Civil Engineering 2 Mathematics Autumn 2011

Exercise Sheet 6

1. Test whether the following functions can be the real (or imaginary) part of an analytic function ; if yes, then use the Cauchy-Riemann relations to find the general analytic function, $f(z) = u + iv$, associated with them:

$$(i)\ u = xy + x,\ (ii)\ v = 2xy + 2y,\ (iii)\ v = \cos x \sinh y.$$

2. For what values of a and b are the following functions harmonic?

$$(i)\ u = \exp(ax)\cos(3y), \quad (ii)\ v = \exp(-2x)\sin by.$$

For such values determine in case (i) the analytic function $f(z)$ that has u as real part and in case (ii) the analytic function that has v as imaginary part.

3. Check which of the following functions are harmonic. For those which are, calculate the trajectories orthogonal to either $u(x, y) = c$ or $v(x, y) = c$, where c is a generic constant.

$$(i)\ u = \exp(3x)\cos(3y), \quad (ii)\ u = \exp(2x)\cos(3y),$$
$$(iii)\ v = \exp(-2x)\sin 2y, \quad (iv)\ v = \exp(-2x)\sin y.$$

4. Prove that the complex function $f(z) = \bar{z}$ is not differentiable in complex sense, or, in other words, prove that it is not analytic. (Remember that $\bar{z}$ denotes the complex conjugate of z so $f(x+iy) = x-iy$). Prove it in two ways: first using the definition and then the Cauchy Riemann relations.

5. Using the definition, show that the derivative in complex sense of the function $f(z) = z^3$ is $f'(z) = 3z^2$. Also, use the Cauchy Riemann relations to double-prove that $f(z)$ is holomorphic and that $f'(z) = 3z^2$.

Answers

1.(i) $v = y - \frac{1}{2}x^2 + \frac{1}{2}y^2 + c$; (ii) $u = 2x + x^2 - y^2 + c$; (iii) $u = \sin x \cosh y + c$; (c is real in all cases);

2. (i) $a = \pm 3$, $v = \pm e^{\pm 3x} \sin 3y + c$ (ii) $b = \pm 2$, $u = \mp e^{-2x} \cos 2y + c$

3. (i) and (iii) are harmonic. See answer to question 2.