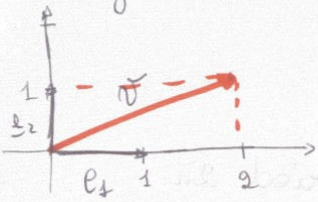
FOURIER SERIES

• To help intuition, let's start with something that has nothing to do with ~~F~~ Fourier Series and that you already know.



Take a vector in $\mathbb{R}^2$, say

$$v = (2, 1).$$

Call $e_1 = (1, 0)$, $e_2 = (0, 1)$ the "fundamental vectors" of $\mathbb{R}^2$. Why "fundamental?" because we can write any vector of $\mathbb{R}^2$ as a sum of e_1 and e_2 with appropriate weights, indeed

$$v = \begin{vmatrix} 2 \\ 1 \end{vmatrix} = \underbrace{2}_{\text{weights}} \begin{vmatrix} 1 \\ 0 \end{vmatrix} + \underbrace{1}_{\text{weights}} \cdot \begin{vmatrix} 0 \\ 1 \end{vmatrix} = 2 e_1 + 1 e_2 .$$

$$w = \begin{vmatrix} -1 \\ 7 \end{vmatrix} = -1 \begin{vmatrix} 1 \\ 0 \end{vmatrix} + 7 \begin{vmatrix} 0 \\ 1 \end{vmatrix} = -1 e_1 + 7 e_2 .$$

Notice that e_1 and e_2 are orthogonal, in fact their scalar product is zero $\begin{vmatrix} 1 \\ 0 \end{vmatrix} \cdot \begin{vmatrix} 0 \\ 1 \end{vmatrix} = 0$.

In the case of v and w we just easily guessed what the right weights are. Though let me remark that

$$v = 2\, e_1 + 1\, e_2$$

$$1 = v \cdot e_2 = \begin{vmatrix} 2 \\ 1 \end{vmatrix} \cdot \begin{vmatrix} 0 \\ 1 \end{vmatrix} = 1$$

$$2 = v \cdot e_1 = \begin{vmatrix} 2 \\ 1 \end{vmatrix} \cdot \begin{vmatrix} 1 \\ 0 \end{vmatrix} = 2$$

so the weight that goes in front of e_1 is the scalar product between v and e_1 and the weight in front of e_2 is the scalar prod. betw. v and e_2.

In this way we can decompose any vector into sums of "fundamental vectors" with appropriate weights.

• Can we do the same thing with functions?
It was Fourier's great idea that we can decompose periodic functions (which we can think of as signals) into sums of "fundamental signals". What do I use as "fundamental signals"?

ORTHOGONALITY RELATIONS

$\cos x$ and $\sin x$ are periodic funct. of period 2π.

$$\int_{-\pi}^{\pi} \sin(ax)\cos(bx)\,dx = 0 \qquad a, b \text{ integers} \quad a \neq b$$

$$\int_{-\pi}^{\pi} \sin(ax)\sin(bx)\,dx = 0$$

$$\int_{-\pi}^{\pi} \cos(ax)\cos(bx)\,dx = 0$$

If a & b are integers $a = b \neq 0$ then

$$\int_{-\pi}^{\pi} \sin(ax)\cos(ax)\,dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} dx \sin^2(ax) = \int_{-\pi}^{\pi} dx \cos^2(ax) = \pi.$$

The integral acts like a kind of a scalar product, so we call the previous relations <u>orthogonality relations</u>, as they tell us that $\sin(ax)$ is orthogonal to $\cos(bx)$ $\forall a, b$ integers.

Hence the idea is that we can use $\{\{\cos mx\}_{m\in\mathbb{N}}, \{\sin mx\}_{m\in\mathbb{N}}\}$

as fundamental signals and decompose functions as sums of sin and cos with appropriate weights.

I won't prove all of the above relations, just a couple:

$$\int_{-\pi}^{\pi} \sin(mx)\cos mx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin(2mx)\,dx = \left.\frac{\cos 2mx}{2m}\right|_{-\pi}^{\pi}$$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$ $\qquad = \frac{1}{2m}\left[\cos(2mx) - \cos(-2mx)\right] = 0$

cos is even

using the identity: $\sin^2\alpha = \frac{1-\cos(2\alpha)}{2}$ we have

$$\int_{-\pi}^{\pi} \sin^2(nx) = \int_{-\pi}^{\pi} \frac{1-\cos(2nx)}{2} = \int_{-\pi}^{\pi} \frac{1}{2} - \int_{-\pi}^{\pi} \frac{2n \cdot \cos(2nx)}{2 \cdot 2n} =$$

$$= \frac{x}{2}\Big|_{-\pi}^{\pi} - \frac{1}{4n} \cancel{\sin(2nx)\Big|_{-\pi}^{\pi}} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$\downarrow$ cos is even ($\cos\alpha = \cos(-\alpha)$)

Def: $f(x)$ is said to be periodic of period $2L$ if $f(x+2L) = f(x) \quad \forall x \in \mathbb{R}$.

- Given a function $f(x)$, continuous and periodic of period $2L$, we can represent it by its Fourier series of period $2L$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] - \quad (*)$$

~~More precisely, given a function, periodic~~

More precisely, given a $2L$-periodic function f, its Fourier series of period $2L$ is $\mathcal{F}_f(x)$,

$$\mathcal{F}_f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) -$$

If the function is continuous and periodic on $\mathbb{R}$ then $\mathcal{F}_f(x)$ actually represents the function, i.e. $\mathcal{F}_f(x) = f(x) \quad \forall x \in \mathbb{R}$, i.e. formula $(*)$ holds.

How do we calculate the coefficients a_n and b_n?

Let's say that our function is periodic of period 2π $(\pi = L)$. Then we can use the orth. rel. in order to find a_n's and b_n's, which are the "scalar product" between f and the "fundamental functions".

Let's see how we do this: if formula (*) holds then, for $q \in \mathbb{N}$:

$$\int_{-\pi}^{\pi} f(x) \cos qx \, dx = \int_{-\pi}^{\pi} \frac{a_0}{2} \cos qx + \int_{-\pi}^{\pi} \sum_{m=1}^{\infty} a_m \cos mx \cos qx$$

$$+ \int_{-\pi}^{\pi} \sum_{u=1}^{\infty} b_u \sin mx \cos qx$$

$$\int_{-\pi}^{\pi} \frac{a_0}{2} \cos qx = -\frac{a_0}{2q} \sin qx \Big|_{-\pi}^{\pi} = 0$$

$= 0 \ \forall m, q$ except when $m = q$

$$\Rightarrow \int_{-\pi}^{\pi} f(x) \cos qx \, dx = \sum_{u=1}^{\infty} a_m \int_{-\pi}^{\pi} \cos mx \cos qx$$

$$+ \sum_{u=1}^{\infty} b_u \underbrace{\int_{-\pi}^{\pi} \sin mx \cos qx}_{= 0 \ \forall m, q}$$

$\to$ so this sum reduces to just one term, the term that we obtain when $m = q$; such term is equal to π.

$$\Rightarrow \int_{-\pi}^{\pi} f(x) \cos qx = a_q \pi$$

$$\Rightarrow a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \qquad \forall m \geq 1$$

If we look at $\int_{-\pi}^{\pi} f(x) \sin(qx) \, dx = b_q \cdot \pi$

so again $b_u = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx) \, dx \quad \forall m \geq 1$.

How do we find a_0?

$$\int_{-\pi}^{\pi} f(x) \, dx = \int_{-\pi}^{\pi} \frac{a_0}{2} dx + \sum_{u=1}^{\infty} a_u \overbrace{\int_{-\pi}^{\pi} \cos mx}^{0} + \sum_{u=1}^{\infty} b_u \overbrace{\int_{-\pi}^{\pi} \sin mx}^{0}$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x) \, dx = \frac{a_0}{2} \cdot x \Big|_{-\pi}^{\pi} = \frac{a_0}{2} \cdot (\pi + \pi) = a_0 \cdot \pi$$

$$\Rightarrow a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx.$$

• REM : $\mathcal{F}_f(x)$ is periodic of period $2L$
(This is an obvious remark but please bear it in mind)

~~What if we want to try and represent something that is not periodic? For example~~ :

• In general the Fourier series of period $2L$ is

$$\mathcal{F}_f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

with $$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx \qquad n \geq 1$$

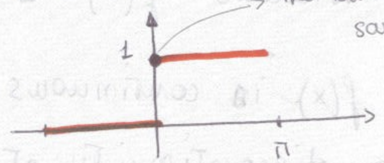
$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx \qquad n \geq 1.$$

• Suppose that instead of wanting to represent a function that is periodic on $\mathbb{R}$, I want to represent a function on the interval $[-L, L]$, for example $[-\pi, \pi]$.

Example : Consider the function

$$f(x) = \begin{cases} 1 & 0 \leq x < \pi \\ 0 & -\pi < x < 0 \end{cases}$$



the bullet here is to say that $f(0) = 1$

Let's calculate $\mathcal{F}_f(x)$.

$$\mathcal{F}_f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\cancel{\pi} x}{\cancel{\pi}}\right) + b_n \sin\left(\frac{n\cancel{\pi} x}{\cancel{\pi}}\right)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\int_0^{\pi} 1\, dx = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx) = \frac{1}{\pi}\int_0^{\pi}\cos(nx)\, dx = \frac{1}{n\pi}\sin(nx)\Big|_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin(nx) = \frac{1}{\pi}\int_0^{\pi} \sin nx\,dx =$$

$$= \frac{1}{n\pi}\left(-\cos(nx)\right)\Big|_0^{\pi} = \frac{1}{n\pi}\left[(-1)\cdot(-1)^n + 1\right] = \begin{cases} 0 & n \text{ even} \\ \frac{2}{n\pi} & n \text{ odd} \end{cases}$$

FACT (without proof): $\displaystyle\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty} \frac{\sin nx}{n} = \begin{cases} \frac{\pi}{4} & 0 < x < \pi \\ -\frac{\pi}{4} & -\pi < x < 0 \\ 0 & x = 0 \end{cases}$

So $a_0 = 1$, $a_n = 0\ \forall n \geq 1$, $b_n = \begin{cases} 0 & n \text{ even} \\ \frac{2}{n\pi} & n \text{ odd} \end{cases}$. Therefore

$$\mathcal{F}_f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} b_n \sin(nx) = \frac{1}{2} + \sum_{\substack{n=1\\ n\text{ odd}}}^{\infty} \frac{2}{\pi}\,\frac{\sin(nx)}{n} =$$

$$= \begin{cases} \frac{1}{2} + \frac{\pi}{4}\cdot\frac{2}{\pi} = 1 & 0 < x < \pi \\ \frac{1}{2} - \frac{\pi}{4}\cdot\frac{2}{\pi} = 0 & -\pi < x < 0 \end{cases}$$

$\Rightarrow \mathcal{F}_f(x) = \begin{cases} 1 & 0 < x < \pi \\ 0 & -\pi < x < 0 \end{cases}$ Well done! Except...

$\mathcal{F}_f(0) = \frac{1}{2}$ whereas $f(0) = 1$! Why???

The function $f(x)$ is continuous on $(-\pi, 0)$ and on $(0, \pi)$ but it has a discontinuity at $x = 0$.

Let $f^+(0) = \lim_{x\to 0^+} f(x)$ and $f^-(0) = \lim_{x \to 0^-} f(x)$.

Then $f^+(0) = 1$ and $f^-(0) = 0$.

At the point where f is discontinuous we have

$$\mathcal{F}_f(0) = \frac{f^+(0) + f^-(0)}{2} = \frac{1}{2}.$$

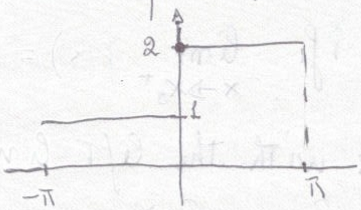
What have we learnt?

REM: Suppose I want to represent the function $f(x)$ defined on $(-L, L)$ through its Fourier series of period $2L$, $\mathcal{F}_f(x)$.

At the points where $f(x)$ is continuous we have $f(x) = \mathcal{F}_f(x)$ whereas if x_0 is a point of discontinuity for $f(x)$ then at x_0 we have $\mathcal{F}_f(x_0) = \dfrac{f^+(x_0) + f^-(x_0)}{2}$

where $f^+(x_0) = \lim\limits_{x \to x_0^+} f(x)$ and $f^-(x_0) = \lim\limits_{x \to x_0^-} f(x)$

REM Notice that the value of $\mathcal{F}_f(x_0)$ (x_0 is a point of discontinuity) does not depend on the value of f at x_0, it only depends on the values of the right limit $f^+(x_0)$ and of the left limit $f^-(x_0)$. To illustrate this point, consider

$f(x) =$ 

$$f(x) = \begin{cases} 2 & 0 \leq x < \pi \\ 1 & -\pi < x < 0 \end{cases}$$

so that $f(0) = 2$.

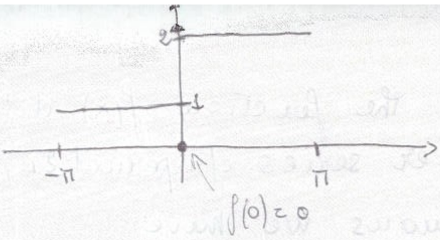
Then $f(x)$ is cont on $(-\pi, 0)$ and on $(0, \pi)$ and it is discont at $x_0 = 0$. So $f(x) = \mathcal{F}_f(x)$ for $x \in (-\pi, 0) \cup (0, \pi)$. But

$f^+(0) = \lim\limits_{x \to 0^+} f(x) = 2$ and $f^-(0) = \lim\limits_{x \to 0^-} f(x) = 1$

so $\mathcal{F}_f(0) = \dfrac{2+1}{2} = \dfrac{3}{2} \neq f(0) = 2$.

Now modify f at $x = 0$: $f(x) = \begin{cases} 2 & 0 < x < \pi \\ 1 & -\pi < x < 0 \\ 0 & x = 0 \end{cases}$

i.e.



Even if $f(0)=0$, we still have $\lim_{x\to 0^+} f(x) = 2 = f^+(0)$

and $f^-(0) = \lim_{x \to 0^-} f(x) = 1$, so

$\mathcal{F}_f(0) = \frac{2+1}{2} = \frac{3}{2} \neq f(0) = 0$ again!

REMINDER (In case you don't remember some basic facts about continuity)

Loosely speaking, a continuous function is a function without jumps. The correct definition is the following: let $f(x)$ be a function on the interval $[a,b]$ =

f is continuous at $x_0 \in (a,b)$ if $\lim_{x \to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = f(x_0)$

i.e. if the right limit coincides with the left limit and with the value of the function at x_0.

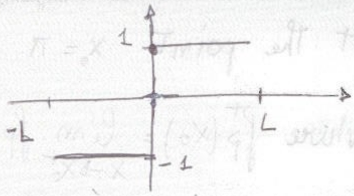
At the extrema a and b we can only talk about right continuity and left continuity, respectively:

$f(x)$ is right continuous at a if $\lim_{x \to a^+} f(x) = f(a)$ and

$f(x)$ is left continuous at b if $\lim_{x \to b^-} f(x) = f(b)$.

In conclusion a function f is continuous in $[a,b]$ if it is continuous $\forall x \in (a,b)$ and it is right cont. at a and left cont. at b.

REM: if $f(x) = \begin{cases} 1 & 0 \le x < L \\ -1 & -L < x < 0 \end{cases}$ i.e.



then $\lim_{x \to 0^+} = 1$ and $\lim_{x \to 0^-} = -1$. So the function is not cont. at $x = 0$. Notice also that changing the value of $f(x)$ at $x=0$ does not change the value of the right and left limit! Indeed if

$f(x) = \begin{cases} 1 & 0 < x < L \\ -1 & -L < x < 0 \\ 0 & x = 0 \end{cases}$ we still have that $\lim_{x \to 0^+} f = 1$ and $\lim_{x \to 0^-} f = -1$.

Going back to our Fourier series, now that we have seen how $\mathcal{F}_f(x)$ behaves at discontinuities of f, let's see what happens at the extrema. Consider the following

EXAMPLE:

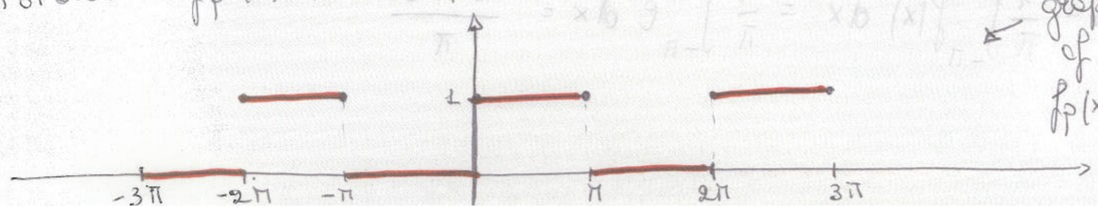
$f(x) = \begin{cases} 1 & 0 \le x \le \pi \\ 0 & -\pi < x < 0 \end{cases}$ What is the value of $\mathcal{F}_f(x)$ at $x = \pi$?

At $x = \pi$ we can only talk about left continuity so a priori we cannot say anything. Though we had found

$\mathcal{F}_f(x) = \frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{2}{\pi} \frac{\sin(nx)}{n}$ hence $\mathcal{F}_f(\pi) = \frac{1}{2} \neq f(\pi) = 1$!

Why is that? The reason is the following:

Consider $f_p(x)$, the periodic extention of $f(x)$, i.e.



graph of $f_p(x)$

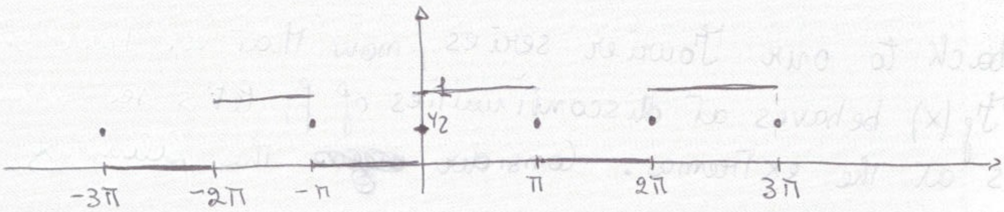
At the point $x_0 = \pi$ we have $\mathcal{F}_{f_p}(x_0) = \frac{f_p^+(x_0) + f_p^-(x_0)}{2}$

where $f_p^+(x_0) = \lim_{x \to x_0^+} f_p(x)$ and $f_p^-(x_0) = \lim_{x \to x_0^-} f_p(x_0)$,

Indeed $f_p^+(\pi) = 0$, $f_p^-(\pi) = 1$ so $\mathcal{F}_f(\pi) = \frac{1+0}{2} = \frac{1}{2}$.

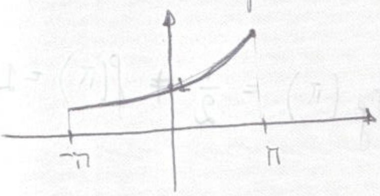
So, what is the graph of $\mathcal{F}_f(x)$, $\forall x \in \mathbb{R}$?

We know that $\mathcal{F}_f = f$ for $x \in (-\pi, 0) \cup (0, \pi)$ and that $\mathcal{F}_f(\pi) = \frac{1}{2}$. Also, we know that $\mathcal{F}_f$ is periodic of period 2π so once we know how to represent it in $(-\pi, \pi]$ we also know how to represent it everywhere.



Notice that by periodicity we know $\mathcal{F}_f(x) = \mathcal{F}_f(x + 2\pi)$ $\forall x$, so $\mathcal{F}_f(-\pi) = \mathcal{F}_f(-\pi + 2\pi) = \mathcal{F}_f(\pi)$.

<u>EXAMPLE</u>. Find the Four series expansion of $f(x) = e^x$ for $-\pi < x \leq \pi$.



$$\mathcal{F}_f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)$$

here $L = \pi$ so

$$\mathcal{F}_f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(mx) + b_m \sin(mx), \text{ where}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x\, dx = \frac{e^{\pi} - e^{-\pi}}{\pi}$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos(mx)\,dx$$

$$\int_{-\pi}^{\pi} e^x \left(\frac{\sin(mx)}{m}\right)' dx = \left[\cancel{e^x \frac{\sin mx}{m}\Big|_{-\pi}^{\pi}} - \int_{-\pi}^{\pi} (e^x)' \frac{\sin(mx)}{m}\,dx \right] =$$

$$= \frac{1}{m^2}\int_{-\pi}^{\pi} e^x (\cos(mx))'\,dx = \frac{1}{m^2}\left[e^x \cos mx \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} e^x \cos(mx)\,dx \right]$$

$$\Rightarrow \left(1 + \frac{1}{m^2}\right)\int_{-\pi}^{\pi} e^x \cos(mx)\,dx = \frac{1}{m^2}\left[e^{\pi}\cos m\pi - e^{-\pi}\cos(-m\pi) \right]$$

$$\Rightarrow \int_{-\pi}^{\pi} e^x \cos(mx)\,dx = \frac{1}{m^2+1}(-1)^m\left(e^{\pi} - e^{-\pi}\right) \Rightarrow a_m = \frac{(-1)^m}{\pi(m^2+1)}\left(e^{\pi} - e^{-\pi}\right)$$

with similar calculations

$$b_m = \frac{1}{\pi}\,\frac{(-1)^{m+1}}{1+m^2}\cdot m\left(e^{\pi} - e^{-\pi}\right).$$ So we have

$$\mathcal{F}_f(x) = \frac{e^{\pi} - e^{-\pi}}{2\pi} + \sum_{m=1}^{\infty}\left[\frac{(-1)^m}{\pi(m^2+1)}\left(e^{\pi} - e^{-\pi}\right)\cos(mx) + \frac{(-1)^{m+1}}{\pi(1+m^2)}\cdot m\left(e^{\pi} - e^{-\pi}\right)\sin(mx) \right]$$

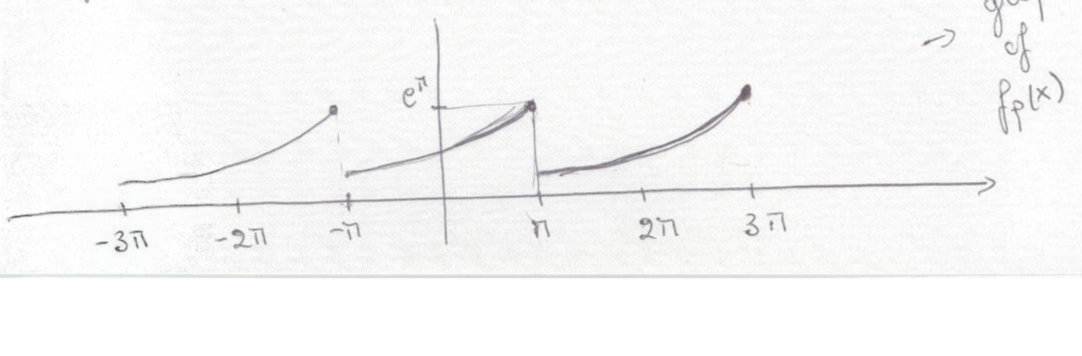
We know that $f(x)$ is continuous on $(-\pi, \pi)$, so we know that $f(x) = \mathcal{F}_f(x)$ $\forall x \in (-\pi, \pi)$. What about $x = \pi$?

Let's try with a direct calculation

$$\mathcal{F}_f(\pi) = \frac{e^{\pi} - e^{-\pi}}{2\pi} + \sum_{m=1}^{\infty} \frac{e^{\pi} - e^{-\pi}}{\pi(m^2+1)} \quad (**)$$ a bit complicated, isn't it?

The strategy we saw before is gonna give an easier answer.

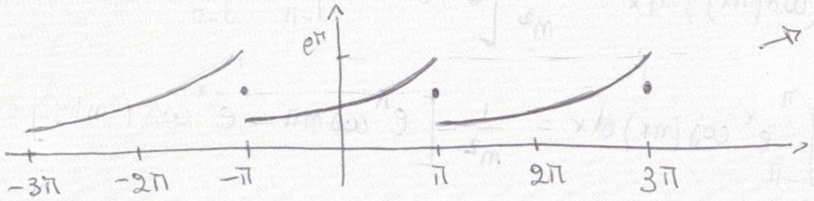
Extend $f(x)$ by periodicity:



We know that $\mathcal{F}_f(\pi) = \dfrac{f_p^+(\pi) + f_p^-(\pi)}{2}$

$f_p^+(\pi) = e^{-\pi}$, $f_p^-(\pi) = e^{\pi}$ so $\mathcal{F}_f(\pi) = \dfrac{e^{\pi} + e^{-\pi}}{2} \; (*)$

So the graph of $\mathcal{F}_f(x)$ is



→ graph of $\mathcal{F}_f(x)$

Notice that equating $(*)$ and $(**)$ we obtain

$$\frac{e^{\pi} - e^{-\pi}}{2\pi} + \sum_{m=1}^{\infty} \frac{e^{\pi} - e^{-\pi}}{\pi(m^2+1)} = \frac{e^{\pi} + e^{-\pi}}{2}$$

$$\Rightarrow \sum_{m=1}^{\infty} \frac{1}{m^2+1} = \left(\frac{e^{\pi} + e^{-\pi}}{2} - \frac{e^{\pi} - e^{-\pi}}{2\pi} \right) \frac{\pi}{e^{\pi} - e^{-\pi}} = \left[\left(\frac{\pi - 1}{2} \right) e^{\pi} + \left(\frac{\pi + 1}{2} \right) e^{-\pi} \right] \frac{1}{(e^{\pi} - e^{-\pi})}$$

This is if you wandered what is $\sum_{m=1}^{\infty} \frac{1}{m^2+1}$.

REM : suppose again that I want to represent $f(x)$ over $(-L, L)$ using Fourier series.

(•) If $f(x)$ is even about 0 then $b_n = 0 \quad \forall n \geq 1$

so the Four series of period $2L$ reduces to $\mathcal{F}_f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\left(\frac{m\pi x}{L}\right)$

(••) If $f(x)$ is odd about 0 then $a_0 = 0$ and $a_m = 0 \quad \forall m \geq 1$.

So $\mathcal{F}_f(x) = \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{L}\right)$

Proof of (••): $f(x)$ odd about zero means $f(x) = -f(-x)$.

$a_0 = \frac{1}{L}\int_{-L}^{L} f(x) = 0$ (you already know that the integral of an odd function over $(-L, L)$ is zero).

let $g(x) = f(x)\cos\left(\frac{m\pi x}{L}\right)$.

$\cos\left(\frac{m\pi x}{L}\right) = \cos\left(-\frac{m\pi x}{L}\right)$ (cos is even)

so $g(-x) = f(-x)\cos\left(-\frac{m\pi x}{L}\right) = -f(x)\cos\left(\frac{m\pi x}{L}\right) = -g(x)$

so $a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right) = \frac{1}{L}\int_{-L}^{L} g(x) = 0$

$g(x)$ is odd

The proof of (•) is anologous. ∎

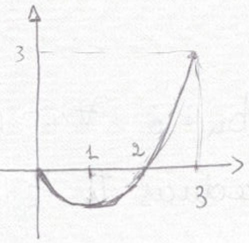
HALF RANGE SINE and COSINE SERIES

- So far we have tried to represent $f(x)$ on $[-L, L]$ through its Fourier series of period $2L$.

Suppose I have a function $f(x)$ defined on $[0, L]$ and I want to represent it only on $[0, L]$.

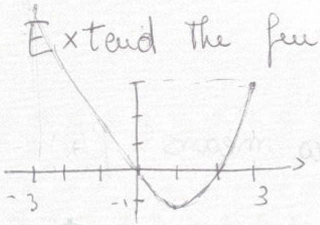
EXAMPLE: Represent the function $f(x)$ on the interval $[0,3]$.

$f(x) = (x-1)^2 - 1$



We have three choices:

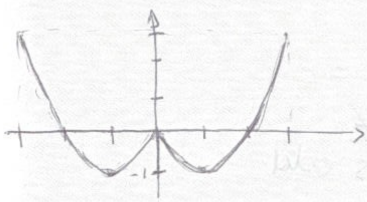
CHOICE 1: Extend the function to the interval $(-3,3)$, so consider



. Find the Fourier series of $f(x)$ of period 6.

$f(x)$ is continuous in $(-3,3)$ so we know that $\mathcal{F}_f(x) = f(x)$ for any x in $(-3,3)$. So in particular $\mathcal{F}_f(x) = f(x)$ $\forall x \in (0,3)$ hence we have found one possible representation of $f(x)$ in $(0,3)$. But there are easier ways...

CHOICE 2: Take the even extention of $f(x)$ to the interval $(-3,3)$, which we will call $f_e(x)$.



Clearly $f_e(x) = f(x)$ $\forall x \in (0,3)$.

Finding the Fourier series of period 6 of $f_e(x)$ is easy because $f_e(x)$ is even so $b_n = 0$ and we need to calculate only a_0 and a_n.

$$\mathcal{F}_{f_e}(x) = \frac{a_{0e}}{2} + \sum_{n=1}^{\infty} a_{n,e} \cos\left(\frac{n\pi x}{3}\right)$$

where $a_{ne} = \dfrac{1}{3}\displaystyle\int_{-3}^{3} f_e(x) \cos\left(\frac{n\pi x}{3}\right) = \frac{2}{3}\int_0^3 f(x) \cos\left(\frac{n\pi x}{3}\right) dx$

and $a_{0e} = \dfrac{1}{3}\displaystyle\int_{-3}^{3} f_e(x)\, dx = \frac{2}{3}\int_0^3 f(x)\, dx$

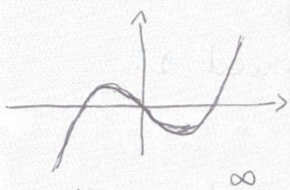
$f_e(x)$ is continuous on $(-3,3)$ so $\mathcal{F}_{f_e}(x) = f_e(x) \; \forall x \in (-3,3)$.
But $f_e(x) = f(x)$ in $(0,3)$. So $\mathcal{F}_{f_e}(x)$ is a representation of $f(x)$ in $(0,3)$.

Briefly: given $f(x)$ in $(0,L)$ its <u>Half Range Fourier cosine series</u> is given by

$$\mathcal{F}_f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\left(\frac{m\pi x}{L}\right)$$ where this time

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx \quad \text{and} \quad a_m = \frac{2}{L}\int_0^L dx\, f(x) \cos\left(\frac{m\pi x}{L}\right) \; \forall m \geq 1.$$

<u>CHOICE 3</u>: Consider the <u>ODD</u> extension of $f(x)$ and obtain the <u>half range Fourier sine series of $f(x)$</u>:



$$\mathcal{F}_f(x) = \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{L}\right) \quad \text{where} \quad b_m = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx$$

<u>Rem</u>: The half range Fourier sine series and the half range Fourier cosine series are periodic functions of period $2L$.

The half range cosine series is an even function on $(-L,L)$ and it coincides with $f(x)$ on $(0,L)$.

The half range sine series is an odd function (on $(-L,L)$) and it coincides with $f(x)$ on $(0,L)$

• Summary: If I want to represent $f(x)$ on $(0,L)$ I can do it either by its Four. cosine series of period $2L$

$$\mathcal{F}_f^c(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\left(\frac{m\pi x}{L}\right) \quad \left(\text{often denoted by } \mathcal{F}_f(x)\right)$$

with $a_0 = \frac{2}{L}\int_0^L f(x)$, $a_m = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx$

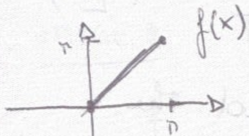
or by its Four sine series of period $2L$

$$\mathcal{F}_f^s(x) = \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{L}\right) \quad \text{with } b_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)$$

(often simply denoted by $\mathcal{F}_f(x)$)

EXAMPLE

Given the function $f(x) = x$, $x \in [0,\pi]$,
find the Fourier cosine series of $f(x)$, of period 2π,
find " " sine " " ",
sketch the graph of both $\mathcal{F}_f^c$ and $\mathcal{F}_f^s$ for $-3\pi \le x \le 3\pi$.



Start with $\mathcal{F}_f^c$:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\frac{x^2}{2}\Big|_0^{\pi} = \frac{2}{\pi}\frac{\pi^2}{2} = \pi$$

$$a_m = \frac{2}{\pi}\int_0^{\pi} x\cos(mx)\,dx = \frac{2}{\pi}\int_0^{\pi} x\left(\frac{\sin(mx)}{m}\right)' dx =$$

$$= \cancel{\frac{2}{\pi}\, x\frac{\sin(mx)}{m}\Big|_0^{\pi}} - \frac{2}{\pi}\int_0^{\pi}\frac{\sin mx}{m}\,dx =$$

$$= \frac{2}{\pi}\frac{\cos(mx)}{m^2}\Big|_0^{\pi} = \frac{2}{m^2\pi}\left[(-1)^m - 1\right] = \begin{cases} -\dfrac{4}{\pi m^2} & m \text{ odd} \\ 0 & m \text{ even} \end{cases}$$

so $\mathcal{F}_f^c(x) = \frac{\pi}{2} + \sum_{\substack{m=1\\ m\,\text{odd}}}^{\infty} \frac{-4}{m^2\pi}\cos(mx) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{\substack{m=1\\ m\,\text{odd}}}^{\infty} \frac{\cos(mx)}{m^2}$

<u>As for $\mathcal{F}_f^s(x)$:</u> $\quad b_m = \frac{2}{\pi}\int_0^\pi f(x)\sin(mx)\,dx =$

$$= \frac{2}{\pi}\int_0^\pi x\sin(mx)\,dx = \frac{2}{\pi}\int_0^\pi x\left(-\frac{\cos(mx)}{m}\right)' =$$

$$= -\frac{2}{\pi}x\,\frac{\cos(mx)}{m}\Big|_0^\pi + \frac{2}{\pi}\int_0^\pi \frac{\cos mx}{m}\,dx =$$

$$= -\frac{2}{\cancel{\pi}}\cdot\cancel{\pi}\,\frac{(-1)^m}{m} + \frac{2}{\pi}\cancel{\frac{\sin mx}{m^2}}\Big|_0^\pi = (-1)^{m+1}\cdot\frac{2}{m}$$

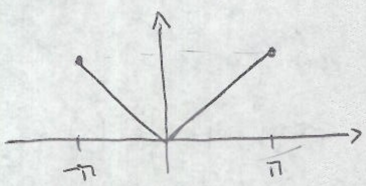
so $\mathcal{F}_f^s(x) = \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\cdot 2\sin(mx)$

We now want to draw the graph of $\mathcal{F}_f^c(x)$.

To this end we know that $\mathcal{F}_f^c(x) = f(x) \quad \forall x \in (0,\pi)$

because $f(x)$ is continuous on $(0,\pi)$ -

What happens at the extrema? Consider $f_{ep}(x)$, the periodic extension of $f_e(x)$:



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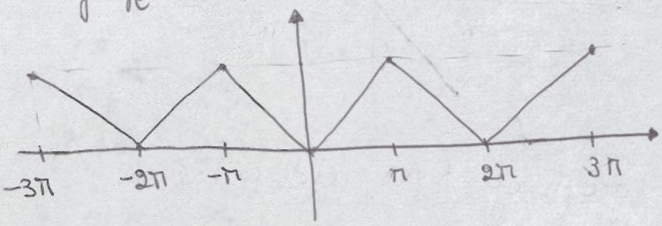
graph of $f_e(x)$, even extension of $f(x)$ to the interval $(-\pi,\pi)$



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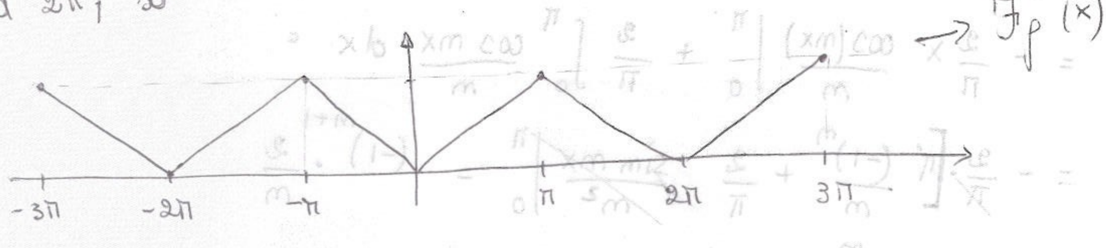
graph of $f_{ep}(x)$, periodic extension of the even extension

$f_{ep}(x)$ is continuous both at zero and at π, so

$\mathcal{F}_f^c(0) = 0$ and $\mathcal{F}_f^c(\pi) = \pi$.

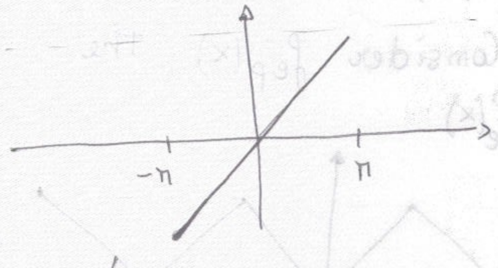
Also, we know that $\mathcal{F}_f^c(0)$ is even on $(-\pi, \pi)$ and it coincides with $f(x)$ on $[0, \pi]$. We also know that it is periodic of period 2π, so



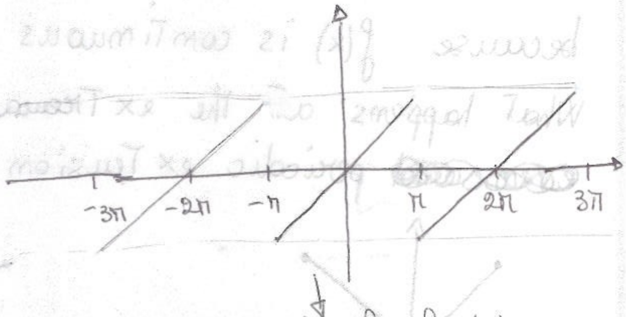
Let us now come to $\mathcal{F}_f^s(x)$.

Again we know that $f(x) = \mathcal{F}_f^s(x) \quad \forall x \in (0, \pi)$.

To find out what happens at the extrema, consider f_{op}, the periodic extension of the odd extension



graph of $f_o(x)$, odd extension of $f(x)$ to $(-\pi, \pi)$



graph of $f_{op}(x)$, periodic extension of the odd extension.

At $x = 0$ $f_{op}(x)$ is continuous so $\mathcal{F}_f^s(0) = f(0) = 0$.

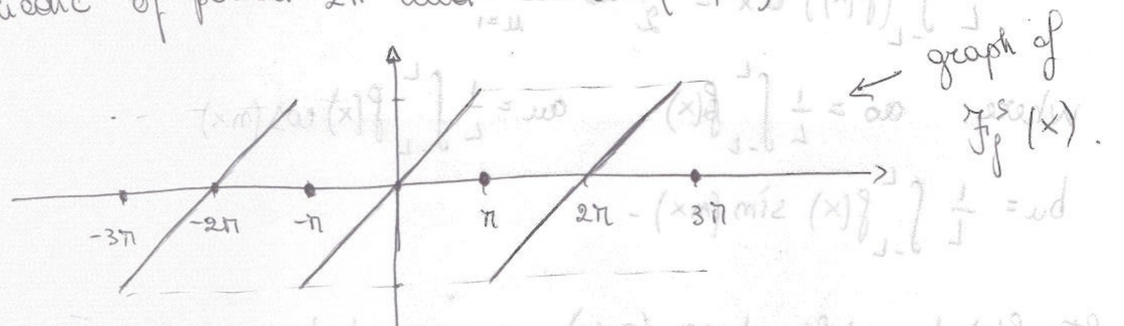
At $x = \pi$ we have that $\mathcal{F}_f^s(\pi) = \dfrac{f_{op}^+(\pi) + f_{op}^-(\pi)}{2}$

where $f_{op}^+(\pi) = \lim_{x \to \pi^+} f_{op}(x)$ and $f_{op}^-(\pi) = \lim_{x \to \pi^-} f_{op}(x)$

so $f_{op}^{+}(\pi) = -\pi$, $f_{op}^{-}(\pi) = \pi$ and

$F_f^s(\pi) = \frac{-\pi+\pi}{2} = 0$ so $F_f^s(\pi) \neq f(\pi)$!!

For the graph of $F_f^s(x)$, again notice that $F_f^s(x)$ is periodic of period 2π and odd on $(-\pi, \pi)$:



• WHAT DO WE USE FOURIER SERIES FOR?

(i) Find the value of series (the sum of ∞ terms)

(ii) Solve PDEs.

(ii) is gonna be the topic of the next few lectures, let us now focus on (i).

We had seen some pages ago that

$$e^x = \frac{e^{\pi} - e^{-\pi}}{2\pi} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{\pi(n^2+1)} (e^{\pi} - e^{-\pi}) \cos(nx) + \frac{(-1)^{n+1} n}{\pi(1+n^2)} (e^{\pi} - e^{-\pi}) \sin(nx) \right]$$

for $-\pi < x < \pi$.

choosing $x = 0$ gives $\quad 1 = \frac{e^{\pi} - e^{-\pi}}{2\pi} + \frac{(e^{\pi} - e^{-\pi})}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n^2+1)}$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} = \left(1 - \frac{e^{\pi} - e^{-\pi}}{2\pi}\right) \cdot \frac{\pi}{e^{\pi} - e^{-\pi}}$$

We can also use the following

PARSEVAL'S THEOREM: • let f be defined on $(-L, L)$; suppose f has a finite number of discontinuities, then

$$\frac{1}{L}\int_{-L}^{L}(f(x))^2\,dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \quad (PP)$$

where $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)$ $\quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(nx)$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(nx)$.

• let $f(x)$ be defined on $(0, L)$; suppose L has a finite number of discontinuities, then

$$\frac{2}{L}\int_0^L f(x)^2\,dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n$$

and $\frac{2}{L}\int_0^L f(x)^2\,dx = \sum_{n=1}^{\infty} b_n$

where $a_0 = \frac{2}{L}\int_0^L f(x)$, $a_n = \frac{2}{L}\int_0^L f(x)\cos(nx)$, $b_n = \frac{2}{L}\int_0^L f(x)\sin(nx)$.

So for example for $f(x) = \begin{cases} 1 & 0 \leq x < \pi \\ 0 & -\pi < x < 0 \end{cases}$

we had found $a_0 = 1$, $a_n = 0$, $b_n = \begin{cases} 0 & n \text{ even} \\ \frac{2}{n\pi} & n \text{ odd} \end{cases}$

So using (PP) we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 = \frac{1}{\pi}\int_0^{\pi} 1\,dx = 1 = \frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty}\left(\frac{2}{n\pi}\right)^2$$

$$\Rightarrow \frac{1}{2} = \frac{4}{\pi^2}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty}\frac{1}{n^2} \Rightarrow \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{8}$$