Civil Engineering 2 Mathematics Autumn 2011

Exercise Sheet 3

1. Find the Fourier series of period 2π of the following functions, defined on the interval $-\pi < x \leq \pi$:

$$\text{(i) } f(x) = x; \;\; \text{(ii) } f(x) = x^2; \;\; \text{(iii) } f(x) = \sinh x.$$

In each case, does the Fourier series $F_f(x)$ converge to $f(x)$ for $-\pi < x < \pi$? Find the value to which F_f converges at $x = \pi$ and sketch the graph of $F_f(x)$ for $-3\pi < x < 3\pi$.

2. For $f(x) = \cos(\alpha x)$ obtain the Fourier expansion

$$\cos \alpha x = \frac{\sin \alpha\pi}{\alpha\pi} + \sum_{n=1}^{\infty} (-1)^n \frac{2\alpha \sin \alpha\pi}{\pi(\alpha^2 - n^2)} \cos nx, \;\; |x| \leq \pi,$$

when α is not an integer. What happens to the terms of the series when $\alpha \to m$, an integer?

3. Find the half range cosine series that represents the function $(x-1)^2$ over the range $0 \leq x \leq 2$. Using Parseval's theorem deduce the value of $\sum_{n=1}^{\infty} 1/n^4$.

4. Show that the Fourier series representation on $[-\pi, \pi]$ of the function

$$f(x) = \begin{cases} 1 + (x/\pi), & -\pi \leq x < 0, \\ 1 - (x/\pi), & 0 \leq x \leq \pi, \end{cases}$$

is given by

$$F_f(x) = \frac{1}{2} + \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{\cos((2k+1)x)}{(2k+1)^2}.$$

Explain why $F_f(x) = f(x) \, \forall x \in [-\pi, \pi]$. Deduce that

$$\sum_{k=0}^{\infty} 1/(2k+1)^2 = \pi^2/8.$$

5. For the function $f(x) = x(\pi - x)$, $0 \leq x \leq \pi$, derive the Fourier half-range sine and cosine expansions

$$\text{(i) } f(x) = \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{\sin((2k-1)x)}{(2k-1)^3}, \;\; \text{(ii) } f(x) = \frac{\pi^2}{6} - \sum_{k=1}^{\infty} \frac{\cos(2kx)}{k^2},$$

Sketch the graph of the sine expansion and of the cosine expansion for $-\pi \leq x \leq \pi$. Explain why both expansions are valid for $0 \leq x \leq \pi$. By evaluating the series at appropriate points, find:

$$\text{(a)} \sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \text{(b)} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}, \quad \text{(c)} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3}.$$

Use Parseval's theorem to show that $\sum_{n=1}^{\infty} 1/n^6 = \pi^6/945$.

6. Show that the half-range Fourier sine series for $f(x) = 1 + (x/L)$, $0 < x < L$, is:

$$\sum_{n=1}^{\infty} \frac{2}{n\pi} \left(1 - 2(-1)^n\right) \sin\left(\frac{n\pi x}{L}\right).$$

Sketch the function represented by the series in the range $-L < x < L$.

Answers

1. (i) $x = \sum_{n=1}^{\infty} \left(2(-1)^{n+1}/n\right) \sin(nx)$; zero at $x = \pi$;
(ii) $x^2 = \pi^2/3 + 4\sum_{n=1}^{\infty} \left((-1)^n/n^2\right) \cos(nx)$. $f = \pi^2$ at $x = \pi$;
(iii) $\sinh x = ((2\sinh\pi)/\pi) \sum_{n=1}^{\infty} (-1)^{n+1} n \sin(nx)/(1+n^2)$; zero at $x = \pi$.

2. As $\alpha \to m$, all terms of series tend to zero except A_m which tends to 1. Thus the Fourier series tends to $\cos mx$ as $\alpha \to m$.

3. $(x-1)^2 = (1/3) + (4/\pi^2) \sum_{n=1}^{\infty} \cos(n\pi x)/n^2$; $\sum_{n=1}^{\infty} (1/n^4) = \pi^4/90$.

5. (a) $\pi^2/6$; (b) $\pi^2/12$; (c) $\pi^3/32$.