Civil Engineering 2 Mathematics Autumn 2011

Solutions 7

1. Here are the regions of integration for the 4 integrals (notice that the y axis is the horizontal one and the x axis is the vertical one)

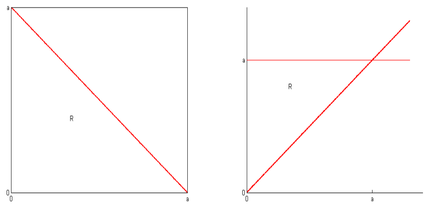

Figure 1: (i) (ii)

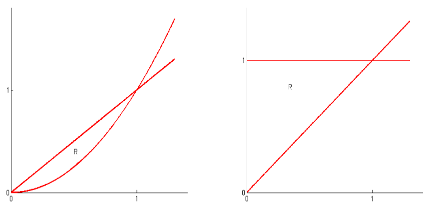

Figure 2: (iii) (iv)

(i)(b) $I = \int_0^a (a-x)\,dx = \underline{a^2/2}$. (c)&(d) $\int_0^a (\int_0^{a-y} dx)\,dy = \int_0^a (a-y)\,dy = \underline{a^2/2}$.

(ii)(b) $\int_0^a \left[x^2y + \frac{1}{3}y^3\right]_0^x dx = \int_0^a \frac{4}{3}x^3 dx = \underline{a^4/3}$.

(c)&(d) $\int_0^a (\int_y^a x^2 + y^2 dx)\,dy = \int_0^a \left[\frac{1}{3}x^3 + y^2x\right]_y^a dy = (\frac{a^4}{3} + \frac{a^4}{3} - \frac{a^4}{12} - \frac{a^4}{4}) = \underline{a^4/3}$.

(iii) (b) $\int_0^1 \left[xy^3/3\right]_x^{\sqrt{x}} = \frac{1}{3}\left[\frac{2}{7}x^{7/2} - \frac{1}{5}x^5\right]_0^1 = \underline{1/35}$.

(c)&(d) $\int_0^1 (\int_{y^2}^y xy^2 dx)\,dy = -\int_0^1 \left[x^2y^2/2\right]_y^{y^2} dy = -\frac{1}{2}\int_0^1 (y^6 - y^4)\,dy = \underline{1/35}$.

(iv) (b) $\int_0^1 xe^{-x^2} dx = -\frac{1}{2}\left[e^{-x^2}\right]_0^1 = \underline{\frac{1}{2}(1-e^{-1})}$.

(c)&(d) Changing order of integration: $\int_0^1 (\int_y^1 e^{-x^2} dx)\,dy$. The inner integral can't be done!

2. If we integrate in dx first then

$$\int\int_B \frac{y}{(1+x)(1+y^2)} dx\,dy = \int_0^1 dy \int_0^{y^2} \frac{y}{(1+x)(1+y^2)} dx$$

$$= \int_0^1 \frac{y}{1+y^2} dy \int_0^{y^2} \frac{dx}{1+x} = \int_0^1 \frac{y}{1+y^2} \log(1+y^2)$$

$$= \frac{1}{4}[\log^2(1+y^2)]\,|_0^1 = \frac{1}{4}\log^2(2).$$

3. First derive $x = (u+v)/2$ and $y = (v-u)/2$. Then

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

and $\det J = 1/2$. In the new coordinate system the boundaries transform to $u = -v, u = v, v = 0, v = 1$. So the integral becomes $I = \int_0^1 \int_{-v}^{v} \frac{1}{2} v^2 \cos(vu)\, dudv = \frac{1}{2}\int_0^1 2v \sin v^2 dv = \frac{1}{2}\int_0^1 \sin t\, dt$ (putting $t = v^2$) $= \underline{\frac{1}{2}(1-\cos(1))}$.

4. Let $u = x - y, v = y + x$. Then $x^2 + y^2 = \frac{1}{2}(u^2+v^2)$ and J is as above.
The boundaries transform to $u = 2, u = 0, v = 0, v = 2$ and the integral becomes
$I = \int_{v=0}^{v=2}\int_{u=0}^{u=2} \frac{1}{2}(u^2+v^2)\frac{1}{2} dudv = \frac{1}{4}\int_{v=0}^{v=2} \left[\frac{1}{3}u^3 + v^2 u\right]_{u=0}^{u=2} dv = \frac{1}{4}\int_{v=0}^{v=2} (\frac{8}{3} + 2v^2)\, dv = \underline{8/3}$.

5. The change of coordinates $x = \rho\cos\theta,\ y = \rho\sin\theta$ has Jacobian

$$J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

and $\det J = \rho$. $I = \int_0^{2\pi} d\theta \int_0^1 d\rho \rho^4 (\,\rho(\sin\theta))\rho\, d\rho\, d\theta =$
$\frac{1}{7}\int_0^{2\pi} \cos^4\theta\, \sin\theta\, d\theta = 0$ (the integral is of the type $\int f^\alpha(x) f'(x)$).

6. Let's integrate w.r. to x first:

$$\int_0^1 dy \int_{\sqrt{y}}^{2-\sqrt{y}} 2dx = \int_0^1 dy 2x \Big|_{x=\sqrt{y}}^{x=2-\sqrt{y}} = \int_0^1 dy(4 - 4\sqrt{y}) = \frac{4}{3}.$$

If we integrate w.r. to y first, we have to split things up:

$$\int_0^1 dx \int_0^{x^2} 2dy + \int_1^2 dx \int_0^{(x-2)^2} 2dy$$

$$= \int_0^1 dx (2y \Big|_{y=0}^{y=x^2}) + \int_1^2 dx (2y \Big|_{y=0}^{y=(x-2)^2}) = \cdots = \frac{4}{3}.$$