omega t \\ 0 & -\sin \omega t & \cos \omega t \end{array} \right| \left| \begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array} \right|$. Imposing the initial conditions we have $x(0) = 1 \Rightarrow c_1 = 1$, $y(0) = 1 \Rightarrow c_2 = 1$, $z(\pi/\omega) = -1 \Rightarrow c_3 = 1$.