Civil Engineering 2 Mathematics Autumn 2011

Solutions 6

1. (i) $u_x = y+1 \Rightarrow u_{xx} = 0$; $u_y = x \Rightarrow u_{yy} = 0 \Rightarrow u_{xx} + u_{yy} = 0 \Rightarrow$ $\underline{u \text{ harmonic}}$.

C-R $\Rightarrow v_y = u_x = y + 1 \Rightarrow v = \frac{1}{2}y^2 + y + h(x)$; $v_x = -u_y = -x \Rightarrow v = -\frac{1}{2}x^2 + g(y)$

$\Rightarrow v = \frac{1}{2}y^2 + y - \frac{1}{2}x^2 + c \Rightarrow f = u + iv = xy + x + i(\frac{1}{2}y^2 + y - \frac{1}{2}x^2 + c)$

(ii) $v_y = 2x + 2 \Rightarrow v_{yy} = 0$; $v_x = 2y \Rightarrow v_{xx} = 0 \Rightarrow v_{xx} + v_{yy} = 0 \Rightarrow$ $\underline{v \text{ harmonic}}$.

C-R $\Rightarrow u_x = v_y = 2x + 2 \Rightarrow u = x^2 + 2x + g(y)$; $u_y = -v_x = -2y \Rightarrow u = -y^2 + h(x)$

$\Rightarrow u = x^2 - y^2 + 2x + c \Rightarrow f = u + iv = x^2 - y^2 + 2x + c + i(2xy + 2y)$

(iii) $v_y = \cos x \cosh y \Rightarrow v_{yy} = \cos x \sinh y; v_x = -\sin x \sinh y \Rightarrow v_{xx} = -\cos x \sinh y$

$\Rightarrow v_{xx} + v_{yy} = 0 \Rightarrow$ $\underline{v \text{ harmonic}}$. C-R: $u_x = v_y = \cos x \cosh y \Rightarrow u = \sin x \cosh y + g(y)$;

$u_y = -v_x = \sin x \sinh y \Rightarrow u = \sin x \cosh y + h(x) \Rightarrow u = \sin x \cosh y + c$

$\Rightarrow f = u + iv = \sin x \cosh y + i(\cos x \sinh y) + c$

2. (i) $u_{xx} = a^2 e^{ax} \cos 3y$, $u_{yy} = -9e^{ax} \cos 3y$

$\Rightarrow u_{xx} + u_{yy} = (a^2 - 9)e^{ax} \cos 3y \Rightarrow \underline{a = \pm 3}$.

Then, using C-R, $v_y = u_x = \pm 3e^{\pm 3x} \cos 3y$ and $v_x = -u_y = 3e^{\pm 3x} \sin 3y$

$\Rightarrow v = \pm e^{\pm 3x} \sin 3y + c \Rightarrow f = u + iv = e^{\pm 3x} \cos 3y \pm ie^{\pm 3x} \sin 3y + ic$

(ii) $v_{xx} = 4e^{-2x} \sin by$, $v_{yy} = -b^2 e^{-2x} \sin by$

$\Rightarrow v_{xx} + v_{yy} = (4 - b^2)e^{-2x} \sin by \Rightarrow \underline{b = \pm 2}$.

Then, using C-R, $u_y = -v_x = 2e^{-2x} \sin by$ and $u_x = v_y = be^{-2x} \cos by$

$\Rightarrow u = \mp e^{-2x} \cos 2y + c \Rightarrow f = u + iv = \mp(e^{-2x} \cos 2y - ie^{-2x} \sin 2y) + c.$

3. The functions in (ii) and (iv) ar not harmonic the others are. For thoese which are harminic we can use the Cauchy-Riemann relations to determine a family of orthogonal curves. If you look at question 2 you already have the answer!

4. Using the definition, for $h = h_1 + ih_2$

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{f(x + h_1 + i(y + h_2)) - f(x + iy)}{h}$$

$$= \lim_{h\to 0} \frac{x + h_1 - iy - ih_2 - x + iy}{h} = \lim_{h\to 0} \frac{h_1 - ih_2}{h_1 + ih_2}$$

If we calculate this limit when h approaches 0 along the x axis, i.e. if we choose $h_1 = \delta, h_2 = 0$ we have

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{\delta\to 0} \frac{\delta}{\delta} = 1$$

whereas approaching 0 from the y axis, i.e. $h = (0, \delta)$ then

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{\delta \to 0} \frac{-i\delta}{i\delta} = -1$$

hence the function is not differentiable. Using C-R: $f(x+iy) = x - iy$ hence $u(x,y) = x$ and $v(x,y) = -y$; $u_x = 1 \neq v_y = -1$ so the function is not differentiable.

5. Using the definition: for $h = h_1 + ih_2$

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{(z+h)^3 - z^3}{h}$$

$$= \lim_{h \to 0} \frac{h^3 + 3z^2 h + 3zh^2}{h} = \lim_{h \to 0} (h^2 + 3z^2 + 3zh) = 3z^2.$$

Using C-R: first rewrite f in terms of x and y. $z = x + iy$ so $f(z) = z^3 = x^3 - iy^3 + 3x^2yi - 3xy^2 = x^3 - 3xy^2 + i(3x^2y - y^3)$ so $u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$. Hence $u_x = 3x^2 - 3y^2$, $u_y = -6xy$ and $v_x = 6xy$, $v_y = 3x^2 - 3y^2$. Therefore the C-R relations hold. Also, we noticed during the lectures that, when $C - R$ hold, we simply have $f' = u_x + iv_x = 3x^2 - 3y^2 + i6xy = 3z^2$.