Civil Engineering 2 Mathematics Autumn 2011

Solutions 4

1. Let $u = X(x)T(t) \Rightarrow X''/X = T''/T = -\lambda^2$ for an oscillatory (periodic) solution in x. Hence $X = A_1 \sin \lambda x + A_2 \cos \lambda x$, $T = B_1 \sin \lambda t + B_2 \cos \lambda t$. b.c's $\Rightarrow X(0) = X(\pi) = 0 \Rightarrow \underline{A_2 = 0}$ & $\sin(\lambda\pi) = 0 \Rightarrow \underline{\lambda = n}$ (an integer). Condition on $\partial u/\partial t \Rightarrow T'(0) = 0 \Rightarrow \underline{B_1 = 0}$. General solution is therefore $\underline{u = \sum_{n=1}^{\infty} a_n \sin nx \cos nt}$. Initial condition on $u \Rightarrow \sin x + 2\sin 7x = \sum_{n=1}^{\infty} a_n \sin nx \ (*)$. Could proceed by using half-range Fourier sine series formula or (easier) by equating coefficients of $\sin nx$ in $(*)$. This gives $a_1 = 1$, $a_7 = 2$ and all other $a_n's$ zero. Thus the solution is $\underline{u = \sin x \cos t + 2 \sin 7x \cos 7t}$.

2. Let $u = X(x)T(t) \Rightarrow X''/X = T'/T = -\lambda^2$ for a solution tht decays exponentially in time. $\Rightarrow X = A_1 \sin \lambda x + A_2 \cos \lambda x$, $T = B_1 \exp(-\lambda^2 t)$. b.c's $\Rightarrow X'(0) = X'(L) = 0 \Rightarrow \underline{A_1 = 0}$ & $\sin \lambda L = 0 \Rightarrow \underline{\lambda = n\pi/L}$. Then

$$u = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L)\exp(-n^2\pi^2 t/L^2).$$

Initial condition $\Rightarrow f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L) \ (0 < x < L)$. Half-range FCS $\Rightarrow \underline{a_n = (2/L)\int_0^L f(x)\cos(n\pi x/L)\,dx}$, for $n = 0, 1, 2, \ldots$.

(i) $f(x) = x^2$. Then $a_n = (2/L)\int_0^L x^2 \cos(n\pi x/L)\,dx =$ (by parts twice) $= \underline{(\frac{2L}{n\pi})^2(-1)^n}$, and $\underline{a_0 = 2L^2/3}$. Therefore solution is

$$u = \frac{L^2}{3} + \left(\frac{2L}{\pi}\right)^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2}{L^2}t\right)$$

(ii) In this case $a_n = (2/L)\int_0^{L/2} \cos(n\pi x/L)\,dx = \underline{(2/n\pi)\sin(n\pi/2)}$ and $\underline{a_0 = 1}$. So the solution for u is $\underline{u = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n} \cos\left(\frac{n\pi x}{L}\right) \exp\left(-n^2\pi^2 t/L^2\right)}$.

(iii) $a_n = (2/L)\int_0^L (1/2)\cos(2\pi x/L)\cos(n\pi x/L)\,dx = \underline{1/2 \text{ if } n = 2 \text{ and zero otherwise}}$ (by evaluating the integral or otherwise). Thus we have $\underline{u = \frac{1}{2}\cos\left(\frac{2\pi x}{L}\right)\exp\left(-\frac{4\pi^2}{L^2}t\right)}$.

3. Let $u = X(x)Y(y)$. If $X''/X = -Y''/Y = \lambda^2$ we obtain contradiction. Similar for $\lambda = 0$. Also, note that for both (i) and (ii) $u = 0$ on $x = 0$ & $x = 2$ so we require a solution periodic in x. In other words, let $u = X(x)Y(y) \Rightarrow X''/X = -Y''/Y = -\lambda^2$ for periodicity in x. Then $X = A\cos\lambda x + B\sin\lambda x$, $Y = C\cosh\lambda y + D\sinh\lambda y$. b.c.'s $X(0) = X(2) = 0 \Rightarrow \underline{A = 0}$ & $\sin 2\lambda = 0 \Rightarrow \underline{\lambda = n\pi/2}$. $Y(0) = 0 \Rightarrow \underline{C = 0}$. Thus: $\underline{u = \sum_{n=1}^{\infty} B_n \sin(n\pi x/2)\sinh(n\pi y/2)}$ for both (i) & (ii).

(i) Condition on $y = 4 \Rightarrow 1 = \sum_{n=1}^{\infty} B_n \sin(n\pi x/2)\sinh(2n\pi)$, $(0 < x < 2)$. Half-range FS $\Rightarrow B_n \sinh(2n\pi) = \int_0^2 \sin(n\pi x/2)\,dx = \cdots = \underline{(2/n\pi)(1 - (-1)^n)} =$

0 if n is even & $4/(n\pi)$ if n odd. Solution is $u = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin(\frac{n\pi x}{2})\sinh(\frac{n\pi y}{2})}{n\sinh(2n\pi)}$. Put $n = 2m-1$ to get answer on question sheet.

(ii) This time we have $\sin(\pi x/2) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/2)\sinh(2n\pi)$, $(0 < x < 2)$. Equating coefficients of $\sin(n\pi x/2)$ we have $B_1\sinh(2\pi) = 1$, $B_n = 0$ otherwise. Thus, $u = \frac{\sin(\frac{\pi x}{2})\sinh(\frac{\pi y}{2})}{\sinh(2\pi)}$.

4. b.c's $\Rightarrow X(0) = X''(0) = 0 \Rightarrow A + C = 0$ and $-A + C = 0$ and also $A = C = 0$. Then $X(L) = X''(L) = 0 \Rightarrow B\sin\beta L + D\sinh\beta L = 0$ and $-\beta^2 B\sin\beta L + \beta^2 D\sinh\beta L = 0 \Rightarrow D\sinh\beta L = 0$ and $B\sin\beta L = 0 \Rightarrow D = 0$ and $\beta L = n\pi$. Thus: $X = B\sin(n\pi x/L)$. $\partial u/\partial t = 0$ at $t = 0 \Rightarrow T'(0) = 0 \Rightarrow b = 0 \Rightarrow T = a\cos(c\beta^2 t) = a\cos(cn^2\pi^2 t/L^2)$. So,

$$u = \sum_{n=1}^{\infty} B_n \sin(n\pi x/L)\cos(n^2\pi^2 ct/L^2) \ (*).$$

At $t = 0 : f(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/L)$ $(0 < x < L)$

$$\Rightarrow B_n = (2/L)\int_0^L f(x)\sin(n\pi x/L)\,dx \ (+).$$

So, solution is $(*)$ with B_n given by $(+)$. The frequency is the coefficient of t and is proportional to n^2. In Q1 the frequency $\propto n$.

5. **(i)** $a = 2, b = 1, c = 0 \Rightarrow b^2 - 4ac = 1 > 0 \Rightarrow$ hyperbolic; **(ii)** $a = 0, b = 0, c = 1 \Rightarrow b^2 - 4ac = 0 \Rightarrow$ parabolic; **(iii)** $a = -1, b = 5, c = 4 \Rightarrow b^2 - 4ac = 41 > 0 \Rightarrow$ hyperbolic; **(iv)** $a = 6, b = -1, c = 5 \Rightarrow b^2 - 4ac = -119 < 0 \Rightarrow$ elliptic.

6. Here $a = 2, b = -1, c = -1 \Rightarrow b^2 - 4ac = 9 > 0 \Rightarrow$ hyperbolic, as required. Then let $\xi = x + \beta y$, $\eta = x + \delta y$. Then $u_x = u_\xi + u_\eta$, $u_y = \beta u_\xi + \delta u_\eta$, $u_{xy} = \beta u_{\xi\xi} + \delta u_{\eta\eta} + (\beta + \delta)u_{\xi\eta}$, $u_{xx} = u_{\xi\xi} + u_{\eta\eta} + 2u_{\xi\eta}$, $u_{yy} = \beta^2 u_{\xi\xi} + \delta^2 u_{\eta\eta} + 2\beta\delta u_{\xi\eta}$. Substitute into pde to get $(2 - \beta - \beta^2)u_{\xi\xi} + (2 - \delta - \delta^2)u_{\eta\eta} + (4 - 2\beta\delta - \beta - \delta)u_{\xi\eta} = 0$. Let β, δ be the roots of $2 - x - x^2 = 0 \Rightarrow (x + 2)(x - 1) = 0 \Rightarrow \beta = -2, \delta = 1$. So new variables are $\xi = x - 2y$, $\eta = x + y$ and the transformed pde is $9\partial^2 u/\partial\xi\partial\eta = 0$. Integrating both sides twice, once w.r.t. ξ and another time w.r.to η (see lecture), we get $u = F(\eta) + G(\xi)$, i.e. $u = F(x + y) + G(x - 2y)$. Apply b.c. $u = 0$ on $y = 0 \Rightarrow 0 = F(x) + G(x)$. Thus: $u = F(x + y) - F(x - 2y)$. Then $\partial u/\partial y = F'(x + y) + 2F'(x - 2y)$. Apply condition on $y = 0 : 3F'(x) = 2x\exp(-x^2) \Rightarrow F(x) = \frac{2}{3}\int xe^{-x^2}dx = -\frac{1}{3}e^{-x^2} + C$. Therefore solution is $u = \frac{1}{3}(e^{-(x-2y)^2} - e^{-(x+y)^2})$.

7. As in Q6 let $\xi = x + \beta y$, $\eta = x + \delta y$. After some work the pde becomes $(-6-\beta+\beta^2)u_{\xi\xi} + (-6-\delta+\delta^2)u_{\eta\eta} + (2\beta\delta-(\beta+\delta)-12)u_{\xi\eta} + (3-\beta)u_\xi + (3-\delta)u_\eta =$

$-25y$. Let β, δ be roots of $-6-x+x^2=0 \Rightarrow (x-3)(x+2)=0 \Rightarrow \underline{\beta=3,\ \delta=-2}$. So $\xi = x+3y$, $\eta = x-2y$ and pde becomes $\underline{-25u_{\xi\eta}+5u_\eta = -25y} \Rightarrow u_{\xi\eta} - \frac{1}{5}u_\eta = y = \frac{1}{5}(\xi-\eta)$. Let $\tau = u_\eta$. Then $\underline{\tau_\xi - \frac{1}{5}\tau = (\xi-\eta)/5}$. Integrating factor is $e^{-\xi/5} \Rightarrow e^{-\xi/5}\tau = \frac{1}{5}\int(\xi-\eta)e^{-\xi/5}\,d\xi = \text{(by parts)} = (\eta-\xi-5)e^{-\xi/5} + \tilde{G}(\eta) \Rightarrow \partial u/\partial\eta = \eta-\xi-5+\tilde{G}(\eta)e^{\xi/5} \Rightarrow \underline{u = \frac{\eta^2}{2} - \xi\eta - 5\eta + G(\eta)e^{\xi/5} + H(\xi)}$. In terms of x and y the general solution is therefore:

$$u = \frac{1}{2}(x-2y)^2 - (x+3y)(x-2y) - 5(x-2y) + G(x-2y)e^{(x+3y)/5} + H(x+3y).$$

Applying bc on u at $y=0$ we have $-x^2/2 = x^2/2 - x^2 - 5x + G(x)e^{x/5} + H(x) \Rightarrow \underline{H(x) = 5x - G(x)e^{x/5}} \Rightarrow \underline{H'(x) = 5 - (G' + \frac{1}{5}G)e^{x/5}}$. Now at $y=0$, $\partial u/\partial y = -2x - x + 10 - 2G'(x)e^{x/5} + \frac{3}{5}G(x)e^{x/5} + 3H'(x)$. Using expression for H' from above we have $(\partial u/\partial y)_{y=0} = -3x + 25 - 5G'(x)e^{x/5}$. Applying bc we have $G'(x) = 1 \Rightarrow \underline{G(x) = x + C}$. Then $\underline{H(x) = 5x - xe^{x/5} - Ce^{x/5}}$ and the solution for u is therefore $u = \frac{1}{2}(x-2y)^2 - (x+3y)(x-2y) - 5(x-2y) + (x-2y)e^{(x+3y)/5} + Ce^{(x+3y)/5} + 5(x+3y) - (x+3y)e^{(x+3y)/5} - Ce^{(x+3y)/5}$. The terms involving the arbitrary constant cancel, and after some simplification we obtain $\underline{u = -\frac{1}{2}x^2 + 25y + 8y^2 - 3xy - 5ye^{(x+3y)/5}}$.