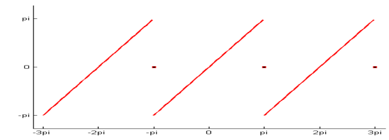
Civil Engineering 2 Mathematics Autumn 2011

Solutions 3

1.(i) $f(x) = x$, odd about $x = 0$, therefore $a_n = 0 \Rightarrow F_f = \sum_{n=1}^{\infty} b_n \sin nx$. $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \ dx = \frac{1}{\pi n}\{[-x\cos nx]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\cos nx\, dx\} = -\frac{2\cos n\pi}{n} = \frac{2(-1)^{n+1}}{n}$. $f(x)$ is continuous for $-\pi < x < \pi$ so

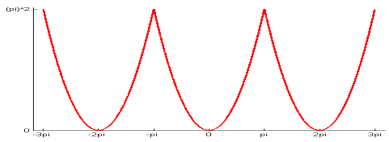
$$\underline{F_f(x) = x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin \ nx, |x| < \pi; \ .}$$

Though from the expression of F_f we have $F_f(\pi) = F_f(-\pi) = 0$ so $F_f = f$ only for $|x| < \pi$.



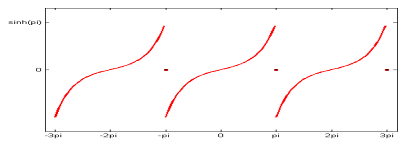
(ii) $f(x) = x^2$, even about $x = 0 \Rightarrow b_n = 0 \Rightarrow F_f = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \ dx = \frac{2\pi^2}{3}$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx \ dx =$ (by parts twice) $= \frac{4(-1)^n}{n^2}$. $f(x)$ is continuous for $-\pi < x < \pi$ and $F_f(\pi) = F_f(-\pi) = \pi^2$, so series is valid for $|x| \leq \pi$

$$\underline{F_f(x) = \ x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \ nx} \quad \text{for } |x| \leq \pi.$$



(iii) $f(x) = \sinh x$, odd about $x = 0 \Rightarrow a_n = 0 \Rightarrow F_f(x) = \sum_{n=1}^{\infty} b_n \sin nx$. $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sinh x \sin nx \ dx = \frac{1}{\pi}\{[\cosh x \sin nx]_{-\pi}^{\pi} - n\int_{-\pi}^{\pi} \cosh x \cos nx dx\} = -\frac{n}{\pi}[\sinh x \cos nx]_{-\pi}^{\pi} - \frac{n^2}{\pi}\int_{-\pi}^{\pi} \sinh x \sin nx dx \Rightarrow (1+n^2)b_n = -\frac{2n}{\pi}(-1)^n \sinh \pi \Rightarrow b_n = (-1)^{n+1}\frac{2n}{\pi(1+n^2)} \sinh \pi$. $f(x)$ is continuous for $-\pi < x < \pi$ and $F_f(\pi) = F_f(-\pi) = 0$, so series is valid for $|x| < \pi$:

$$\underline{F_f(x) = \sinh \ x = \frac{2\sinh \pi}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{1+n^2} \sin \ nx, \quad \text{for } |x| < \pi.}$$



2. $\cos\alpha x$ is even about $x = 0$ so $b_n = 0$ and $f(x) = F_f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$. The expansion is valid for $\mid x \mid \leq \pi$ by continuity and by looking at the periodic extension of $\cos(\alpha x)$.

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\alpha x \, dx = \frac{2\sin\alpha\pi}{\alpha\pi}$.

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\alpha x \cos nx \, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(n+\alpha)x + \cos(n-\alpha)x \, dx$

$= \frac{1}{2\pi}\left[\frac{\sin(n+\alpha)x}{n+\alpha} + \frac{\sin(n-\alpha)x}{n-\alpha}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left(\frac{\sin(\alpha\pi+n\pi)}{n+\alpha} - \frac{\sin(\alpha\pi-n\pi)}{n-\alpha}\right) = \frac{\cos n\pi}{\pi}\left(\frac{\sin\alpha\pi}{n+\alpha} - \frac{\sin\alpha\pi}{n-\alpha}\right)$

$= (-1)^n 2\alpha\frac{\sin\alpha\pi}{\pi(\alpha^2-n^2)}$. So $\cos\alpha x = \frac{\sin\alpha\pi}{\pi} + \sum_{n=1}^{\infty}(-1)^n\frac{2\alpha\sin\alpha\pi}{\pi(\alpha^2-n^2)}\cos nx$,

for $|x| \leq \pi$. As $\alpha \to m$, all terms of the series $\to 0$ except for a_m :

$a_m = \lim_{\alpha\to m}\frac{(-1)^m 2\alpha\sin\alpha\pi}{\pi(\alpha^2-m^2)}$ To evaluate limit above you can use L'Hopital's rule, so $\lim_{\alpha\to m}\frac{(-1)^m 2\alpha\sin\alpha\pi}{\pi(\alpha^2-m^2)} = \frac{(-1)^m 2}{\pi}\lim_{\alpha\to m}\frac{\sin\alpha\pi+\alpha\pi\cos\alpha\pi}{2\alpha} = \frac{(-1)^m 2}{\pi}\frac{m\pi(-1)^m}{2m} = 1$. Thus, $\boxed{a_m = 1.}$ This means that the Fourier Series converges to $\cos mx$ as $\alpha \to m$.

3. The cosine series converges to $f(x)$ for $0 \leq x \leq 2$ (because $(x-1)^2$ is continuous and also the periodic extension of its even extension is continuous), so $(x-1)^2 = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x$, $0 \leq x \leq 2$.

$a_0 = \int_0^2 (x-1)^2 \, dx = 2/3$,

$a_n = \int_0^2 (x-1)^2 \cos\frac{n\pi x}{2} \, dx =$ (by parts twice) $= \frac{16}{n^2\pi^2}$ when n is even and 0 when n is odd.

$\Rightarrow (x-1)^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\cos(n\pi x)\frac{1}{n^2}$, $0 \leq x \leq 2$ where we used the fact that even numbers can be written as $2n$.

Parseval $\Rightarrow \int_0^2 [f(x)]^2 \, dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2 \Rightarrow 2/5 = 2/9 + \sum_{n=1}^{\infty} 16/(n^4\pi^4)$

$$\underline{\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{16}\left(\frac{2}{5} - \frac{2}{9}\right) = \frac{\pi^4}{90}.}$$

4. $f(x)$ even about $x = 0$ and continuous. Also its periodic extension is continuous. $\Rightarrow F_f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$.

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0}(1+\frac{x}{\pi}) \, dx + \int_0^{\pi}(1-\frac{x}{\pi}) \, dx\right\} = 1$,

$a_n = \frac{1}{\pi}\left\{\int_{-\pi}^{0}(1+\frac{x}{\pi})\cos nx \, dx + \int_0^{\pi}(1-\frac{x}{\pi})\cos nx \, dx\right\} = \frac{2}{\pi}\int_0^{\pi}(1-\frac{x}{\pi})\cos nx$

$=$ (by parts) $= -\frac{2}{n^2\pi^2}((-1)^n - 1)$, so $a_n = 0$ (n even), $4/(n\pi)^2$ (n odd).

Therefore $F_f = f(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos(2k+1)x$ for $x \in [-\pi, \pi]$. Putting $x = 0 : f(0) = 1 = \frac{1}{2} + \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$

$$\underline{\Rightarrow \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.}$$

5. Sine series: $F_f = \sum_{n=1}^{\infty} b_n \sin nx$.

$b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx\, dx = \text{by parts} = \frac{4}{\pi n^3}(1-(-1)^n) = \begin{cases} 0 & n \text{ even} \\ \frac{8}{\pi n^3} & n \text{ odd} \end{cases}$

$F_f(\pi) = F_f(0) = 0$ so

$$\underline{\Rightarrow\ x(\pi - x) = \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{\sin\ (2k-1)x}{(2k-1)^3},\ 0 \le x \le \pi.}$$

The expansion is valid on $[0,\pi]$ because the odd and periodic (of period 2π) extension of f is everywhere continuous.

Cosine series: $a_0 = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\ dx = \frac{\pi^2}{3}$,

$a_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\cos nx\, dx = (\text{by parts}) = -\frac{2}{n^2}(1+(-1)^n) = \begin{cases} -4/n^2 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

$F_f(x) = \frac{\pi^2}{6} - \sum_{k=1}^{\infty}\frac{4}{(2k)^2}\cos 2kx$

$$\underline{\Rightarrow\ x(\pi - x) = \frac{\pi^2}{6} - \sum_{k=1}^{\infty}\frac{\cos 2kx}{k^2},\ 0 \le x \le \pi.}$$

The expansion is valid on $[0,\pi]$ because the even and periodic (of period 2π) extension of f is everywhere continuous.

(a) put $x = 0$ in FCS to get $\boxed{\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}.}$

(b) put $x = \pi/2$ in FCS to get $\frac{\pi^2}{4} = \frac{\pi^2}{6} - \sum_{k=1}^{\infty}\frac{(-1)^k}{k^2} \Rightarrow \boxed{\frac{\pi^2}{12} = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}.}$

(c) put $x = \pi/2$ in FSS to get $\frac{\pi^2}{4} = \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k-1)\pi/2}{(2k-1)^3} \Rightarrow \boxed{\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^3} = \frac{\pi^3}{32}}.$

Applying Parseval's formula to the FSS we have:

$\frac{2}{\pi}\int_0^{\pi} x^2(\pi - x)^2 dx = \sum_{n=1}^{\infty} B_n^2 = \left(\frac{8}{\pi}\right)^2\sum_{k=1}^{\infty}\frac{1}{(2k-1)^6} \Rightarrow \sum_{k=1}^{\infty}\frac{1}{(2k-1)^6} = \frac{\pi^6}{960}$

because $\pi^5/30 = \int_0^{\pi} x^2(\pi - x)^2 dx$.

$\therefore \sum_{n=1}^{\infty}\frac{1}{n^6} \equiv \sum_{n=1}^{\infty}\frac{1}{(2n)^6} + \sum_{n=1}^{\infty}\frac{1}{(2n-1)^6} = \frac{1}{2^6}\left(\sum_{n=1}^{\infty}\frac{1}{n^6}\right) + \frac{\pi^6}{960}$.

$\Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{1}{n^6} = (\pi^6/960)/(1 - \frac{1}{2^6}) = \frac{\pi^6}{945}.}$

6. Sine series is $\sum_{n=1}^{\infty} b_n \sin(n\pi x/L)$ where $b_n = \frac{2}{L}\int_0^L (1 + \frac{x}{L})\sin\left(\frac{n\pi x}{L}\right)\, dx$.

Integrating by parts we find: $b_n = \frac{2}{n\pi}(1 - 2(-1)^n)$. The function is continuous and $F_f(0) = F_f(L) = 0$, so

$$\underline{1 + \frac{x}{L} = \sum_{n=1}^{\infty}\frac{2}{n\pi}(1 - 2(-1)^n)\sin\left(\frac{n\pi x}{L}\right)}$$

only for $x \in (0, L)$.