Civil Engineering 2 Mathematics Autumn 2011

Solutions 5

1. $\nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & -2xz & 2yz \end{vmatrix} = (2x+2z)\,\mathbf{i} - (x^2+2z)\,\mathbf{k}.$

So $\nabla\times(\nabla\times\mathbf{A}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x+2z & 0 & -x^2-2z \end{vmatrix} = \underline{(2x+2)\,\mathbf{j}}.$

Now, $\nabla\cdot\mathbf{A} = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(-2xz) + \frac{\partial}{\partial z}(2yz) = 2xy + 2y.$

Then $\nabla(\nabla\cdot\mathbf{A}) = \mathbf{i}\frac{\partial}{\partial x}(2xy+2y) + \mathbf{j}\frac{\partial}{\partial y}(2xy+2y) + \mathbf{k}\frac{\partial}{\partial z}(2xy+2y) = 2y\,\mathbf{i} + (2x+2)\,\mathbf{j}.$

Also, $\tilde{\nabla}^2\mathbf{A} = \partial^2\mathbf{A}/\partial x^2 + \partial^2\mathbf{A}/\partial y^2 + \partial^2\mathbf{A}/\partial z^2 = 2y\,\mathbf{i}.$

Thus, $\nabla(\nabla\cdot\mathbf{A}) - \tilde{\nabla}^2\mathbf{A} = 2y\mathbf{i} + (2x+2)\,\mathbf{j} - 2y\,\mathbf{i} = \underline{(2x+2)\,\mathbf{j}} = \nabla\times(\nabla\times\mathbf{A})$, as required.

2. (i) $(\mathbf{A}\cdot\nabla) = 2yz\,\partial/\partial x - x^2y\,\partial/\partial y + xz^2\,\partial/\partial z.$

Then $(\mathbf{A}\cdot\nabla)\phi = 2yz\frac{\partial}{\partial x}(2x^2yz^3) - x^2y\frac{\partial}{\partial y}(2x^2yz^3) + xz^2\frac{\partial}{\partial z}(2x^2yz^3) = \underline{8xy^2z^4 - 2x^4yz^3 + 6x^3yz^4}$;

(ii) $\nabla\phi = \mathbf{i}\frac{\partial}{\partial x}(2x^2yz^3) + \mathbf{j}\frac{\partial}{\partial y}(2x^2yz^3) + \mathbf{k}\frac{\partial}{\partial z}(2x^2yz^3) = 4xyz^3\,\mathbf{i} + 2x^2z^3\mathbf{j} + 6x^2yz^2\mathbf{k}.$

Then $\mathbf{A}\cdot(\nabla\phi) = (2yz)(4xyz^3) - (x^2y)(2x^2z^3) + (xz^2)(6x^2yz^2) = \underline{8xy^2z^4 - 2x^4yz^3 + 6x^3yz^4}.$

(iii) $(B\cdot\nabla)\mathbf{A} = x^2\frac{\partial}{\partial x}(2yz\,\mathbf{i} - x^2y\,\mathbf{j} + xz^2\mathbf{k}) + yz\frac{\partial}{\partial y}(2yz\,\mathbf{i} - x^2y\,\mathbf{j} + xz^2\mathbf{k}) - xy\frac{\partial}{\partial z}(2yz\,\mathbf{i} - x^2y\,\mathbf{j} + xz^2\mathbf{k})$
$= \cdots = \underline{(2yz^2 - 2xy^2)\,\mathbf{i} - (x^2yz + 2x^3y)\,\mathbf{j} + (x^2z^2 - 2x^2yz)\,\mathbf{k}}$;

(iv) $\mathbf{A}\times(\nabla\phi) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2yz & -x^2y & xz^2 \\ 4xyz^3 & 2x^2z^3 & 6x^2yz^2 \end{vmatrix}$

$= \underline{(-6x^4y^2z^2 - 2x^3z^5)\,\mathbf{i} - (12x^2y^2z^3 - 4x^2yz^5)\,\mathbf{j} + (4x^2yz^4 + 4x^3y^2z^3)\,\mathbf{k}}$;

(v) $\mathbf{A}\times\nabla = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2yz & -x^2y & xz^2 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \end{vmatrix} = \mathbf{i}\,(-x^2y\frac{\partial}{\partial z} - xz^2\frac{\partial}{\partial y}) - \mathbf{j}\,(2yz\frac{\partial}{\partial z} - xz^2\frac{\partial}{\partial x}) + \mathbf{k}\,(2yz\frac{\partial}{\partial y} + x^2y\frac{\partial}{\partial x}).$

Thus, $(\mathbf{A}\times\nabla)\,\phi = \mathbf{i}\,(-x^2y\frac{\partial\phi}{\partial z} - xz^2\frac{\partial\phi}{\partial y}) - \mathbf{j}\,(2yz\frac{\partial\phi}{\partial z} - xz^2\frac{\partial\phi}{\partial x}) + k\,(2yz\frac{\partial\phi}{\partial y} + x^2y\frac{\partial\phi}{\partial x})$

$= \cdots = \underline{(-6x^4y^2z^2 - 2x^3z^5)\mathbf{i} - (12x^2y^2z^3 - 4x^2yz^5)\,\mathbf{j} + (4x^2yz^4 + 4x^3y^2z^3)\mathbf{k}}.$

Note that the answers to (i)&(ii), and (iv)&(v) are the same - this is always true.

3. (i) $\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = \underline{3}.$

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = 0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k} = 0.$$

(ii) Note that if $r^2 = x^2 + y^2 + z^2$ then $\partial r/\partial x = x/r, \partial r/\partial y = y/r$ and $\partial r/\partial z = z/r$.

So $\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}(x/r) + \frac{\partial}{\partial y}(y/r) + \frac{\partial}{\partial z}(z/r)$
$= (1/r) - (x/r^2)\,(\partial r/\partial x) + (1/r) - (y/r^2)\,(\partial r/\partial y) + (1/r) - (z/r^2)\,(\partial r/\partial z)$
$= (3/r) - (x^2 + y^2 + z^2)/r^3 = \underline{2/r}.$

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x/r & y/r & z/r \end{vmatrix}$$

$= \mathbf{i}\,(-(z/r^2)(\partial r/\partial y) + (y/r^2)(\partial r/\partial z)) - \mathbf{j}\,(-(z/r^2)(\partial r/\partial x) + (x/r^2)(\partial r/\partial z)) + \mathbf{k}\,(-(y/r^2)(\partial r/\partial x) + (x/r^2)(\partial r/\partial y))$
$= \mathbf{i}\,(-yz/r^3 + yz/r^3) - \mathbf{j}\,(-zx/r^3 + zx/r^3) + \mathbf{k}\,(-xy/r^3 + xy/r^3) = 0.$

(iii) $\mathbf{v} = \omega\,\mathbf{k} \times (x\,\mathbf{i} + y\,\mathbf{j}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ x & y & 0 \end{vmatrix} = -\omega y\,\mathbf{i} + \omega x\,\mathbf{j}.$

Then $\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}(-\omega y) + \frac{\partial}{\partial y}(\omega x) = 0.$

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\omega y & \omega x & 0 \end{vmatrix} = 0\mathbf{i} - 0\,\mathbf{j} + (\omega + \omega)\,\mathbf{k} = \underline{2\omega\,\mathbf{k}}.$$

(iv) $\mathbf{v} = \omega\,\mathbf{k} \times (\mathbf{r}/r) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ x/r & y/r & 0 \end{vmatrix} = -(\omega y/r)\,\mathbf{i} + (\omega x/r)\,\mathbf{j}.$

Then $\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}(-\omega y/r) + \frac{\partial}{\partial y}(\omega x/r) = (\omega y/r^2)\,(\partial r/\partial x) - \big(\omega x/r^2\big)\,(\partial r/\partial y) = \omega xy/r^3 - \omega xy/r^3 = 0.$

$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\omega y/r & \omega x/r & 0 \end{vmatrix} = \mathbf{k}\,(\omega/r - (\omega x/r^2)(\partial r/\partial x) + \omega/r - (\omega y/r^2)(\partial r/\partial y))$

$= \mathbf{k}\,(2\omega/r - \omega x^2/r^3 - \omega y^2/r^3) = \underline{(\omega/r)\ \mathbf{k}}.$

4. $\nabla\cdot(\phi\mathbf{G}) = \frac{\partial}{\partial x}(\phi G_1) + \frac{\partial}{\partial y}(\phi G_2) + \frac{\partial}{\partial z}(\phi G_3)$
$= (\partial\phi/\partial x)\,G_1 + \phi(\partial G_1/\partial x) + (\partial\phi/\partial y)\,G_2 + \phi(\partial G_2/\partial y) + (\partial\phi/\partial z)\,G_3 + \phi(\partial G_3/\partial z)$
$= \underline{\phi(\nabla\cdot\mathbf{G}) + (\nabla\phi)\cdot\mathbf{G}}$, as required.

Using this result $\nabla\cdot((y/xr^3)\mathbf{r}) = (y/xr^3)\nabla\cdot\mathbf{r} + \nabla(y/xr^3)\cdot\mathbf{r}$. Now $\nabla\cdot\mathbf{r} = 3$ (fromQ3(i)), and

$\nabla(y/xr^3) = \mathbf{i}\,(-y/x^2r^3 - (3y/xr^4)(\partial r/\partial x)) + \mathbf{j}\,(1/xr^3 - (3y/xr^4)(\partial r/\partial y)) + \mathbf{k}\,((-3y/xr^4)(\partial r/\partial z))$. Because $r = \sqrt{x^2+y^2+z^2}$,

use $\partial r/\partial x = x/r$, $\partial r/\partial y = y/r$, $\partial r/\partial z = z/r$ to get

$\nabla(y/xr^3)\cdot\mathbf{r} = -y/xr^3 + y/xr^3 - (3y/xr^5)(x^2+y^2+z^2) = -3y/xr^3.$

Thus $\nabla\cdot((y/xr^3)\mathbf{r}) = 3y/xr^3 - 3y/xr^3 = 0.$

5. (i) $\nabla\times(\nabla\phi) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \end{vmatrix} = \mathbf{i}\,(\phi_{zy} - \phi_{yz}) - \mathbf{j}\,(\phi_{zx} - \phi_{xz}) + \mathbf{k}\,(\phi_{yx} - \phi_{xy}) = 0.$

(ii) $\nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix} = \mathbf{i}\,(A_{3y} - A_{2z}) - \mathbf{j}\,(A_{3x} - A_{1z}) + \mathbf{k}\,(A_{2x} - A_{1y}).$

Then $\nabla\cdot(\nabla\times\mathbf{A}) = \frac{\partial}{\partial x}(A_{3y} - A_{2z}) - \frac{\partial}{\partial y}(A_{3x} - A_{1z}) + \frac{\partial}{\partial z}(A_{2x} - A_{1y})$
$= A_{3yx} - A_{2zx} - A_{3xy} + A_{1zy} + A_{2xz} - A_{1yz} = 0.$