$\boxed{Q1}$ • Eigenvalues $\lambda_1 = 2 \quad \lambda_2 = -1 \quad \lambda_3 = -2$

• Eigenvectors: $\lambda_1 = 2 \rightsquigarrow v = (1, 0, 0)$

$\lambda_2 = -1 \rightsquigarrow w = (0, 1, 0)$

$\lambda_3 = -2 \rightsquigarrow u = (0, 2, 1)$

$$C = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix}.$$

To calculate C^{-1} you can use either Gauss elimination or the method of the minors. In the latter case:

$$C_M = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{vmatrix} \qquad C_M^T = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{vmatrix} \qquad \det C = 1$$

matrix of the minors

$$\text{so } C^{-1} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{vmatrix}$$

(any other procedure to calculate the inverse is ok)

check that $A = C \Delta C^{-1}$ with $\Delta = \begin{vmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{vmatrix}$.

• $A^{22} = C \Delta^{22} C^{-1} = \ldots$ do your maths!

• $e^{A} = C e^{\Delta} C^{-1}$ where $e^{\Delta} = \begin{vmatrix} e^{2} & 0 & 0 \\ 0 & e^{-1} & 0 \\ 0 & 0 & e^{-2} \end{vmatrix}$

• The solution of the system is $y(t) = e^{At} D$ where $D = \begin{vmatrix} d_1 \\ d_2 \\ d_3 \end{vmatrix}$

$At = C(\Delta t) C^{-1}$ with $\Delta t = \begin{vmatrix} 2t & 0 & 0 \\ 0 & -t & 0 \\ 0 & 0 & -2t \end{vmatrix}$

$$e^{At} D = C e^{\Delta t} \underbrace{C^{-1} D}_{\text{still a vector of generic constants, call it } F} = C e^{\Delta t} F \qquad F = \begin{vmatrix} f_1 \\ f_2 \\ f_3 \end{vmatrix} \qquad e^{\Delta t} = \begin{vmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-2t} \end{vmatrix}$$

$\boxed{Q2}$ We want to look at $\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$

where $h = h_1 + i h_2$.

i) when $f(z) = f(x+iy) = x+y$ we have

$$f(z+h) = f((x+h_1) + i(y+h_2)) = x + h_1 + y + h_2$$

so $$\frac{f(z+h) - f(z)}{h} = \frac{h_1 + h_2}{h_1 + i h_2}$$

choose $h = (0, \delta)$ then $\frac{h_1 + h_2}{h_1 + i h_2} = \frac{\delta}{i\delta} \xrightarrow{\text{as } \delta \to 0} \frac{1}{i}$

choose $h = (\delta, 0)$ then $\frac{h_1 + h_2}{h_1 + i h_2} = \frac{\delta}{\delta} \xrightarrow{\text{as } \delta \to 0} 1$

hence f is not holo.

ii) if $f(z) = f(x+iy) = y^2$ then

$$\frac{f(z+h) - f(z)}{h} = \frac{(y+h_2)^2 - y^2}{h_1 + i h_2} = \frac{h_2^2 + 2y h_2}{h_1 + i h_2}$$

choose $h = (\delta, \delta)$ Then $\frac{h_2^2 + 2y h_2}{h_1 + i h_2} = \frac{\delta^2 + 2y\delta}{\delta + i\delta} \xrightarrow{\delta \to 0} \frac{2y}{1+i}$

choose $h = (0, \delta)$ then $\frac{h_2^2 + 2y h_2}{h_1 + i h_2} = \frac{\delta^2 + 2y\delta}{i\delta} \xrightarrow{\delta \to 0} \frac{2y}{i}$

and these two limits do not coincide (unless $y = 0$)

so f is not holo.

Q5 $A = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix}$. The solution is $y(t) = e^{At} D$ $\quad D = \begin{vmatrix} d_1 \\ d_2 \end{vmatrix}$.

$At = \begin{vmatrix} t & 2t \\ -2t & t \end{vmatrix}$. From lectures we know that

if $J = \begin{vmatrix} a & b \\ -b & a \end{vmatrix}$ then $e^{J} = e^{a} \begin{vmatrix} \cos b & \sin b \\ -\sin b & \cos b \end{vmatrix}$. Hence

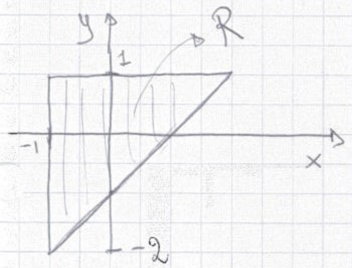
$e^{At} = e^{t} \begin{vmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{vmatrix}$. Now just write

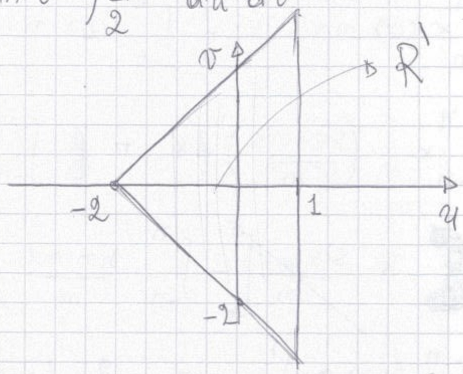
explicitly $y(t) = e^{At} D$.

Q4 we know that the Jacobian determinant is $\frac{1}{2}$.

$$I = \iint_R \Big(\sin(x-y)\cos(x+y) + \cos(x-y)\sin(x+y) \Big)\, dx\, dy$$

$$= \iint_{R'} \Big(\sin u \cos v + \cos u \sin v \Big) \frac{1}{2}\, du\, dv$$





$$\begin{cases} u = x - y \\ v = x + y \end{cases} \Rightarrow \begin{cases} y = \frac{v-u}{2} \\ x = \frac{u+v}{2} \end{cases}$$

$y = 1 \Rightarrow v - u = 2 \Rightarrow v = 2 + u$

$x = -1 \Rightarrow v = -u - 2$

$y - x = -1 \Rightarrow +u = +1$

Recall that $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$, Therefore

$$I = \iint_{R_1} \frac{1}{2} \sin(u+v)\, du\, dv =$$

$$= \int_{-2}^{1} du \int_{-u-2}^{u+2} dv \; \frac{1}{2} \sin(u+v)$$

$$= \int_{0}^{2} dv \int_{v-2}^{1} du \; \frac{1}{2} \sin(u+v) + \int_{-2}^{0} dv \int_{-v-2}^{1} du \; \frac{1}{2} \sin(u+v) = \dots$$

$\boxed{Q6}$ First calculate coefficients a_n, b_n, a_0.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x+1)\, dx = \frac{1}{\pi}\left(\frac{x^2}{2} + x\right)\Bigg|_{x=-\pi}^{x=\pi} = \frac{1}{\pi}\left[\frac{\pi^2}{2} + \pi - \left(\frac{\pi^2}{2} - \pi\right)\right] =$$

$$= \frac{1}{\cancel{\pi}} \cdot 2\cancel{\pi} = 2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x+1) \cos(nx)\, dx =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx + \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx =$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x \left(\frac{\sin nx}{n}\right)' dx + \int_{-\pi}^{\pi} \left(\frac{\sin nx}{n}\right)' dx\right] =$$

$$= \frac{1}{\pi}\left[\cancel{x \; \frac{\sin nx}{n}\Bigg|_{-\pi}^{\pi}} - \int_{-\pi}^{\pi} \frac{\sin nx}{n} + 0\right] =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \left(\frac{\cos nx}{n^2}\right)' dx = \frac{2}{\pi}\left(\frac{\cos n\pi}{n^2}\right) = \frac{2}{\pi}\,\frac{(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (1+x) \sin nx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi}\left(-\frac{\cos nx}{n}\right)' + \int_{-\pi}^{\pi} x\left(-\frac{\cos nx}{n}\right)' dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{\cos n\pi}{\cancel{n}} + \frac{\cos n\pi}{\cancel{n}} - x\,\frac{\cos nx}{n}\Bigg|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos nx}{n}\, dx\right]$$

$$= \frac{1}{\pi}\left[-\left(\pi\,\frac{(-1)^n}{n} + \pi\,\frac{(-1)^n}{n}\right) + \int_{-\pi}^{\pi}\left(\frac{\sin nx}{n^2}\right)' dx\right] =$$

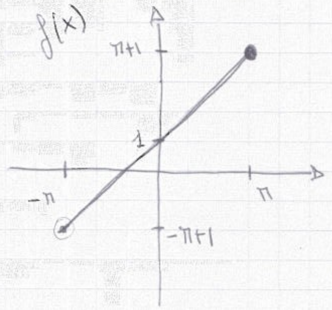
$$= \frac{1}{\pi}\left(\frac{2(-1)^{n+1}\pi}{n} + 0\right) = 2\,\frac{(-1)^{n+1}}{n}$$

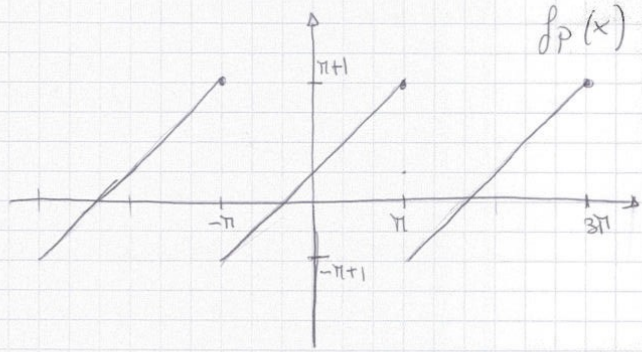
so, hoping I didn't make mistakes in the calculation,

we get $a_0 = 2$ $\quad a_n = \frac{2}{\pi}\frac{(-1)^n}{n^2}$ $\quad \forall n \geq 1$

and $b_n = \frac{2}{n}(-1)^{n+1}$ hence

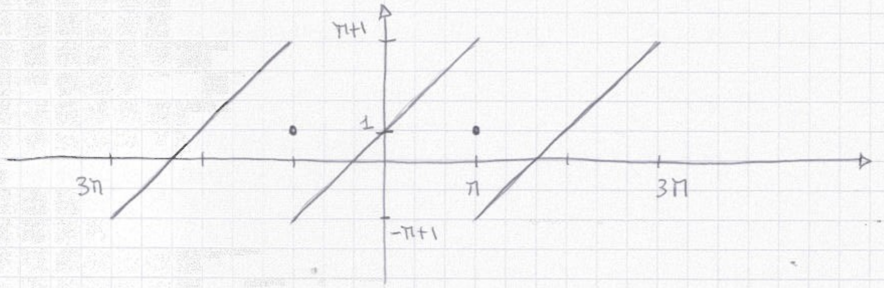
$$\mathcal{F}_f(x) = 2 + \sum_{n=1}^{\infty} \frac{2}{\pi}\frac{(-1)^n}{n^2}\cos nx + \frac{2}{n}(-1)^{n+1}\sin nx$$





$$F_f(\pi) = \frac{f_P^+(\pi) + f_P^-(\pi)}{2} = \frac{\pi+1 - \pi+1}{2} = 1$$

$$F_f(-\pi) = \frac{f_P^+(-\pi) + f_P^-(-\pi)}{2} = 1$$ So



If we calculate $\mathcal{F}_f(x)$ at $x = 0$ then we have

$$1 = 2 + \sum_{n=1}^{\infty} \frac{2}{\pi}\frac{(-1)^n}{n^2}\cdot 1$$

$$\Rightarrow \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -1 \quad \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi}{2}.$$

$\boxed{Q7}$ $\quad x^2 y'' - xy' - 3y = 0 \quad (\#)$

$y_1(x) = x^3$ so $y_1'(x) = 3x^2$, $y_1'' = 6x$

Substituting in (#) we have

$x^2 \cdot 6x - x \cdot 3x^2 - 3x^3 = 0$. So $y_1(x)$ is a sol. of (#).

The general solution of the inhomog. equation

$$x^2 y'' - xy' - 3y = \frac{1}{x^5} \quad (IE)$$

is of the form $\underbrace{y = y_1 \int v}_{(*)}$ with v to be determined

Substitute (*) into (IE) and we get:

(use $y_1 = x^3$, $y = x^3 \int v$, $y' = 3x^2 \int v + x^3 v$

and $y'' = 6x \int v + 3x^2 v + 3x^2 v + x^3 v'$)

$$x^2 \left(\cancel{6x \int v} + 6x^2 v + x^3 v' \right) - x \left(\cancel{3x^2 \int v} + x^3 v \right) - \cancel{3x^3 \int v} = \frac{1}{x^5}$$

$$6x^4 v + x^5 v' - x^4 v = \frac{1}{x^5} \qquad x^5 v' + 5x^4 v = \frac{1}{x^5}$$

$$v' + \frac{5}{x} v = \frac{1}{x^{10}} \quad (\bullet) \qquad A(x) = \int \frac{5}{x} = 5 \log(x) \qquad (x > 0)$$

so $e^{A(x)} = x^5$. Multiply both sides of $(\bullet)$ by x^5:

$$(x^5 v)' = \frac{1}{x^5} \Rightarrow x^5 v = \int \frac{1}{x^5} + C = -\frac{x^{-4}}{4} + C$$

with C generic constant.

Hence $x^5 v = -\frac{1}{4x^4} + C \quad \Rightarrow \quad v = -\frac{1}{4x^9} + \frac{C}{x^5}$

and $\int v = \frac{1}{32x^8} - \frac{C}{4x^4} + D$. Therefore

$y = x^3 \int v = \frac{1}{32x^5} - \frac{C}{4x} + Dx^3 \Big] \longrightarrow$ general solution

$$\left.\begin{aligned} y(1) &= \frac{1}{32} - \frac{C}{4} + D = 0 \\ y(2) &= \frac{1}{2^{10}} - \frac{C}{8} + D8 = \frac{1}{2^{10}} \end{aligned}\right\} \rightsquigarrow$$ hence the values of C & D

(N.B: my calculations aren't perfect, we all know that :). The important thing is the method).