Civil Engineering 2 Mathematics Autumn 2011

Exercise Sheet 4

1. The wave equation (with a wave-speed of unity) is

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \;\; t \geq 0, \;\; 0 \leq x \leq \pi.$$

Use the method of separation of variables to find the oscillatory (periodic) solution of the wave equation with boundary conditions

$$u(0,t) = 0 \text{ and } u(\pi, t) = 0 \text{ for all } t > 0$$

and initial conditions

$$u(x,0) = \sin x + 2\sin 7x \text{ and } \frac{\partial u}{\partial t}(x,0) = 0 \text{ for all } 0 \leq x \leq \pi.$$

2. Use separation of variables to find the solution to the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

over the range $0 < x < L$, $t > 0$, that decays exponentially in time and with the perfectly-insulated boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0,$$

and the initial condition $u(x,0) = f(x), \; 0 < x < L$.

Find the particular solutions for the cases

$$\text{(i) } f(x) = x^2, \;\text{(ii) } f(x) = \begin{cases} 1, & 0 < x < L/2 \\ 0, & L/2 < x < L. \end{cases} \quad \text{(iii) } f(x) = \frac{1}{2}\cos\left(\frac{2\pi x}{L}\right).$$

3. Find the solution of Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

in the rectangle $0 \leq x \leq 2, \; 0 \leq y \leq 4$, by the method of separation of variables, where:

(i) u = 1 on the upper side and zero on the other three sides;

(ii) u = $\sin(\pi x/2)$ on the upper side and zero on the other three sides.

4. It can be shown that the small vertical vibrations of a uniform beam are governed by

$$\frac{\partial^2 u}{\partial t^2} + c^2\frac{\partial^4 u}{\partial x^4} = 0. \qquad (1)$$

Let $u(x,t) = X(x)T(t)$ where

$$\begin{aligned} X &= A\cos\beta x + B\sin\beta x + C\cosh\beta x + D\sinh\beta x, \\ T &= a\cos c\beta^2 t + b\sin c\beta^2 t, \end{aligned}$$

where A, B, C, D, a, b, β are arbitrary constants and $\beta \neq 0$. Verify that u is a solution of equation (1). Hence find the solution subject to the boundary conditions

$$u(0,t) = u(L,t) = \frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(L,t) = 0,$$

and the initial conditions

$$u(x,0) = f(x), \;\; \frac{\partial u}{\partial t}(x,0) = 0, \;\; (0 < x < L).$$

Find the dependence on n of the frequency of the n^{th} mode - how does this compare with that for the wave equation in Q1? (The frequency is the coefficient of the time variable t.)

5. Classify the following partial differential equations:

$$\text{(i) } 2u_{xx} + u_{xy} - 4u_x + u = 0; \;\; \text{(ii) } u_{yy} + u_x = \exp(x);$$
$$\text{(iii) } -u_{xx} + 5u_{xy} + 4u_{yy} + u_x - 7u_y = 0; \;\; \text{(iv) } 6u_{xx} - u_{xy} + 5u_{yy} - 3u = \sin x.$$

6. Show that the partial differential equation

$$2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} = 0$$

is hyperbolic and transform the equation to canonical form. Obtain the general solution of the transformed equation in terms of two arbitrary functions. Deduce the general solution of the original equation and find the particular solution which satisfies the following boundary conditions at $y = 0$:

$$u = 0, \;\; \frac{\partial u}{\partial y} = 2x\exp(-x^2) \quad \text{for all } x.$$

7. Transform the PDE

$$-6u_{xx} - u_{xy} + u_{yy} + 3u_x - u_y = -25y$$

to canonical form and hence find the general solution of this equation. Find also the particular solution corresponding to the boundary conditions:

$$u(x,0) = -\frac{1}{2}x^2, \;\; \frac{\partial u}{\partial y}(x,0) = 25 - 3x - 5e^{\frac{1}{5}x}.$$

Answers

1. $u = \sin x \cos t + 2 \sin 7x \cos 7t$.

2. $u = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L) \exp(-n^2\pi^2 t/L^2)$
with $a_n = (2/L) \int_0^L f(x) \cos(n\pi x/L)\, dx$ for $n = 0, 1, 2, \ldots$

(i) $u = \frac{L^2}{3} + \left(\frac{2L}{\pi}\right)^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2}{L^2} t\right)$;

(ii) $u = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n} \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2}{L^2} t\right)$;

(iii) $u = \frac{1}{2} \cos\left(\frac{2\pi x}{L}\right) \exp\left(-\frac{4\pi^2}{L^2} t\right)$.

3. **(i)** $u = \frac{4}{\pi} \sum_{m=1}^{\infty} \sin((2m-1)\frac{\pi x}{2}) \sinh((2m-1)\frac{\pi y}{2})/((2m-1)\sinh(2(2m-1)\pi))$;

(ii) $u = \sin\left(\frac{\pi x}{2}\right) \sinh\left(\frac{\pi y}{2}\right) / \sinh(2\pi)$.

4. $u = \sum_{n=1}^{\infty} B_n \sin(n\pi x/L) \cos(n^2\pi^2 ct/L^2)$
with $B_n = (2/L) \int_0^L f(x) \sin(n\pi x/L)\, dx$ for $n = 1, 2, \ldots$.
The frequency $\propto n^2$ while in Q1 the frequency is proportional to n.

5. **(i)** hyperbolic; **(ii)** parabolic; **(iii)** hyperbolic; **(iv)** elliptic.

6. General solution $u = F(x + y) + G(x - 2y)$; particular solution: $u = \frac{1}{3}(e^{-(x-2y)^2} - e^{-(x+y)^2})$.

7. Gen. sol: $u = \frac{1}{2}(x - 2y)^2 - (x + 3y)(x - 2y) - 5(x - 2y) + G(x - 2y)e^{(x+3y)/5} + H(x + 3y)$;
particular solution: $u = -\frac{1}{2}x^2 + 25y + 8y^2 - 3xy - 5ye^{(x+3y)/5}$.