Civil Engineering 2 Mathematics Autumn 2011

Warning: I will soon start putting the solutions to these exercises on my webpage. The solutions are not too detailed so if you have problems/doubts come and see me (send an email and we'll arrange to meet). Most important: NEVER TRUST MY CALCULATIONS!

Recap

1. Find eigenvalues and eigenvectors of the matrix

$$A = \begin{vmatrix} 2 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -2 \end{vmatrix}.$$

Determine whether A is diagonalizable. If yes, write the matrix C and the diagonal matrix Δ such that $A = C\Delta C^{-1}$. Using the expression $A = C\Delta C^{-1}$, calculate A^{22} and e^A. Solve the system of ODEs $y'(t) = Ay(t)$, where $y(t) = (y_1(t), y_2(t), y_3(t))^T$ and $'$ denotes derivative with respect to t.

2. Let $z = x + iy$ and let $f(z) = f(x + iy)$ be the function $f(x + iy) = x + y$. Using the definition of holomorphic function, show that f is not holomorphic. Repeat the same exercise for the function $f(z) = (Im(z))^2$, i.e. the function $f(x + iy) = y^2$.

3. Given the function $u(x, y) = ax^2 - by^2 + cy$, where $a, b, c \in \mathbb{R}$, determine the values of a, b and c such that u can be taken to be the real part of an analytic function. For such values, determine the most general analytic function f that has u as real part.

4. Using the change of variable $u = x-y, v = x+y$, evaluate the double integral

$$I = \int\int_R (\sin(x - y)\cos(x + y) + \cos(x - y)\sin(x + y))dxdy,$$

where R is the region of the (x, y)-plane enclosed by the lines $y = 1, x = -1$, and $y = x - 1$. In particular calculate I by integrating v first; check that you get the same result when you integrate u first.

5. Write the matrix A such that the system of ODEs

$$\begin{cases} y_1'(t) = y_1(t) + 2y_2(t) \\ y_2'(t) = -2y_1(t) + y_2(t). \end{cases}$$

can be written as $y'(t) = Ay(t)$, where $y(t) = (y_1(t), y_2(t))^T$. Hence solve the above system.

6. Write the Fourier series of period 2π, $F_f(x)$, of the function $f(x) = x + 1$, for $-\pi < x \leq \pi$. Find the value of $F_f(\pi)$ and $F_f(-\pi)$. Sketch the graph of $F_f(x)$ for $-3\pi < x < 3\pi$. Evaluate the series at an appropriate point in order to calculate

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

7. Verify that $y_1 = x^3$ is a solution of the homogeneous equation

$$x^2 y'' - xy' - 3y = 0.$$

Hence find the general solution of the inhomogeneous equation

$$x^2 y'' - xy' - 3y = \frac{1}{x^5}.$$

(you can assume $x > 0$). Find the solution with initial conditions

$$y(1) = 0, \quad y(2) = \frac{1}{2^{10}}.$$

8. Revise everything we have done about PDEs (both the method of separation of variables and the reduction to canonical form).

9. Revise how to calculate matrix inverse. Any method is fine, as long as you get the right inverse!

10. Given the vector field $\mathbf{A} = xyz\mathbf{i} - x^2\mathbf{j} + \sin z\mathbf{k}$ and the scalar field $\phi = x^2y^2z$, verify that

$$\nabla(\nabla \cdot \mathbf{A}) = z\mathbf{j} + (y - \sin z)\mathbf{k}$$
$$\nabla \times \mathbf{A} = xy\mathbf{j} - (2x + xz)\mathbf{k}$$
$$\nabla\phi \cdot \mathbf{A} = 2x^2y^3z^2 - 2x^4yz + x^2y^2 \sin z.$$

Good luck for the exam!