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Recap on Fourier series

A function $f(x)$ is called "$2L$-periodic" if $f(x) = f(x + 2L)$ for all x. A continuous $2L-$periodic function can be represented by (or, in other words, it coincides with) its **Fourier series** of period $2L$, which we will denote $F_f(x)$:

$$F_f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] \quad (1)$$

where the constants a_n and b_n are called the **Fourier coefficients** of $f(x)$. These can be calculated from the formulae

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx\ . \quad (2)$$

The factor of $\frac{1}{2}$ in (1) is included so that the formula (2) holds for $n = 0$ also. It is easiest to deal with 2π-periodic functions ($L = \pi$). The formulae for the coefficients can be derived using the following **orthogonality relations:**

$$\int_{-\pi}^{\pi} \sin mx \cos nx\, dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin mx \sin nx\, dx = 0 = \int_{-\pi}^{\pi} \cos mx \cos nx\, dx$$

where m and n are integers with $m \neq n$. If $m = n \neq 0$, then

$$\int_{-\pi}^{\pi} \sin nx \cos nx\, dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin^2 nx\, dx = \int_{-\pi}^{\pi} \cos^2 nx\, dx = \pi.$$

So, as we said, if $f(x)$ is continuous and $2L$ periodic then $F_f(x) = f(x)\ \forall x \in \mathbb{R}$. What happens if the function f is not continuous or not periodic? Let us start with an example.

Example 1. Consider the "square-wave" function

$$h(x) = \begin{cases} +1 & \text{for} \quad 0 < x < \pi \\ -1 & \text{for} \quad -\pi < x < 0 \end{cases}$$

Then

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{\pi} \cos nx\, dx + \frac{1}{\pi}\int_{-\pi}^{0} (-1)\cos nx\, dx = 0 \\ b_n &= \frac{1}{\pi}\int_0^{\pi} \sin nx\, dx + \frac{1}{\pi}\int_{-\pi}^{0} (-1)\sin nx\, dx \\ &= \frac{1}{\pi}\left([-\cos nx]_0^{\pi} - [-\cos nx]_{-\pi}^{0}\right) = \frac{1}{\pi}(1 - \cos n\pi)\ . \end{aligned}$$

Now $\cos n\pi = (-1)^n$, and so $b_n = 0$ if n is even, and $b_n = 4/(\pi n)$ if n is odd. The Fourier series for $h(x)$, which we will denote $F_h(x)$, is thus

$$F_h(x) = \frac{4}{\pi} \sum_{\substack{n=1 \\ n \ odd}}^{\infty} \frac{\sin nx}{n} = \begin{cases} +1 & \text{for} \quad 0 < x < \pi \\ -1 & \text{for} \quad -\pi < x < 0, \end{cases} \tag{3}$$

so the Fourier series of $h(x)$ coincides with (or better, converges to) $h(x), \forall x \in (-\pi, 0) \cup (0, \pi)$. So far we have not defined $h(x)$ at $x_0 = 0$. Suppose we define

$$h(x) = \begin{cases} +1 & \text{for} \quad 0 \leq x < \pi \\ -1 & \text{for} \quad -\pi < x < 0, \end{cases}$$

so that $h(0) = 1$. Though from (3) we have $F_h(0) = 0$ so in 0 the value of the Fourier series of h does not coincide with the value of h Why is that? Notice that $h(x)$ has a discontinuity at 0.

Behaviour at discontinuities: If the function $f(x)$ is discontinuous at the point $x = x_0$, taking the value $f(x_0)^+$ as x approaches x_0 from the right, and the value $f(x_0)^-$ as x approaches x_0 from the left, then the series converges to the average, $\frac{1}{2}(f^+(x_0) + f^-(x_0))$. Let's be more precise: also at the points of continuity we have $F_f(x) = \frac{1}{2}(f^+(x) + f^-(x))$ but, if $f(x)$ is continuous at x then $f^+(x) = f^-(x) = f(x)$ so $F_f(x) = \frac{1}{2}(f^+(x) + f^-(x)) = \frac{2f(x)}{2} = f(x)$. Indeed, in the previous example, $h^+(0) = 1, h^-(0) = -1$ therefore $\frac{1}{2}(h^+(0) + h^-(0)) = 0$, which is precisely the value of $F_h(0)$ that we had found. However, we will come back to this problem later on.

Odd and even functions: Wy were the cosine coefficients $a_n = 0$ in the above example? This was because $h(x)$ was an *odd* function: $h(-x) = -h(x)$. Thus as cos is an *even* function, $h(x)\cos(n\pi x/L)$ is an odd function. If we integrate an odd function between $-L$ and $+L$ the areas under the curve obviously cancel, and we are left with zero. Similarly, suppose $f(x)$ were an even function. Then $f(x)\sin(n\pi x/L)$ would be an odd function and thus $b_n = 0$ in that case. **Even functions only have cosines and odd functions only have sines** in their Fourier series.

Half-Range Sine & Cosine Series: Suppose $f(x)$ is defined only in $0 < x < L$. Then if we assume $f(x)$ is even, we can extend the definition to $-L < x < L$ and find its Fourier series which will have cosines only. Likewise, if we assume $f(x)$ is odd, then we can find a Fourier series with sines only. These are called **half-range series**.

Differentiation and Integration of Fourier series: If we differentiate (1) with respect to x, we find that

$$F'_f(x) = \sum_{n=1}^{\infty} \left[\left(\frac{-n\pi}{L} \right) a_n \sin\left(\frac{n\pi x}{L} \right) + \left(\frac{n\pi}{L} \right) b_n \cos\left(\frac{n\pi x}{L} \right) \right] \tag{4}$$

which gives us another Fourier series for $f'(x)$. Note that differentiating brings down a factor of n, so that the coefficients of the new Fourier series are larger for large values of n, and the new series may not converge. If it does converge, however, it converges to the right answer. Likewise we can integrate (1). In that case the new series always converges.

Parseval's Theorem: What happens if we take the Fourier series for $f(x)$ and square both sides and then integrate over a period? We can use the orthogonality relations to evaluate the integrals of the product of the two series, and all these integrals are zero except when multiplying like terms together. This gives the result

$$\frac{1}{L}\int_{-L}^{L}[f(x)]^2\,dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) \tag{5}$$

which is known as Parseval's theorem, which is in some sense a generalisation of Pythagoras' theorem! The way to think about it physically is as follows: A Fourier series decomposes a signal into a sum of independent fundamental signals each with an "energy" given by the coefficient squared. Equation (5) then states that the total energy of the original signal is equal to the sum of the energies in the component parts. If we apply (5) to the square wave function, we obtain the strange-looking result

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(1)\,dx = \sum_{\substack{n=1\\ n\,odd}}^{\infty}\left(\frac{4}{\pi n}\right)^2 \quad \Rightarrow \quad \frac{\pi^2}{8} = 1^2 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \dots$$

Fourier series often lead to surprising formulae such as this.

Again on the convergence of Fourier Series. Because *nihil recte sine exemplo docetur*, let's look at some examples.
In Example 1 we saw that at the points of discontinuity of f we need to be careful, because $F_f(x)$ might not converge to $f(x)$. We now want to investigate a bit more carefully what happens at the extrema L and $-L$.

Example 2. Consider the function $f(x) = |x|$, $x \in [-\pi, \pi]$. Using formulae (1) and (2) we find that the Fourier series of $f(x)$ is

$$F_f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos[(2k+1)x]}{(2k+1)}. \tag{6}$$

We know that if f is continuous at x then $f(x) = F_f(x)$. So we already know that $f(x) = F_f(x)$ for $x \in (-\pi, \pi)$. What happens at $x = \pi$ and $x = -\pi$?
Step 1: extend the graph of $f(x)$ by periodicity. We will call f_p the periodic extension of f.
Step 2: the values $F_f(\pi)$ and $F_f(-\pi)$ are given by $F_f(\pi) = \frac{1}{2}\left(f_p^+(\pi) + f_p^-(\pi)\right) = \pi$ and $F_f(-\pi) = \frac{1}{2}\left(f_p^+(-\pi) + f_p^-(-\pi)\right) = \pi$. So in this case $F_f = f$ on $[-\pi, \pi]$.

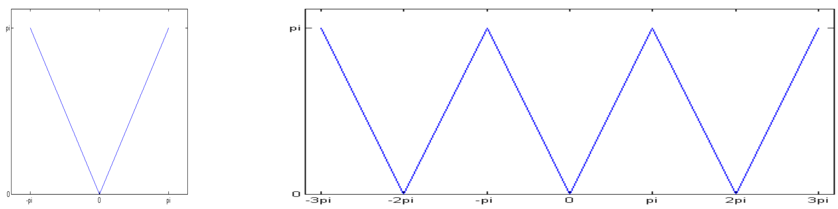


Figure 1: $f(x)$ on the left, $f_p(x)$ on the right.

In this case it is also true that $F_f(x) = f_p(x) \forall x \in \mathbb{R}$ (but this is not true in general). Indeed the graph of F_f can be found in the following way: for $x \in (-\pi, \pi)$ we know that $F_f = f$; for $x = \pi, -\pi$ we have just found the value of F_f and outside $[-\pi, \pi]$ we can use that fact that F_f is periodic of period 2π.

Example 3. Let

$$f(x) = \begin{cases} 0 & \text{if} \quad -\pi < x \leq 0 \\ x & \text{if} \quad 0 < x \leq \pi. \end{cases}$$

Again using formulae (1) and (2) we find that the Fourier series of $f(x)$ is

$$F_f(x) = \frac{\pi}{4} - \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{\cos[(2k+1)x]}{(2k+1)^2} - \sum_{k=1}^{\infty} \frac{(-1)^k \sin kx}{k}.$$

Again, f is continuous in $(-\pi, \pi)$ so $F_f = f$ in $(-\pi, \pi)$. At $x = -\pi$ the function is not defined. To check what happens at $x = \pi$ let us sketch the graph of $f_p(x)$, obtained extending f by periodicity:

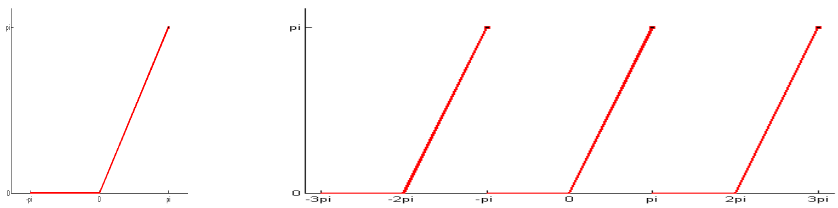
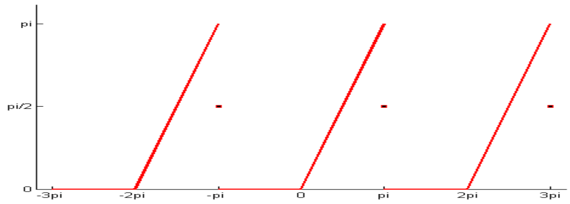


Figure 2: $f(x)$ on the left, $f_p(x)$ on the right. Notice the little marker at the points $(-\pi, \pi)$, (π, π), $(3\pi, \pi)$.

$F_f(\pi) = \frac{1}{2}\left(f_p^+(\pi) + f_p^-(\pi)\right) = \frac{1}{2}(0+\pi) = \frac{\pi}{2}$. So $f(\pi) \neq F_f(\pi)$. Indeed the graph of F_f is



and notice that in this case the graphs of F_f and f_p do not coincide!

In the lucky case in which you can calculate $F_f(L)$ and $F_f(-L)$ directly from the expression for F_f you don't need to follow the procedure that we presented in Example 2 and 3, you can just compare the values of $F_f(L)$ and $F_f(-L)$ with those of $f(L)$ and $f(-L)$.

SUMMARY

• Given a function $f(x)$, continuous on the interval $-L < x < L$,

$$\boxed{F_f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]}$$

where

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

and $f(x) = F_f(x)$ at least for $-L < x < L$.
In general the convergence at the extrema needs to be checked case by case.
If f has a finite number of discontinuities, then at the continuity points we have $f(x) = F_f(x)$ and at the discontinuities $F_f(x_0) = \frac{1}{2}(f^+(x_0) + f^-(x_0))$.
Remember that $F_f(x)$ is a periodic function of period $2L$.

• If we want to represent the function just on $(0, L)$ we can either represent it as a sum of cosines:

$$\boxed{F_f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)} \quad \rightsquigarrow \text{Half range cosine series}$$

with

$$a_0 = \frac{2}{L}\int_{0}^{L} f(x)dx, \quad a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx,$$

or as a sum of sines:

$$\boxed{F_f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)} \quad \rightsquigarrow \text{Half range sine series}$$

with

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

Again, we will have $f(x) = F_f(x)$ at the points of continuity of f. At the points of discontinuity of f and at the extrema of the interval we have to check case by case.

- Parseval's theorem. If $F_f(x) = f(x)$ on $(-L, L)$ then

$$\boxed{\frac{1}{L}\int_{-L}^{L} [f(x)]^2\,dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)}$$

In the case of half range series this simplifies to

$$\boxed{\frac{2}{L}\int_0^L [f(x)]^2\,dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2} \quad \text{for half range cosine series}$$

or to

$$\boxed{\frac{2}{L}\int_0^L [f(x)]^2\,dx = \sum_{n=1}^{\infty} b_n^2} \quad \text{for half range sine series.}$$