WELL-POSEDNESS AND STATIONARY SOLUTIONS OF MCKEAN-VLASOV (S)PDES

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Abstract. This paper is composed of two parts. In the first part we consider McKean-Vlasov Partial Differential Equations (PDEs), obtained as thermodynamic limits of interacting particle systems (i.e. in the limit $N \to \infty$, where N is the number of particles). It is well-known that, even when the particle system has a unique invariant measure (stationary solution), the limiting PDE very often displays a phase transition: for certain choices of (coefficients and) parameter values, the PDE has a unique stationary solution, but as the value of the parameter varies multiple stationary states appear. In the first part of this paper, we add to this stream of literature and consider a specific instance of a McKean-Vlasov type equation, namely the Kuramoto model on the torus perturbed by a symmetric double-well potential, and show that this PDE undergoes the type of phase transition just described, as the diffusion coefficient is varied. In the second part of the paper, we consider a rather general class of McKean-Vlasov PDEs on the torus (which includes both the original Kuramoto model and the Kuramoto model in double well potential of part one) perturbed by (strong enough) infinite-dimensional additive noise. To the best of our knowledge, the resulting Stochastic PDE, which we refer to as the *Stochastic McKean-Vlasov equation*, has not been studied before, so we first study its well-posedness. We then show that the addition of noise to the PDE has the effect of restoring uniqueness of the stationary state in the sense that, irrespective of the choice of coefficients and parameter values in the McKean-Vlasov PDE, the Stochastic McKean-Vlasov PDE always admits at most one invariant measure.

Keywords. McKean Vlasov PDE, Stochastic McKean Vlasov equation, Stochastic Partial Differential equations, Ergodic theory for SPDEs, Stationary solutions of PDEs.

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1. Introduction

Consider the following system of N interacting particles

$$dX_t^{i,N} = -V'(X_t^{i,N}) + \frac{1}{N}\sum_{j=1}^{N} F'(X_t^{j,N} - X_t^{i,N})dt + \sqrt{2\sigma}d\beta_t^i, \tag{1.1}$$

where, for every $i = 1 \dots N$, $X_t^{i,N} \in \mathbb{T}$ represents the position of the $i-$th particle on the torus $\mathbb{T}$ of length 2π, $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, the potentials V and F, $V, F : \mathbb{T} \to \mathbb{R}$, are, respectively the *environmental* and *inter-particle potential*, $'$ is the derivative with respect to the argument of the function and the β_t^i's are independent one-dimensional standard Brownian motions.

It is well known that, as $N \to \infty$, the particle system (1.1) converges to the non-local PDE

$$\partial_t \rho_t(x) = \sigma \partial_{xx}\rho_t(x) + \partial_x \left[\left(V'(x) + (F' * \rho_t)(x) \right) \rho_t(x) \right] , \tag{1.2}$$

for the unknown $\rho_t(x) = \rho(t,x) : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}$,[1] in the sense that the empirical measure $\mu_t^N := \frac{1}{N}\sum \delta_{X_t^{i,N}}$ of the particle system (1.1), which is, for each $t > 0$, a random probability measure, converges weakly to the (deterministic) function ρ_t, provided this is true for the corresponding initial data, i.e provided μ_0^N converges to ρ_0 [23, 29]. Another way of seeing this is the following: as $N \to \infty$, the particles become independent (*propagation of chaos*) and, in the limit, the motion of each of them is described by the following SDE

$$dX_t = -\left(V'(X_t) + \int_{\mathbb{T}} F'(y - X_t)\rho_t(dy)\right) dt + \sqrt{2\sigma} d\beta_t\,, \quad X_t \in \mathbb{T}\,, \tag{1.3}$$

where β_t is a one-dimensional standard Brownian motion and ρ_t is the law of X_t at time t, so that the above evolution is non-linear in the sense that the process depends on its own law, i.e. it is *non-linear in the sense of McKean.* By Itô's formula, the law ρ_t of X_t is a solution of the PDE (1.2) and invariant measures of the SDE (1.3) are precisely the stationary solutions of (1.2).

In this paper we will consider (specific instances of) the PDE (1.2) as well as the following SPDE

$$\begin{aligned} &\partial_t u = \partial_{xx} u + \partial_x \left[V^{'} u + (F^{'} * u)u\right] + Q^{1/2}\partial_t W, \quad (0,T)\times\mathbb{T} \\ &u(t,0) = u(t,2\pi), \qquad t \in [0,T] \\ &u(0,x) = u_0(x), \qquad x \in \mathbb{T}, \end{aligned} \tag{1.4}$$

for the unknown $u(t,x) : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}$ (having omitted, as customary, dependence on the realization ω). The evolution (1.4) is obtained from (1.2) by adding infinite dimensional noise, as in the above $W(t,x)$ is cylindrical Wiener noise while Q is a positive, symmetric and trace class operator (precise notation and assumptions in Section 2). Evolutions of the type (1.2) are often called McKean-Vlasov PDEs (or also granular media or aggregation equation) and for this reason we refer to the SPDE (1.4) as to *McKean-Vlasov SPDE* or, more accurately, *Stochastic McKean-Vlasov equation* (SMKV). The former name could be misleading so we clarify that the solution to (1.4), seen as a function-space-valued process, is *not* an (infinite dimensional) McKean-Vlasov SDE, as the process does not depend on its own law. The investigation of infinite-dimensional McKean-Vlasov SDEs has been recently tackled in [27, 28]. However, to the best of our knowledge, the McKean-Vlasov SPDE (1.4) that we consider here has not been studied in the literature, so this paper constitutes a first work on the topic. We will give more detail on comparison between the infinite dimensional McKean-Vlasov SDEs of [27, 28] and the evolution (1.4) in Note 2.7.

The evolution (1.2) and many of its variants have been extensively studied in the PDE, statistical physics, stochastic analysis and modelling literature. In particular well-posedness for (1.2) and (1.3) have been studied in a number of works, see e.g. [6, 26], for a PDE and probabilistic perspective, respectively. So we will not discuss this aspect in the present work. Beyond an intrinsic theoretical interest, McKean-Vlasov evolutions emerge naturally as models in opinion formation, animal navigation, in the study of rating systems and of neural

[1]Throughout the paper, for any quantity, say Y, that depends on time, we use interchangeably the notation Y_t or $Y(t)$ to denote time-dependence

networks, to mention just a few application fields where this equation plays a central role, and for this reason they have attracted growing attention for a few decades now, see [23, 37, 38] and references therein for modelling aspects.

When $V \equiv 0$ and $F(x) = -K \cos x$, for some $K > 0$, F acts as an attractive force between particles and the parameter $K > 0$ modulates the strength of the force; for these choices of V and F, system (1.1) and the related PDE are often referred to as the *Kuramoto model* (or also the mean field classical XY model), which has been subject of careful study, see e.g. [4, 9, 10].[2] In particular, the asymptotic behaviour of this model has been described in detail, using various approaches, see [6, 22, 43], and references therein and, more recently, [5]. One of the phenomena of interest is the following: while (for each N fixed) the Kuramoto particle system has a unique invariant measure (uniqueness being straightforward in view of ellipticity), the Kuramoto PDE undergoes a phase transition. Namely, there exists $K_c > 0$ (depending on the noise strength σ) such that for $K < K_c$ the PDE has a unique stationary solution – the uniform distribution on the torus – while for $K > K_c$ the equation has a whole manifold of stationary states. This can be intuitively understood as follows: under the effect of the force $F(x) = -K \cos x$, particles are attracted to each other; however, as soon as $\sigma > 0$, there is a competition between such an attractive force and the effect of the noise, which makes particles diffuse on the torus (and in this sense it can be seen as having a formally 'repulsive' effect). If the noise is strong enough then the particles spread homogeneously around the torus; if this is not the case then a non-homogeneous steady state appears, concentrated say at zero; then, by rotational symmetry, a whole manifold of steady states follows. Indeed, the Kuramoto model enjoys a rotational symmetry which is key in the study of the dynamics: it is easy to see that, if $X_t^{i,N}$ is a solution of the Kuramoto particle system, then also $X_t^{i,N} + \theta$ is a solution, for any $\theta \in [0, 2\pi]$; accordingly, if ρ_t solves the Kuramoto PDE, then also $\rho_t(x + \theta)$ solves the same PDE, for any $\theta \in [0, 2\pi]$.

For a general $V \neq 0$ the rotational symmetry of the Kuramoto model no longer holds; this is the case on which we focus in the first part of this paper. In particular, in Section 3 we consider the McKean-Vlasov PDE (1.2) when

$$V(x) = \cos(2x)\,, \quad F(x) = -\cos x, \quad x \in \mathbb{T}\,, \tag{1.5}$$

and show that, in this setting, there exists a critical value of the noise, σ_c, such that when $\sigma > \sigma_c$ the PDE (1.2) has a unique stationary solution, whereas when $\sigma < \sigma_c$ (and small enough) there exist exactly three stationary solutions (see Theorem 2.2 for a precise statement), namely the homogeneous distribution and the other two concentrated at either minima of the double-well potential V.[3] This is due to the fact that, with respect to the case when $V = 0$, the particles are subject not only to the competition between attractive force and noise, but also to the environmental potential, which introduces a tendency for the particles to converge towards the minima of the potential.

[2]We point out for completeness that in the physics literature the Kuramoto model also includes the effect of an intrinsic oscillation frequency for each particle, see [33, 34].

[3]In the discussion of the Kuramoto model, to be consistent with the cited literature, we implicitly fixed the value of σ and discussed the phase transition as K varies. In this paper we (equivalently) fix the value of K ($K = 1$, see (1.5)) and study the behaviour as σ varies.

Overall, the above discussion should serve the purpose of showing that the behaviour of the PDE (1.2) can be rather complex, and in particular the number of stationary solutions of the PDE depends on the detailed properties of the potential V and of the inter-particle force F, as well as on the value of σ. This is certainly not the only PDE that has a complicated set of stationary solutions, and indeed similar observations could be made e.g. for the Allen-Cahn and Navier-Stokes equations, to mention just a few examples; for such equations it has been observed that addition of (appropriately strong) noise to the PDE 'restores' uniqueness of the equilibrium state, in the sense that the Stochastic Allen-Cahn and the Stochastic Navier-Stokes equations have a unique invariant measure (stationary state), see [3, 20, 21, 32, 44]. In the second part of this paper we add to this stream of literature and show that the SMKV equation (1.4) admits at most one invariant measure, irrespective of the choice of V, F and σ. We will discuss more the technical aspects involved in proving this result below and in the next section, however we point out that we work here under the assumption that the added noise is 'strong enough', see Theorem 2.6 and comments afterwards. It is not a priori obvious what is the 'minimum amount of noise' one can add to the PDE (1.2) so that the resulting SPDE has a unique invariant measure. This is a question that requires more sophisticated tools than those we use in this paper, such as those developed in [24], and we will tackle such a question in future work.
The fact that the set of stationary solutions of the PDE is generally more complex than the set of stationary solutions of the corresponding SPDE can be seen as the infinite dimensional analogue of what is well known to happen in finite dimension: to fix ideas, let $V(x)$ be a multi-well potential on the torus (but clearly the same is true in $\mathbb{R}$ for multi-well confining potentials); then the deterministic ODE

$$dx_t = -V'(x_t)dt, \quad x_t \in \mathbb{T}, t \geq 0, \tag{1.6}$$

has multiple steady states (as many as the critical points of V). However the Langevin equation

$$dx_t = -V'(x_t)dt + d\beta_t,$$

has a unique invariant measure, as the (elliptic, hence 'strong enough') Brownian noise β_t allows full exploration of state space. If we see the steady states of (1.6) as invariant measures (by considering Dirac deltas concentrated at the critical points), then we can say that the addition of noise has 'restored' uniqueness of the invariant measure.

To summarise, this paper is divided in two parts: in the first part (Section 3) we study the PDE (1.2) when V and F are as in (1.5) and show that the number of steady states depends on the strength of the noise σ. In the second part of the paper, we first prove the well-posedness in mild sense of the SPDE (1.4) (Section 4) and then we show that such an SPDE admits at most one invariant measure. In order to do so, one needs to prove that the semigroup associated with the evolution (1.4) is irreducible and Strong Feller. We prove irreducibility in Section 5 and Strong Feller property in Section 6. The next section, Section 2, contains precise statements of the main results and more thorough relation to literature. The proofs of Section 3 are independent of the proofs Section 4 and following sections.

Further Motivation. As we have mentioned, we are not aware of any works on either well-posedness or ergodic properties of the SMKV equation (1.4), so this paper is, primarily, a first contribution towards establishing properties of such an evolution. In contrast, there is a large literature on the following SPDE

$$\partial_t u_t(x) = \partial_x \left[\left(V'(x) + (F' * u_t)(x)\right) u_t(x) + (\sigma + \tilde{\sigma})\partial_x u_t(x)\right] - \sqrt{2\tilde{\sigma}}(\partial_x u_t)\partial_t \beta_t \,, \tag{1.7}$$

which can be viewed, for the purposes of this discussion, as a different stochastic perturbation of the PDE (1.2), [35] and references therein. The difference between (1.4) and (1.7) is that in the former the (infinite dimensional) noise is additive, while in the latter noise is multiplicative; more importantly, (1.7) has transport (gradient) structure, while (1.4) does not - fact that is source of many complications. The SPDE (1.7) has attracted a lot of attention as it can be obtained in the limit $N \to \infty$ of the following particle system

$$dX_t^{i,N} = -V'(X_t^{i,N}) - \frac{1}{N}\sum_{j=1}^{N} F'(X_t^{i,N} - X_t^{j,N})dt + \sqrt{2\sigma}\beta_t^i + \sqrt{2\tilde{\sigma}}d\beta_t, \tag{1.8}$$

where, crucially, the Brownian noise β_t is the same for each particle, and $\tilde{\sigma} > 0$, [35]. Because the noise same β_t acts on all the particles, the limit of the particle system is no longer deterministic, and it is stochastic instead. In upcoming work we will investigate one possible interpretation of (1.4) in relation to interacting particle limits and some of the results of this work form a basis for future work in this direction.

2. Notation and Main Results

In this section we first introduce some notation and recall some basic facts; we then state the main results of this paper and comment on them in turn.

2.1. **Notation.** In what follows $L^2(\mathbb{T};\mathbb{R})$ denotes the separable Hilbert space of 2π-periodic real-valued square-integrable functions, endowed with the scalar product

$$\langle f, g\rangle_{L^2(\mathbb{T};\mathbb{R})} := \int_{\mathbb{T}} f(x)g(x)\,dx, \qquad f, g \in L^2(\mathbb{T};\mathbb{R}).$$

We fix $\{e_k\}_{k\in\mathbb{Z}}$ to be the following orthonormal Fourier basis of $L^2(\mathbb{T};\mathbb{R})$

$$\begin{cases} e_k(x) = \frac{1}{\sqrt{\pi}}\sin(kx), \quad k > 0, \\ e_0(x) = \frac{1}{\sqrt{2\pi}}, \quad k = 0, \\ e_k(x) = \frac{1}{\sqrt{\pi}}\cos(kx), \quad k < 0\,, \end{cases} \tag{2.1}$$

and for any $f \in L^2(\mathbb{T};\mathbb{R})$, we denote by $f_k := \langle f, e_k\rangle_{L^2(\mathbb{T};\mathbb{R})}$, $k \in \mathbb{Z}$ the k-th Fourier coefficient of f, so that

$$f = \sum_{k\in\mathbb{Z}} f_k e_k,$$

where the equality holds in $L^2(\mathbb{T};\mathbb{R})$.

We denote by A the one-dimensional Laplacian, i.e. the unbounded linear operator $A : \mathcal{D}(A) \subset L^2(\mathbb{T};\mathbb{R}) \to L^2(\mathbb{T};\mathbb{R})$, $A = \partial_{xx}$, which acts on the elements of the basis (2.1) as

$$Ae_k = -k^2 e_k, \quad k \in \mathbb{Z}.$$

With this notation we rewrite the problem (1.4) as

$$\begin{cases} \partial_t u = Au + \partial_x \left[V' u + (F' * u) u \right] + Q^{1/2} \partial_t W, & (0,T) \times \mathbb{T}, \\ u(t,0) = u(t,2\pi), & t \in [0,T], \\ u(0,x) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{2.2}$$

for the unknown $u = u(t) \in L^2(\mathbb{T};\mathbb{R})$ for every $t > 0$ and initial datum $u_0 \in L^2(\mathbb{T};\mathbb{R})$. As a *standing assumption*, throughout the paper $Q : L^2(\mathbb{T};\mathbb{R}) \to L^2(\mathbb{T};\mathbb{R})$ denotes a positive and symmetric operator which we assume to be diagonal w.r.t. the basis $\{e_k\}_{k\in\mathbb{Z}}$; that is,

$$Qe_k = \lambda_k^2 e_k\,, \quad k \in \mathbb{Z}. \tag{2.3}$$

Its eigenvalues $\{\lambda_k\}_{k\in\mathbb{Z}}$ satisfy the following property:

$$\text{there exists a } \delta \in \left(0, \frac{1}{2}\right) \text{ such that } \sum_{k\in\mathbb{Z}} |k|^{4\delta} \lambda_k^2 < +\infty.^{4} \tag{2.4}$$

Moreover, W is an $L^2(\mathbb{T};\mathbb{R})$-valued cylindrical Wiener process defined over a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. That is, W can be represented as

$$W(t,x) = \sum_{k\in\mathbb{Z}} e_k(x) \beta_t^k,$$

where $\{e_k\}_{k\in\mathbb{Z}}$ is the orthonormal basis given in (2.1) and $\{\beta_k\}_{k\in\mathbb{Z}}$ is a family of standard real-valued independent Brownian motions.

We will work with mild solutions of (2.2), so we recall that a continuous $L^2(\mathbb{T};\mathbb{R})$-valued stochastic process $u(t)$, $t \in [0,T]$, is said to be a *mild solution* to (2.2) if the following holds

$$\begin{aligned} u(t) &= e^{tA} u_0 + \int_0^t e^{(t-s)A} \partial_x \left[V' u(s) \right] ds \\ &\quad + \int_0^t e^{(t-s)A} \partial_x \left[(F' * u)(s) u(s) \right] ds + W_A(t), \qquad t \in [0,T],\ \mathbb{P}\text{-a.s.}, \end{aligned} \tag{2.5}$$

where in the above W_A denotes the *stochastic convolution*, namely

$$W_A(t) := \int_0^t e^{(t-s)A} Q^{\frac{1}{2}} \, dW(s), \qquad t \geq 0. \tag{2.6}$$

From (2.4) it follows that the stochastic convolution is differentiable and, for each $t > 0$, $W_A(t)$ belongs to $H^1(\mathbb{T};\mathbb{R})$, see Lemma 5.4 for the statement and Appendix B for the proof of this fact. We will denote by $\{\mathcal{P}_t\}_{\geq 0}$ the semigroup associated with the evolution (2.2), namely

$$(\mathcal{P}_t \psi)(u_0) := \mathbb{E}\left(\psi(u(t;u_0)) \right),\ u_0 \in L^2(\mathbb{T};\mathbb{R}),\ \psi \in \mathcal{B}_b(L^2(\mathbb{T};\mathbb{R});\mathbb{R}), \tag{2.7}$$

where $\mathcal{B}_b(L^2(\mathbb{T};\mathbb{R});\mathbb{R})$ is the class of real-valued bounded Borel measurable functions on $L^2(\mathbb{T};\mathbb{R})$ and $u(t;u_0)$ denotes the mild solution to (2.2) with initial datum u_0. We will use

[4] Let us point out that condition (2.4) implies that Q is trace class. The converse is not true; as a counterexample it suffices to consider the sequence $\lambda_k = \frac{1}{\sqrt{k}\log(k)}$, $k \geq 2$ and $\lambda_k = 0$, $k \leq 1$. An application of the Cauchy's condensation test for infinite series (see e.g. [40, Theorem 3.27]) shows that with this choice of λ_k condition (2.4) doesn't hold for any choice of $\delta \in \left(0, \frac{1}{2}\right)$ while $\sum_{k\geq 2} \lambda_k^2 < +\infty$ does. Therefore, we are imposing a slightly stronger condition on Q than simply being of trace-class.

such a notation every time we want to emphasize the dependence of the solution on the initial datum.

Finally, we denote by $\{e^{tA}\}_{t\geq 0}$ the heat semigroup on $\mathbb{T}$ which acts on $f \in L^2(\mathbb{T};\mathbb{R})$ as follows

$$e^{tA}f = \sum_{k\in\mathbb{Z}} e^{-tk^2} f_k e_k, \quad t \geq 0,\, f \in L^2(\mathbb{T};\mathbb{R}).$$

Equivalently, $e^{tA}f$, $t \geq 0$, can be expressed as the convolution with respect to the space variable on $\mathbb{T}$ between the periodic heat kernel G_t^{per} (defined in Appendix B, equation (B.1)) and f, i.e.

$$e^{tA}f = G_t^{per} * f, \quad t \geq 0. \tag{2.8}$$

2.2. **Statement of Main Results.** As explained in the introduction, in the first part of this paper we study the number of stationary solutions of the PDE (1.2), when V and F are as in (1.5). The main result of this first part is Theorem 2.2 below. In order to state and explain this result, let us start by recalling that stationary solutions of the PDE (1.2) can be characterised as solutions of an appropriate fixed point problem; namely, the following holds.

Lemma 2.1. *Consider the stationary problem associated to the evolution* (1.2)*, i.e.*

$$\sigma\partial_{xx}\rho(x) + \partial_x\left[\left(V'(x) + (F' * \rho)(x)\right)\rho(x)\right] = 0\,, \tag{2.9}$$

with V, F any two functions in $C^\infty(\mathbb{T};\mathbb{R})$. If $\rho \in \mathcal{P}_{ac}(\mathbb{T}) \cap H^1(\mathbb{T};\mathbb{R})$ is a weak solution to (2.9)*, then ρ is smooth, i.e. $\rho \in \mathcal{P}_{ac}(\mathbb{T}) \cap C^\infty(\mathbb{T};\mathbb{R})$ and solves the following fixed point equation*

$$\rho(x) = \frac{1}{Z_\sigma} e^{-\frac{1}{\sigma}\left(V(x) + F*\rho(x)\right)}, \quad x \in \mathbb{T}, \tag{2.10}$$

where Z_σ is the normalization constant so that $\|\rho\|_{L^1(\mathbb{T};\mathbb{R})} = 1$. Conversely, any probability measure whose density satisfies (2.10) *is smooth and it is a solution to* (2.9)*.*

The proof of the above lemma is standard, but we could not find it in the literature for the exact setup we are considering here, so we briefly sketch it in Appendix A.

We are interested in solutions of the problem (2.9) when V and F are as in (1.5), i.e. in solutions of the following problem

$$\partial_{xx}\rho(x) + \partial_x\left[\left(-2\sin(2x) + \int_{\mathbb{T}} \sin(x-y)\rho(y)\,dy\right)\rho(x)\right] = 0, \quad x \in \mathbb{T}, \tag{2.11}$$

so we further specify the fixed point equation (2.10) for such choices of V and F. To this end, by the addition formula for the cosine we have

$$(F * \rho)(x) = -\int_{\mathbb{T}} \cos(x-y)\rho(y)\,dy = -m_1\cos x - m_2\sin x, \tag{2.12}$$

having set

$$m_1 := \int_{\mathbb{T}} \cos y\rho(y)\,dy\,, \quad m_2 := \int_{\mathbb{T}} \sin y\rho(y)\,dy. \tag{2.13}$$

Hence, for our choice of V and F we can rewrite the fixed point problem (2.10) as follows

$$\rho_{m_1,m_2}(x) = \frac{1}{Z_\sigma(m_1,m_2)} e^{-\frac{1}{\sigma}(\cos(2x) - m_1\cos x - m_2\sin x)}, \quad x \in \mathbb{T}, \tag{2.14}$$

where

$$Z_\sigma(m_1,m_2) := \int_{\mathbb{T}} e^{-\frac{1}{\sigma}(\cos(2x)-m_1\cos x - m_2 \sin x)}\,dx$$

is the normalization constant. By multiplying both sides of (2.14) by $\cos x$ ($\sin x$, respectively), we obtain

$$m_1 = \int_{\mathbb{T}} \frac{\cos x}{Z_\sigma(m_1,m_2)} e^{-\frac{1}{\sigma}(\cos(2x)-m_1\cos x - m_2 \sin x)}\,dx, \tag{2.15}$$

$$m_2 = \int_{\mathbb{T}} \frac{\sin x}{Z_\sigma(m_1,m_2)} e^{-\frac{1}{\sigma}(\cos(2x)-m_1\cos x - m_2 \sin x)}\,dx. \tag{2.16}$$

Then, if we define the map

$$\begin{aligned} g_\sigma &: \mathbb{R}^2 \to \mathbb{R}^2 \\ (m_1,m_2) &\to g_\sigma(m_1,m_2) := (g_{1,\sigma}(m_1,m_2), g_{2,\sigma}(m_1,m_2)), \end{aligned} \tag{2.17}$$

where

$$g_{1,\sigma}(m_1,m_2) := \int_{\mathbb{T}} \frac{\cos x}{Z_\sigma(m_1,m_2)} e^{-\frac{1}{\sigma}(\cos(2x)-m_1\cos x - m_2 \sin x)}\,dx, \tag{2.18}$$

$$g_{2,\sigma}(m_1,m_2) := \int_{\mathbb{T}} \frac{\sin x}{Z_\sigma(m_1,m_2)} e^{-\frac{1}{\sigma}(\cos(2x)-m_1\cos x - m_2 \sin x)}\,dx, \tag{2.19}$$

it follows that the density ρ_{m_1,m_2} is stationary solution of (2.11) if and only if (m_1,m_2) is a fixed point of the map g_σ. We have therefore reformulated the problem of finding solutions of equation (2.11) as a two parameter problem of finding fixed points of the map g_σ. This is the basic approach that allows one to prove the following result.

Theorem 2.2. *There exists $\sigma_c > 0$ such that if σ is either sufficiently small (i.e. $\sigma \ll 1$) or $\frac{1}{2} \le \sigma < \sigma_c$ the stationary problem* (2.11) *has exactly three solutions, while for $\sigma > \sigma_c$ it has exactly one solution. Furthermore, the density*

$$\rho_{0,0}(x) = \frac{1}{Z_\sigma(0,0)} e^{-\frac{\cos(2x)}{\sigma}}, \quad x \in \mathbb{T}, \tag{2.20}$$

is always a solution of (2.11)*, irrespective of the value of σ. The two additional stationary solutions for $\sigma \ll 1$ and $\frac{1}{2} \le \sigma < \sigma_c$ are centered around the minima of the double well potential V, i.e. around $x = \pm\frac{\pi}{2}$. Finally, the critical value σ_c can be explicitly characterized as the (unique) zero of the function $f_c : (0,+\infty) \to \mathbb{R}$ defined as*

$$f_c(\sigma) = \frac{1}{\sigma} - 2 + \frac{1}{\sigma} r_0\left(\frac{1}{\sigma}\right), \quad \sigma > 0, \tag{2.21}$$

where r_0 is the modified Bessel function, see (3.29) *for a definition. Analytical computations show that $\sigma_c \simeq 0.7709$.*

The proof of Theorem 2.2 can be found in Section 3.

Note 2.3. Results in the spirit of the above theorem have been known for a long time, see e.g. [5, 9, 16, 16, 19, 22, 25, 42] and references therein and indeed parts of the (long) proof of the above theorem is inspired by [5, 42]. More precisely,

- As we mentioned in the introduction, the setup of Theorem 2.2 can be seen as being a non-rotationally invariant modification of the Kuramoto model considered in [4, 5, 22] (or of the Smoluchowski equation of [9, 10]). Because of this break of rotational invariance, most of the arguments used in [4, 5, 9, 10, 22] cannot be adapted in the current setup.
- Other results similar to Theorem 2.2 are those in [16, 25]: besides the fact that the state space considered in [16, 25] is $\mathbb{R}$ as opposed to the torus, also in [16, 25] the authors consider a double -well potential and an attractive inter-particle force and they reduce the problem of finding stationary solutions of the PDE they consider to a fixed point problem; however, because of the exact analytic form of the attractive force they consider, they can reduce the fixed point problem (2.10) to a one-parameter fixed point problem. In our case, because of our choice of F (1.5), we end up with a two-parameter fixed point problem (see (2.12)), which cannot be a priori further reduced to a one parameter problem, again because of lack of rotational symmetry.
- We further conjecture, based on numerical evidence, that (1.2) has exactly three stationary solutions for any $0 < \sigma < \sigma_c$, but we have been able to prove it only for $\sigma \ll 1$ and $\frac{1}{2} \leq \sigma < \sigma_c$.
- Similar results could be obtained for multi-well potentials (i.e. by considering $V(x) = \cos(mx)$). This is done (for $m = 4$) in [30].

Let us now move on to the second part of the paper, where we study the SPDE (2.2). We clarify that, when studying (2.2), we always consider V, F to be arbitrary coefficients in $C^\infty(\mathbb{T};\mathbb{R})$, i.e. we no longer restrict to the choice (1.5). We first state the main results of part two, Theorem 2.4 and Theorem 2.6 below, and then comment on them.

Theorem 2.4. *Let Q satisfy the standing assumption* (2.3)-(2.4) *and let $V, F \in C^\infty(\mathbb{T};\mathbb{R})$. Then, for any $T > 0$ (independent of $\omega \in \Omega$) and initial datum $u_0 \in L^2(\mathbb{T};\mathbb{R})$ there exists a unique mild solution (in the sense clarified in Subsection 2.1) u to equation* (2.2)*; such a solution is $\mathbb{P}$-a.s. in $C\left([0,T]; L^2(\mathbb{T};\mathbb{R})\right)$.*

The proof of Theorem 2.4 is in Section 4.

Assumption 2.5. *The covariance operator Q satisfies the standing assumption* (2.3)-(2.4) *and, moreover, the eigenvalues $\{\lambda_k^2\}_{k\in\mathbb{Z}}$ of Q satisfy the following growth condition: there exist $c > 0$ and $\gamma < 1$ such that*

$$\lambda_k^2 \geq c(1+k^2)^{-\gamma}, \quad k \in \mathbb{Z}.^5 \tag{2.22}$$

Theorem 2.6. *Let V and F be as in the statement of Theorem 2.4 and let $\mathcal{P}_t$ be the semigroup associated with the SMKV equation* (2.2)*, see* (2.7)*.*

i): *If Q satisfies the standing assumption* (2.3)-(2.4) *then the semigroup is irreducible.*

ii): *If, furthermore, Q satisfies Assumption 2.5, then $\mathcal{P}_t$ is Strong Feller as well, hence the dynamics* (2.2) *admits at most one invariant measure.*

[5]Let us point out that (2.22) combined with (2.4) implies that $\gamma > \frac{1}{2}$ so that when both (2.4) and (2.22) are required we will be working in the $\gamma \in \left(\frac{1}{2}, 1\right)$ regime.

The proof of part i) can be found in Section 5, the proof of ii) in Section 6.

Note 2.7. Some comments on Theorem 2.4 and Theorem 2.6.

- The scheme of proof of Theorem 2.4 uses the trick of reducing the SPDE at hand to a PDE with random coefficients by "subtracting the noise". In particular, we mostly use a combination of the arguments of e.g. [15] developed for the stochastic Burgers equation and of those developed in the PDE literature for the evolution (1.2), with particular reference to [8]. Let us note that, while the PDE (1.2) (which is in gradient form) preserves total mass and positivity, the SPDE (2.2) does not enjoy any of these properties; this is one of the main reasons why in this paper the PDE arguments typically used for (1.2) (which heavily rely on such properties) could be used very sparingly and we rely instead on the methods for Stochastic Burgers'.

  The similarity between the nonlinearity we consider and the Burgers' nonlinearity may become apparent once we observe that, taking the convolution with F' (which is a smooth function on the torus, hence bounded), the nonlinear term $\partial_x((F' * u(x))u(x))$ is, substantially, the derivative of an expression which is quadratic in the unknown u.
- As mentioned in the introduction, the works [27, 28] deal with strong well-posedness of infinite-dimensional McKean-Vlasov SDEs (via Galerkin approximation). Such equations are different from (2.2), as in [27, 28] the solution of the process depends on the law of the process itself, whereas this is not the case for (2.2). In particular the evolution (2.2) enjoys both the Markov Property and the Markov family property (see [35]), while the evolutions in [27, 28] satisfy the Markov property only. Moreover [28] contains some very nice results on averaging for McKean-Vlasov type SPDEs; producing such a result requires studying the ergodic properties of a linearization of (2.2) (namely, in our context, the equation obtained from (2.2) by replacing $F' * u$ with $F' * \nu$, for a given ν), but not of (2.2) itself. So Theorem 2.6 above constitutes, to the best of our knowledge, a first attempt at a partial description of the ergodic behaviour of (2.2).
- The 'basic' method to prove the Strong-Feller property requires the covariance operator Q to have bounded inverse, see [36, 39] and references therein. Our proof of the Strong-Feller property relies on the use of Bismut-Elworthy-Li type of formulas (see Section 6, formula (6.12) and Note 6.3 in particular), where the inverse of the operator Q appears. While we do not require Q to have bounded inverse, we still need to control the growth of Q^{-1}, hence Assumption 2.5. In other words, we need to require that the eigenvalues of Q do not decay too fast, i.e. that the noise is 'strong enough'. This is not unexpected, see e.g. [7] (we will make more comparisons with [7] in Section 6). It is worth mentioning that by using the same technique of Section 6 a strong Feller result can be obtained in the weaker noise setting as well, at the cost of changing space. Namely, it can be shown that when in Assumption 2.5 we take $\gamma \geq 1$ then the semigroup generated by (2.2) is strong Feller in the fractional Sobolev space $H^\gamma(\mathbb{T};\mathbb{R})$(see [2] for its definition), but we don't do this here for brevity.

- The method of proof we use to show the Strong-Feller property relies on proving first that such a property holds for a class of equations with Lipshitz non-linearity, where the nonlinearity depends on the derivative of the solution (and for this reason it is different from the nonlinearities considered in [7, 36]); this general result is contained in Subsection 6.1.
- In this paper we do not cover the purely cylindrical noise case, i.e. $Q = I$, I being the identity operator. While our proofs of irreducibility and Strong Feller property would still hold if $Q = I$, our well-posedness proof requires some smoothness. We leave this further extension for future work.
- Finally and most importantly, in this paper we do not study the *existence* of the invariant measure for (2.2). It turns out that this is not a straightforward task so, to contain the length of this paper, we will do so in separate forthcoming work.

3. Proof of Theorem 2.2

In this section we prove Theorem 2.2. The main argument of the proof is described below and it is divided into four steps. The first three steps are proved in Subsection 3.1 to Subsection 3.3, respectively, step four is proved in Appendix A (as the ideas are completely analogous to those in Subsection 3.2). The proof that the critical value σ_c is a zero of the function f_c (2.21) and the consequent approximation for the numerical value of σ_c can be found in the proof of Theorem 3.3.

Proof of Theorem 2.2. From (2.17), (2.18) and (2.19) it follows that $g_\sigma(0,0) = (0,0)$. Hence, $(m_1, m_2) = (0,0)$ is a fixed point of g_σ for any σ, so that (2.20) is a stationary solution to (1.2) for every $\sigma > 0$ (note that $Z_\sigma(0,0)$ coincides exactly with the normalization constant Z_σ appearing in (2.10)).
Second, exploiting the symmetries of sin and cos functions, it is also easy to see that the axes $\mathbb{M}_1 := \{(m_1, 0)\}_{m_1 \in \mathbb{R}}$ and $\mathbb{M}_2 := \{(0, m_2)\}_{m_2 \in \mathbb{R}}$ are invariant for g_σ i.e. $g_\sigma(m_1, 0) \in \mathbb{M}_1$ and $g_\sigma(0, m_2) \in \mathbb{M}_2$ for all $m_1, m_2 \in \mathbb{R}$.

We now divide the proof into four steps. In Step 1) we show that all fixed points of g_σ lie on either $\mathbb{M}_1$ or $\mathbb{M}_2$. This allows us to reduce the search of fixed points to two one-parameter fixed point problems, one on $\mathbb{M}_1$ and the other on $\mathbb{M}_2$. In Step 2) we consider the fixed point problem on $\mathbb{M}_2$ and show that there exists $\sigma_c \in (0,1)$ such that for $0 < \sigma < \sigma_c$ there are exactly two additional stationary states other than (2.20), while, for $\sigma > \sigma_c$ those two additional solutions collapse into (2.20) which lies on $\mathbb{M}_2$, so for $\sigma > \sigma_c$ there is exactly one steady state of (2.11) which lies on $\mathbb{M}_2$. At this point what one would want to show is that the only fixed point on $\mathbb{M}_1$ is (0,0), irrespective of the value of σ. However, we are able to prove this fact only for $0 < \sigma \ll 1$ (Step 3) and for $\sigma \geq \frac{1}{2}$ (Step 4). Hence, from steps 1 to 4, we deduce the uniqueness for $\sigma > \sigma_c$ and the existence of exactly three stationary states for $\frac{1}{2} \leq \sigma < \sigma_c$ and $0 < \sigma \ll 1$.

Steps 1, 2 and 3 are proven in Subsections 3.1, 3.2, 3.3, respectively, and the details for Step 4 are referred to Subsection A.2 of Appendix A. □

3.1. **Proof of Step 1.** In this section we prove that the fixed points of g_σ belong to either the axis $\mathbb{M}_1$ or the axis $\mathbb{M}_2$. More precisely, the following statement holds.

Proposition 3.1. *Let $(m_1, m_2) \in \mathbb{R}^2$ be a fixed point of the map g_σ different from (0,0); then either $m_1 = 0$ or $m_2 = 0$.*

Proof. If $(m_1, m_2) \in \mathbb{R}^2$ is a fixed point of g_σ then we know (m_1, m_2) satisfies the fixed point equations (2.15) and (2.16). Next, we write $(m_1, m_2) \in \mathbb{R}^2$ in polar coordinates, namely we let $(M, \varphi) \in \mathbb{R}_+ \times [0, 2\pi)$; that is,

$$M = (m_1^2 + m_2^2)^{1/2},\ m_1 = M\cos\varphi,\ m_2 = M\sin\varphi, \qquad \varphi \in [0, 2\pi), M \in \mathbb{R}_+.$$

From the above and from the addition formula for the cosine we can rewrite the fixed point equations (2.15) and (2.16) as

$$m_1 = \frac{1}{Z_\sigma(m_1, m_2)} \int_{\mathbb{T}} \cos x\, e^{-\frac{1}{\sigma}(\cos(2x) - M\cos(x-\varphi))}\, dx, \tag{3.1}$$

$$m_2 = \frac{1}{Z_\sigma(m_1, m_2)} \int_{\mathbb{T}} \sin x\, e^{-\frac{1}{\sigma}(\cos(2x) - M\cos(x-\varphi))}\, dx, \tag{3.2}$$

respectively. Define

$$I(M, \varphi) := \int_{\mathbb{T}} \sin(x - \varphi) e^{-\frac{1}{\sigma}(\cos(2x) - M\cos(x-\varphi))}\, dx. \tag{3.3}$$

If (m_1, m_2) is a fixed point of g_σ then it follows that $I(M, \varphi) = 0$. Indeed, from the addition formula for the sine and from (3.1) and (3.2) we have

$$\begin{aligned} I(M, \varphi) &= \cos\varphi \int_0^{2\pi} \sin x\, e^{-\frac{1}{\sigma}(\cos(2x) - M\cos(x-\varphi)}\, dx \\ &\quad - \sin\varphi \int_0^{2\pi} \cos x\, e^{-\frac{1}{\sigma}(\cos(2x) - M\cos(x-\varphi)}\, dx \\ &= Z_\sigma(m_1, m_2)\,(m_2\cos\varphi - m_1\sin\varphi) \\ &= Z_\sigma(m_1, m_2) M\,(\sin\varphi\cos\varphi - \cos\varphi\sin\varphi) = 0. \end{aligned}$$

Our goal is to show that the equation $I(M, \varphi) = 0$ implies either $M = 0$ or $\varphi \in \{0, \pi/2, \pi, 3\pi/2\}$. By applying the change of variable to $x \to x - \varphi$, $x \in \mathbb{T}$ and using periodicity we obtain

$$I(M, \varphi) = \int_{-\pi}^{\pi} \sin x\, e^{-\frac{1}{\sigma}\cos(2x + 2\varphi)} e^{\frac{1}{\sigma}M\cos x}\, dx.$$

We split the above integral into four parts: $I_{-1}(M, \varphi)$, $I_0(M, \varphi)$, $I_1(M, \varphi)$ and $I_2(M, \varphi)$, namely

$$I(M, \varphi) = \sum_{i=-1}^{2} I_i(M, \varphi) := \sum_{i=-1}^{2} \int_{(i-1)\frac{\pi}{2}}^{i\frac{\pi}{2}} \sin x\, e^{-\frac{1}{\sigma}\cos(2x + 2\varphi)}\, e^{\frac{1}{\sigma}M\cos x}\, dx.$$

We now apply respectively the changes of variable $x \to x + \pi$, $x \to -x$ and $x \to \pi - x$ to $I_{-1}(M, \varphi), I_0(M, \varphi)$ and $I_2(M, \varphi)$, respectively, and then add together the resulting four expressions we obtain the following expression arriving at

$$\begin{aligned}
I(M,\varphi) = \int_0^{\frac{\pi}{2}} \sin x \Big[& - e^{-\frac{1}{\sigma}\cos(2x+2\varphi)} e^{-\frac{1}{\sigma}M\cos x} - e^{-\frac{1}{\sigma}\cos(2x-2\varphi)} e^{\frac{1}{\sigma}M\cos x} \\
& e^{-\frac{1}{\sigma}\cos(2x+2\varphi)} e^{\frac{1}{\sigma}M\cos x} + e^{-\frac{1}{\sigma}\cos(2x-2\varphi)} e^{-\frac{1}{\sigma}M\cos x} \Big] \, dx \\
= \int_0^{\frac{\pi}{2}} \sin x & \left[e^{-\frac{1}{\sigma}\cos(2x+2\varphi)} - e^{-\frac{1}{\sigma}\cos(2x-2\varphi)} \right] \left[e^{\frac{1}{\sigma}M\cos x} - e^{-\frac{1}{\sigma}M\cos x} \right] dx. \qquad (3.4)
\end{aligned}$$

Our first aim is to prove that $I(M,\varphi) = 0$ cannot hold when $M > 0$ and $\varphi \in \left(0, \frac{\pi}{2}\right)$. As a consequence we deduce that there are no fixed points of the map g_σ in the first quadrant of $\mathbb{R}^2$ (i.e. when $m_1 > 0$ and $m_2 > 0$). We will then repeat the procedure on the other quadrants, in turn. In the first place, we note

$$\cos(2x+2\varphi) < \cos(2x-2\varphi), \quad \text{for all } x, \varphi \in \left(0, \frac{\pi}{2}\right). \qquad (3.5)$$

Indeed, from the addition formula for the cosine, equation (3.5) is equivalent to

$$\cos(2x)\cos(2\varphi) - \sin(2x)\sin(2\varphi) < \cos(2x)\cos(2\varphi) + \sin(2x)\sin(2\varphi),$$

which is in turn equivalent to

$$\sin(2x)\sin(2\varphi) > 0. \qquad (3.6)$$

Clearly, (3.6) holds for all $x, \varphi \in \left(0, \frac{\pi}{2}\right)$. Hence,

$$e^{-\frac{1}{\sigma}\cos(2x+2\varphi)} > e^{-\frac{1}{\sigma}\cos(2x-2\varphi)}, \quad \text{for all } x, \varphi \in \left(0, \frac{\pi}{2}\right). \qquad (3.7)$$

Furthermore, since

$$\cos x > 0 > -\cos x, \quad x \in \left(0, \frac{\pi}{2}\right),$$

we deduce

$$e^{\frac{1}{\sigma}M\cos x} > 1 > e^{-\frac{1}{\sigma}M\cos x}, \quad \text{for all } x \in \left(0, \frac{\pi}{2}\right). \qquad (3.8)$$

Using (3.4) we then conclude that $I(M,\varphi) > 0$ for all $\varphi \in \left(0, \frac{\pi}{2}\right)$ and $M > 0$.
Similarly, if we let $M > 0$, $\varphi \in \left(\frac{\pi}{2}, \pi\right)$ then

$$\cos(2x+2\varphi) > \cos(2x-2\varphi), \quad \text{for all } x, \varphi \in \left(\frac{\pi}{2}, \pi\right), \qquad (3.9)$$

which is equivalent to

$$\sin(2x)\sin(2\varphi) < 0. \qquad (3.10)$$

The above holds for all $x \in (0, \frac{\pi}{2})$ and for all $\varphi \in (\frac{\pi}{2}, \pi)$. Hence,

$$e^{-\frac{1}{\sigma}\cos(2x+2\varphi)} < e^{-\frac{1}{\sigma}\cos(2x-2\varphi)}, \quad \text{for all } x, \varphi \in \left(\frac{\pi}{2}, \pi\right). \qquad (3.11)$$

$$e^{\frac{1}{\sigma}M\cos x} > 1 > e^{-\frac{1}{\sigma}M\cos x}, \quad \text{for all } x \in \left(0, \frac{\pi}{2}\right), \qquad (3.12)$$

we have that $I(M,\varphi) < 0$ for all $\varphi \in \left(\frac{\pi}{2}, \pi\right)$ and $M > 0$. The remaining cases i.e. $M > 0$, $\varphi \in \left(\pi, \frac{3\pi}{2}\right)$ and $M > 0$, $\varphi \in \left(\frac{3\pi}{2}, 2\pi\right)$ can be dealt with analogously. □

3.2. **Proof of Step 2.** We recall that the axis $\mathbb{M}_2$ is invariant for g_σ, hence we can define the map $\bar{g}_\sigma : \mathbb{R} \to \mathbb{R}$ to be the restriction of $g_\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ to $\mathbb{M}_2$; namely

$$\bar{g}_\sigma(m) := \frac{1}{Z_\sigma(m)} \int_{\mathbb{T}} \sin x \, e^{-\frac{1}{\sigma}(\cos(2x) - m\sin x)}\, dx, \quad Z_\sigma(m) := \int_{\mathbb{T}} e^{-\frac{1}{\sigma}(\cos(2x)-m\sin x)}\, dx. \tag{3.13}$$

Then m is a fixed point of $\bar{g}_\sigma$ if and only if $(0, m)$ is a fixed point of g_σ. In what follows we prove that the map (3.13) has the following property: there exists $\sigma_c > 0$ such that, when $\sigma > \sigma_c$, $\bar{g}_\sigma$ admits a unique fixed point, while, when $\sigma < \sigma_c$, $\bar{g}_\sigma$ admits exactly three fixed points. The idea of the proof is inspired by [42, Theorem 2.1]. Thus, let us introduce the map $\zeta_\sigma : \mathbb{R} \to \mathbb{R}$ defined as

$$\zeta_\sigma(m) := \int_{\mathbb{T}} (\sin x - m) e^{-\frac{1}{\sigma}\cos(2x) + \frac{m}{\sigma}\sin x}\, dx, \quad m \in \mathbb{R}, \tag{3.14}$$

and note that $m \in \mathbb{R}$ is a fixed point of $\bar{g}_\sigma$ if and only if m is a zero of ζ_σ i.e. $\zeta_\sigma(m) = 0$. Furthermore, $\zeta_\sigma(1) < 0$. The statement is then a consequence of Proposition 3.2 and Theorem 3.3 below, which we first state and then prove in turn.

Proposition 3.2. *The map $\zeta_\sigma : \mathbb{R} \to \mathbb{R}$ defined in* (3.14) *is odd and it admits either exactly one zero or exactly three zeroes, depending on the value of σ. Furthermore, the following holds:*

- *If $\zeta_\sigma'(0) < 0$ then ζ_σ is strictly decreasing on $[0, +\infty)$. Since $\zeta_\sigma(0) = 0$ then we deduce that ζ_σ does not vanish on $(0, +\infty)$. The function ζ_σ is odd so that we can easily deduce that it admits a unique zero on $\mathbb{R}$.*
- *If $\zeta_\sigma'(0) > 0$ then there exists $\bar{m} > 0$ such that ζ_σ is strictly increasing in $[0, \bar{m})$ and then strictly decreasing for $m \geq \bar{m}$. Since $\zeta_\sigma(0) = 0$ and $\zeta_\sigma(1) < 0$, we deduce that ζ_σ has a unique zero on $(0, 1)$. The function ζ_σ is odd, hence it admits exactly three zeroes on $\mathbb{R}$.*

Theorem 3.3. *There exists $\sigma_c > 0$ such that the following holds*

$$\begin{cases} \zeta_\sigma'(0) < 0, & \sigma > \sigma_c, \\ \zeta_\sigma'(0) > 0, & \sigma < \sigma_c. \end{cases} \tag{3.15}$$

In particular, due to Proposition 3.2, we have the following:

- *If $\sigma > \sigma_c$ then ζ_σ admits a unique zero ($m = 0$).*
- *If $\sigma < \sigma_c$ then ζ_σ admits exactly three zeroes (one of which is $m = 0$).*

Furthermore, an analytical approximation of σ_c is given by $\sigma_c \simeq 0.7709$.

Before proving the above statements we state and prove some technical lemmata, which will be needed in the proofs of the above main results. More precisely, we first prove Lemma 3.4 and Lemma 3.5 after that we move on to proving Proposition 3.2 and Theorem 3.3. To this end, let us introduce the sequence $\{s_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ defined as

$$s_k := \int_{\mathbb{T}} (\sin x)^k e^{-\frac{1}{\sigma}\cos(2x)}\, dx, \quad k \in \mathbb{N}. \tag{3.16}$$

Since $\sin x$ is an odd function, $\cos(2x)$ is an even function and $(\sin x)^2 < 1$ for all $x \in \mathbb{T} \setminus \{-\frac{\pi}{2}, \frac{\pi}{2}\}$, the following properties hold for $\{s_k\}_{k\in\mathbb{N}}$:

$$s_{2k+1} = 0, \quad \text{for all } k \in \mathbb{N}, \tag{3.17}$$

$$s_{2k+2} < s_{2k}, \quad \text{for all } k \in \mathbb{N}. \tag{3.18}$$

Lemma 3.4. *For every $\sigma > 0$, the function $\zeta_\sigma : \mathbb{R} \to \mathbb{R}$ admits the following series expansion*

$$\zeta_\sigma(m) = \sum_{k\in\mathbb{N}} \frac{1}{(2k)!} \left(\frac{m}{\sigma}\right)^{2k+1} s_{2k} \Upsilon_k(\sigma), \quad m \in \mathbb{R}, \tag{3.19}$$

where $\{\Upsilon_k(\sigma)\}_{k\in\mathbb{N}}$ is the sequence defined as

$$\Upsilon_k(\sigma) := \frac{s_{2k+2}}{(2k+1)s_{2k}} - \sigma, \quad k \in \mathbb{N},\, \sigma > 0. \tag{3.20}$$

Proof. From the power series expansion of e^x, $e^x = \sum_{k\in\mathbb{N}} \frac{x^k}{k!}$, $\quad x \in \mathbb{R}$, we obtain

$$\begin{aligned}
\zeta_\sigma(m) &= \int_{\mathbb{T}} (\sin x - m) e^{-\frac{1}{\sigma}\cos(2x) + \frac{m}{\sigma}\sin x}\, dx \\
&= \int_{\mathbb{T}} (\sin x - m) e^{-\frac{1}{\sigma}\cos(2x)} \sum_{k\in\mathbb{N}} \frac{1}{k!} \left(\frac{m}{\sigma}\right)^k (\sin x)^k\, dx \\
&= \sum_{k\in\mathbb{N}} \frac{1}{k!} \left(\frac{m}{\sigma}\right)^k (s_{k+1} - m s_k) = \sum_{k\in\mathbb{N}} \frac{1}{k!} \left(\frac{m}{\sigma}\right)^k s_{k+1} - \sum_{k\in\mathbb{N}} \frac{\sigma}{k!} \left(\frac{m}{\sigma}\right)^{k+1} s_k.
\end{aligned}$$

From (3.17) we then deduce

$$\begin{aligned}
\zeta_\sigma(m) &= \sum_{k\in\mathbb{N}} \frac{1}{(2k+1)!} \left(\frac{m}{\sigma}\right)^{2k+1} s_{2k+2} - \sum_{k\in\mathbb{N}} \frac{\sigma}{(2k)!} \left(\frac{m}{\sigma}\right)^{2k+1} s_{2k} \\
&= \sum_{k\in\mathbb{N}} \frac{1}{(2k)!} \left(\frac{m}{\sigma}\right)^{2k+1} s_{2k} \left(\frac{s_{2k+2}}{(2k+1)s_{2k}} - \sigma\right).
\end{aligned}$$

By recalling the definition (3.20) we deduce the assertion. □

Moreover, the following property of the sequence $\{\Upsilon_k(\sigma)\}_{k\in\mathbb{N}}$ holds.

Lemma 3.5. *For any $\sigma > 0$ the sequence $\{\Upsilon_k(\sigma)\}_{k\in\mathbb{N}}$ is strictly decreasing (in k).*

Proof. From the definition of $\{\Upsilon_k(\sigma)\}_{k\in\mathbb{N}}$ (see (3.20)) it suffices to prove that the sequence $\left\{\frac{s_{2k+2}}{(2k+1)s_{2k}}\right\}_{k\in\mathbb{N}}$ is decreasing. Thus, from the definition of s_{2k} and by integrating by parts we obtain

$$\begin{aligned}
s_{2k} &= \int_{\mathbb{T}} \cos x \frac{d}{dx} (\sin x)^{2k-1} e^{-\frac{1}{\sigma}\cos(2x)}\, dx \\
&= (2k-1) \int_{\mathbb{T}} \cos^2 x\, (\sin x)^{2k-2} e^{-\frac{1}{\sigma}\cos(2x)}\, dx \\
&+ \frac{4}{\sigma} \int_{\mathbb{T}} \cos^2 x\, (\sin x)^{2k} e^{-\frac{1}{\sigma}\cos(2x)}\, dx \\
&= (2k-1)s_{2k-2} - (2k-1)s_{2k} + \frac{4}{\sigma} s_{2k} - \frac{4}{\sigma} s_{2k+2}.
\end{aligned}$$

Rearranging, we obtain

$$\frac{(2k+1)s_{2k}}{s_{2k+2}} = \frac{(2k-1)s_{2k-2}}{s_{2k+2}} + \frac{s_{2k}}{s_{2k+2}} + \frac{4}{\sigma}\frac{s_{2k}}{s_{2k+2}} - \frac{4}{\sigma}.$$

From (3.18) we deduce $\frac{s_{2k+2}}{s_{2k}} < 1$ for all $k \in \mathbb{N}$; using this fact in the above expression gives

$$\frac{(2k+1)s_{2k}}{s_{2k+2}} \geq \frac{(2k-1)s_{2k-2}}{s_{2k+2}} = \frac{(2k-1)s_{2k-2}}{s_{2k}}\frac{s_{2k}}{s_{2k+2}} > \frac{(2k-1)s_{2k-2}}{s_{2k}},$$

and this concludes the proof. □

We can now move on to proving Proposition 3.2.

Proof of Proposition 3.2. From (3.19), ζ_σ is an odd function. Moreover, since $0 \leq \frac{s_{2k+2}}{(2k+1)s_{2k}} \leq \frac{1}{2k+1}$ for all $k \in \mathbb{N}$, we deduce from (3.20) that $\Upsilon_k(\sigma) \downarrow -\sigma$ as $k \uparrow +\infty$ for any fixed $\sigma > 0$. Hence, if we set $k_\sigma := \min\{k \in \mathbb{N} : \Upsilon_k(\sigma) \leq 0\}$, since $\{\Upsilon_k(\sigma)\}_{k\in\mathbb{N}}$ is decreasing for any given $\sigma > 0$ we deduce that $\Upsilon_k(\sigma) > 0$ for $k \leq k_\sigma - 1$ and $\Upsilon_k(\sigma) \leq 0$ for $k \geq k_\sigma$. From this and the fact that $\zeta_\sigma^{2k+1}(0) = \frac{(2k+1)}{\sigma^{2k+1}} s_{2k}\Upsilon_k(\sigma)$, $k \in \mathbb{N}$, we obtain that the following power series representation of ζ_σ holds

$$\zeta_\sigma(m) = \sum_{0\leq k\leq k_\sigma-1} \frac{|\zeta_\sigma^{(2k+1)}(0)|}{(2k+1)!} m^{2k+1} - \sum_{k\geq k_\sigma} \frac{|\zeta_\sigma^{(2k+1)}(0)|}{(2k+1)!} m^{2k+1}.$$

Taking out $m^{2k_\sigma+1}$ we have

$$\zeta_\sigma(m) = m^{2k_\sigma+1}\left\{\sum_{k\leq k_\sigma-1} \frac{|\zeta_\sigma^{(2k+1)}(0)|}{(2k+1)!} m^{2k-2k_\sigma} - \sum_{k\geq k_\sigma} \frac{|\zeta_\sigma^{(2k+1)}(0)|}{(2k+1)!} m^{2k-2k_\sigma}\right\}. \tag{3.21}$$

We can now conclude as in [42, cfr. Step 4, Theorem 2.1]. Indeed, since the function $m \to m^{2k-2k_\sigma}$ (resp. $m \to -m^{2k-2k_\sigma}$) is strictly decreasing for all $k \leq k_\sigma - 1$ (resp. for all $k \geq k_\sigma$) then we deduce that the factor between brackets in (3.21) is strictly decreasing for $m > 0$. Hence, again from (3.21) we deduce that if $m > 0$ is a root of ζ_σ then such an m must be a root of the factor between brackets in (3.21)(which we know it admits at most one root because it is strictly decreasing for $m > 0$). Hence, ζ_σ admits at most one zero on $(0, +\infty)$. Furthermore, ζ_σ is odd so that it admits exactly either one or three zeroes on $\mathbb{R}$. Once this is in place, we have to determine which situation occurs. From (3.21) we have

$$\zeta_\sigma'(m) = m^{2k_\sigma}\left\{\sum_{0\leq k\leq k_\sigma-1} \frac{|\zeta_\sigma^{(2k+1)}(0)|}{(2k)!} m^{2k-2k_\sigma} - \sum_{k\geq k_\sigma} \frac{|\zeta_\sigma^{(2k+1)}(0)|}{(2k)!} m^{2k-2k_\sigma}\right\}. \tag{3.22}$$

By the same argument applied to (3.21) for ζ_σ we deduce that the factor between brackets in (3.22) is strictly decreasing. If $\zeta_\sigma'(0) < 0$, then $k_\sigma = 0$ and we conclude that $\zeta_\sigma'(m)$ is strictly decreasing; hence $\zeta_\sigma'(m) < 0$ for all $m \geq 0$ so that ζ_σ admits a unique zero on $\mathbb{R}$. If on the other hand $\zeta_\sigma'(0) > 0$, then, using $\zeta_\sigma(1) < 0$, we know that ζ_σ admits at least one root on $(0, 1)$. Then it admits exactly one root on $(0, +\infty)$ and the proof is thus concluded. □

In order to prove Theorem 3.3 we need to state the following asymptotic expansion results.

Lemma 3.6. *Let U and G be two $C^\infty(\mathbb{T};\mathbb{R})$-continuous functions. Let us define $U_m = U+m\cdot G$ where m is a parameter belonging to some compact interval I of $\mathbb{R}$. Moreover, assume that U_m admits a unique global minimum at x_m, such that $U_m''(x_m) > 0$ and $0 < x_m < 2\pi$. Then, for any $f \in C^3(\mathbb{T};\mathbb{R})$, the following asymptotic result holds (as $\sigma > 0$ tends to 0):*

$$\int_{\mathbb{T}} f(x)e^{-\frac{1}{\sigma}U_m(x)}\,dx = \sqrt{\frac{2\pi\sigma}{\mathcal{U}_2}}e^{-\frac{1}{\sigma}U_m(x_m)}\left(f(x_m) + \gamma_f\sigma + o_m(\sigma)\right), \tag{3.23}$$

with

$$\gamma_f := f(x_m)\left(\frac{5\mathcal{U}_3^2}{24\mathcal{U}_2^3} - \frac{\mathcal{U}_4}{8\mathcal{U}_2^2}\right) - f'(x_m)\frac{\mathcal{U}_3}{2\mathcal{U}_2^2} + \frac{f''(x_m)}{2\mathcal{U}_2}, \tag{3.24}$$

where $\mathcal{U}_k := U_m^{(k)}(x_m)$ for $k \in \mathbb{N}$, the notation $o_m(\sigma)$ is intended to mean that $\frac{o_m(\sigma)}{\sigma} \to 0$ as $\sigma \to 0$ and the convergence holds uniformly in $m \in I$.

Proof of Prop. 3.6. This is obtained by Laplace method and we refer the reader to [25, Lemma A.3, Step 1-Step 2.2.] for further details. □

From Lemma 3.6 we obtain the following lemma, the proof of which is postponed to Appendix A.

Lemma 3.7. *The following two asymptotic expansions hold*

$$s_0 = \sqrt{\frac{\pi\sigma}{2}}e^{\frac{1}{\sigma}}\left(2 + o(1)\right), \tag{3.25}$$

$$s_2 = \sqrt{\frac{\pi\sigma}{2}}e^{\frac{1}{\sigma}}\left(2 + o(1)\right), \tag{3.26}$$

where the notation $o(1)$ means that $o(1) \to 0$ as $\sigma \downarrow 0$ and we recall that the coefficients s_k have been defined in (3.16).

Proof of Theorem 3.3. To establish which case holds, i.e. whether $\zeta_\sigma'(0) > 0$ or $\zeta_\sigma'(0) < 0$, we study the first derivative of ζ_σ at $m = 0$. A straightforward calculation shows that

$$\zeta_\sigma'(0) = \frac{s_0}{\sigma}\left(\frac{s_2}{s_0} - \sigma\right) = \frac{1}{\sigma}\left(s_2 - \sigma s_0\right). \tag{3.27}$$

We note that since $s_2 \leq s_0$, it is clear from (3.27) that for $\sigma > 1$, $\zeta_\sigma'(0) < 0$. While, from Lemma 3.7 and (3.27), it is also clear that for $\sigma \ll 1$, $\zeta_\sigma'(0) > 0$. Since the map $\sigma \to \zeta_\sigma'(0)$ is continuous we deduce that there exists a root $\sigma_c > 0$ of the map $\sigma \to \zeta_\sigma'(0)$. The uniqueness of such a σ_c remains to be proven. To this end, we first note that by using the identity $\cos(2x) = 1 - 2\sin^2 x$, the factor $s_2 - \sigma s_0$ can be rearranged into the following form:

$$\begin{aligned} s_2 - \sigma s_0 &= \int_{\mathbb{T}} \sin^2 x\, e^{-\frac{1}{\sigma}\cos(2x)}\,dx - \sigma\int_{\mathbb{T}} e^{-\frac{1}{\sigma}\cos(2x)}\,dx \\ &= \left(\frac{1}{2} - \sigma\right)\int_{\mathbb{T}} e^{-\frac{1}{\sigma}\cos(2x)}\,dx - \frac{1}{2}\int_{\mathbb{T}}\cos(2x)e^{-\frac{1}{\sigma}\cos(2x)}\,dx. \end{aligned} \tag{3.28}$$

Once this is in place let us introduce the family of functions defined as

$$I_n(z) := \int_{\mathbb{T}} \cos(2nx)e^{z\cos(2x)}\,dx, \quad z \in \mathbb{R},\, n \in \mathbb{N},$$

and, consequently,

$$r_n(z) := \frac{I_{n+1}(z)}{I_n(z)}, \quad z \in \mathbb{R},\, n \in \mathbb{N}, \tag{3.29}$$

the family of functions $\{I_n\}_{n\in\mathbb{N}}$ is commonly referred to as modified Bessel functions of first kind. It is well-known that I_0 is an even function while r_0 is an odd function and, moreover, $r_0(z) > 0$, $r_0'(z) > 0$ for $z > 0$ (see [1, (15)]).
By using (3.27) and (3.28), we obtain the following expression for $\zeta_\sigma'(0)$:

$$\zeta_\sigma'(0) = \frac{\sigma}{2} I_0\left(-\frac{1}{\sigma}\right)\left(\frac{1}{\sigma} - 2 - \frac{1}{\sigma} r_0\left(-\frac{1}{\sigma}\right)\right) = \frac{\sigma}{2} I_0\left(\frac{1}{\sigma}\right)\left(\frac{1}{\sigma} - 2 + \frac{1}{\sigma} r_0\left(\frac{1}{\sigma}\right)\right).$$

Since $\sigma \cdot I_0\left(\frac{1}{\sigma}\right) > 0$ for all $\sigma > 0$, to prove the uniqueness of σ_c it suffices to study the set of zeroes of the function f_c defined in (2.21). By taking the first derivative of f_c we obtain

$$f_c'(\sigma) = -\frac{1}{\sigma^2} - \frac{1}{\sigma^2} r_0\left(\frac{1}{\sigma}\right) - \frac{1}{\sigma^3} r_0'\left(\frac{1}{\sigma}\right) < 0, \quad \sigma > 0. \tag{3.30}$$

Hence, f_c is a strictly decreasing function and, therefore, the critical value $\sigma_c > 0$ must be the unique root of f_c. Lastly, since σ_c is the unique zero of f_c, an approximation to its value can be obtained e.g. via the bisection method. This is the procedure that led to the value $\sigma_c \simeq 0.7709$. This concludes the proof. □

3.3. **Proof of Step 3.** In this subsection we prove that when $0 < \sigma \ll 1$ the unique fixed point of the map $g_\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ restricted to $\mathbb{M}_1$ is the origin. To this end, we introduce the map $h_\sigma : \mathbb{R} \to \mathbb{R}$, which is the restriction of g_σ to $\mathbb{M}_1$, namely

$$h_\sigma(m) := \frac{1}{\hat{Z}_\sigma(m)} \int_{\mathbb{T}} \cos x\, e^{-\frac{1}{\sigma}(\cos(2x) - m\cos x)}\, dx, \;\; \hat{Z}_\sigma(m) := \int_{\mathbb{T}} e^{-\frac{1}{\sigma}(\cos(2x) - m\cos x)}\, dx. \tag{3.31}$$

Then $m \in \mathbb{R}$ is a fixed point of h_σ if and only if $(m, 0)$ is a fixed point of g_σ. We already know that the map h_σ has a fixed point, as $h_\sigma(0) = 0$. Our goal is to show that when $\sigma \ll 1$ the map h_σ does not admit any further fixed point other than $m = 0$.

Theorem 3.8. *When $0 < \sigma \ll 1$ the map $h_\sigma : \mathbb{R} \to \mathbb{R}$ defined in* (3.31) *admits a unique fixed point given by $m = 0$.*

The proof of Theorem 3.8 relies on the asymptotic expansions in Lemma 3.9 below. So we first state Lemma 3.9 and then prove Theorem 3.8. The proof of Lemma 3.9 is in Appendix A.

Lemma 3.9. *The following asymptotic expansions (for σ small) hold, uniformly over $m \in [0, 1]$:*

$$\int_{\mathbb{T}} e^{-\frac{1}{\sigma}(\cos(2x) - m\cos x)}\, dx = \sqrt{\frac{2\pi\sigma}{4 - \frac{m^2}{4}}}\, e^{\frac{1}{\sigma}\left(1 + \frac{m^2}{8}\right)} \left(2 + \left(c(m) + \frac{4}{(4 - \frac{m^2}{2})^2}\right)\sigma + o_m(\sigma)\right), \tag{3.32}$$

$$\int_{\mathbb{T}} \cos x\, e^{-\frac{1}{\sigma}(\cos(2x) - m\cos x)}\, dx = \sqrt{\frac{2\pi\sigma}{4 - \frac{m^2}{4}}}\, e^{\frac{1}{\sigma}\left(1 + \frac{m^2}{8}\right)} \left(\frac{m}{2} + \bar{c}(m)\sigma + o_m(\sigma)\right), \tag{3.33}$$

$$\int_{\mathbb{T}} \cos^2 x\, e^{-\frac{1}{\sigma}(\cos(2x)-m\cos x)}\, dx = \sqrt{\frac{\pi\sigma}{2}} e^{\frac{1}{\sigma}\left(1+\frac{m^2}{8}\right)} \left(\frac{m^2}{8} + \left(\hat{c}(m) + \frac{2}{4-\frac{m^2}{4}}\right)\sigma + o_m(\sigma)\right), \tag{3.34}$$

where $\frac{o_m(\sigma)}{\sigma} \to 0$ *as* $\sigma \downarrow 0$ *and* $c, \bar{c}, \hat{c} : [0,1] \to \mathbb{R}$ *are continuous functions such that* $c(m), \bar{c}(m), \hat{c}(m) \to 0$ *as* $m \downarrow 0$. *In particular, we have* $c(m) = \bar{c}(m) = \hat{c}(m) = O(m)$ *as* $m \in [0,1]$. [6]

Proof of Theorem 3.8. Note that since $|h_\sigma(m)| \leq 1$ for all $m \in \mathbb{R}$, the fixed points of h_σ are in the interval $[-1,1]$. Moreover, since h_σ is a $C^\infty(\mathbb{R};\mathbb{R})$-odd function (continuity and differentiability is meant with respect to $m \in \mathbb{R}$), it is enough to restrict h_σ to the interval $[0,1]$ and prove that $m = 0$ is the unique fixed point for $\sigma > 0$ sufficiently small. We begin with computing the first derivative of h_σ:

$$\begin{aligned} h_\sigma'(m) = &-\frac{1}{\sigma (Z_\sigma(m))^2}\left(\int_{\mathbb{T}} \cos x\, e^{-\frac{1}{\sigma}(\cos(2x)-m\cos x)}\, dx\right)^2 \\ &+ \frac{1}{\sigma Z_\sigma(m)} \int_{\mathbb{T}} \cos^2 x\, e^{-\frac{1}{\sigma}(\cos(2x)-m\cos x)}\, dx. \end{aligned} \tag{3.35}$$

In the first part of the proof we are going to prove that if we fix a $\delta > 0$ small enough then it follows that

$$|h_\sigma'(m)| < 1, \quad \text{for } \sigma \ll 1 \text{ and } m \in [0,\delta). \tag{3.36}$$

Hence,

$$h_\sigma(m) < m, \quad \text{for } \sigma \ll 1 \text{ and } m \in (0,\delta). \tag{3.37}$$

The bound (3.36) is a consequence of the asymptotic expansions of Lemma 3.9. Indeed, from (3.32), (3.33), (3.34) and (3.35) we obtain

$$\begin{aligned} h_\sigma'(m) = \frac{1}{\sigma}&\left(\frac{\frac{m^2}{8} + \left(\hat{c}(m) + \frac{2}{4-\frac{m^2}{4}}\right)\sigma + o_m(\sigma)}{2 + \left(c(m) + \frac{4}{(4-\frac{m^2}{2})^2}\right)\sigma + o_m(\sigma)}\right) \\ -\frac{1}{\sigma}&\left(\frac{\frac{m}{2} + \bar{c}(m)\sigma + o_m(\sigma)}{2 + \left(c(m) + \frac{4}{(4-\frac{m^2}{2})^2}\right)\sigma + o_m(\sigma)}\right)^2. \end{aligned}$$

By expanding the square for the second addend, we have

$$\begin{aligned} h_\sigma'(m) = \frac{1}{\sigma}&\left(\frac{\frac{m^2}{8} + \left(\hat{c}(m) + \frac{2}{4-\frac{m^2}{4}}\right)\sigma + o_m(\sigma)}{2 + \left(c(m) + \frac{4}{(4-\frac{m^2}{2})^2}\right)\sigma + o_m(\sigma)}\right) \\ -\frac{1}{\sigma}&\left(\frac{\frac{m^2}{4} + 2\bar{c}(m)\sigma + o_m(\sigma)}{4 + \left(2c(m) + \frac{8}{(4-\frac{m^2}{2})^2}\right)\sigma + o_m(\sigma)}\right), \end{aligned}$$

[6] We recall that a function $f : [0,1] \to \mathbb{R}$ satisfies $f = O(m)$ with $m \in [0,1]$ if there exists a constant, say $D \in \mathbb{R}$ such that $|f(m)| \leq D \cdot m$, for all $m \in [0,1]$.

hence,

$$h_\sigma^{'}(m) = \frac{2\hat{c}(m) - 2\bar{c}(m) + \frac{4}{4-\frac{m^2}{4}} + \frac{o_m(\sigma)}{\sigma}}{4 + \left(2c(m) + \frac{8}{(4-\frac{m^2}{2})^2}\right)\sigma + o_m(\sigma)}.$$

If we set $\varepsilon, \delta > 0$ small enough then there exists a $\hat{\sigma} > 0$ such that if $\sigma < \hat{\sigma}$ then $|h_\sigma^{'}(m) - \frac{1}{4}| \leq \varepsilon$ for all $m \in [0, \delta)$. This concludes the first part of the proof.
In the remaining part of the proof we are going to show that $h_\sigma(m) < m$ for all $m \in [\delta, 1]$ provided σ is sufficiently small, where $\delta > 0$ is as in (3.37). To be precise, we are going to prove that $h_\sigma(m)$ converges to $\frac{m}{4}$ uniformly over the interval $[\delta, 1]$. Indeed, again from Lemma 3.9 we have

$$h_\sigma(m) = \frac{\frac{m}{2} + \bar{c}(m)\sigma + o_m(\sigma)}{2 + \left(c(m) + \frac{4}{(4-\frac{m^2}{2})^2}\right)\sigma + o_m(\sigma)},$$

with $\frac{o_m(\sigma)}{\sigma} \to 0$ as $\sigma \downarrow 0$ uniformly in $m \in [\delta, 1]$. As a consequence, $h_\sigma(m) < m$ for all $m \in [\delta, 1]$, provided σ is sufficiently small. This concludes the proof. □

The proof of Step 4 is deferred to Appendix A.

4. Proof of Theorem 2.4

In this section we study the well-posedness of the problem (2.2). As we use a combination of the arguments of e.g. [15] developed for the stochastic Burgers' equation, and of those used in the McKean-Vlasov PDE literature, in particular [8], in places we give only essential details.

Proof of Theorem 2.4. The stochastic process $u(t)$ is a mild solution of (2.2) (in the sense (2.5)) if and only if the process $v(t) = u(t) - W_A(t)$ is a mild solution of the following problem

$$\begin{cases} \partial_t v = Av + \partial_x \left[V^{'}\tilde{v} + (F^{'} * \tilde{v})\, \tilde{v}\right], & (0,T) \times \mathbb{T}, \\ v(t,0) = v(t,2\pi), & t \in [0,T], \\ v(0,x) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{4.1}$$

where $\tilde{v}(t) := v(t) + W_A(t)$, $t \in [0,T]$. Therefore, to prove Theorem 2.4, it is enough to show the global existence and uniqueness of a mild solution to (4.1), which is what we do in the following.

As classical, see e.g. [15], global existence and uniqueness (in mild sense) follow directly by local existence and uniqueness, plus appropriate a priori estimates for the (mild) solution. We prove local well-posedness in Proposition 4.2 and the a priori estimates in Proposition 4.4. □

We recall that a continuous $L^2(\mathbb{T};\mathbb{R})$-valued stochastic process $\{v(t)\}_{t\in[0,T]}$, is a mild solution to (4.1) if the following identity holds for every $t \in [0,T]$,

$$v(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}\partial_x \left[V^{'}\tilde{v}(s) + (F^{'} * \tilde{v})(s)\tilde{v}(s)\right] ds, \qquad \mathbb{P}-\text{a.s.}. \tag{4.2}$$

Thus, introducing the linear map $P : C([0,T];H^1(\mathbb{T};\mathbb{R})) \to C([0,T];L^2(\mathbb{T};\mathbb{R}))$ given by

$$P[z](t) := \int_0^t e^{(t-s)A}\partial_x z(s)\,ds, \qquad \text{for } z \in C([0,T];H^1(\mathbb{T};\mathbb{R}))\,, \tag{4.3}$$

we can rewrite equation (4.2) as

$$v(t) = e^{tA}u_0 + P\big[V^{'}\tilde{v} + (F^{'} * \tilde{v})\tilde{v}\big](t), \qquad \mathbb{P}\text{-a.s., } t \in [0,T]\,. \tag{4.4}$$

Hence an $L^2(\mathbb{T};\mathbb{R})$-valued stochastic process v is said to be a *local mild solution* to (4.1) if there exists a stopping time T^* such that equation (4.4) is satisfied for all $t \in [0,T^*)$, $\mathbb{P}$-a.s.

We will make use of the following technical lemma for the operator P, the proof of which is in Appendix B.

Lemma 4.1. *The map P defined in* (4.3) *can be extended to a bounded linear operator over the space $C\left([0,T];L^2(\mathbb{T};\mathbb{R})\right)$,*

$$P : C([0,T];L^2(\mathbb{T};\mathbb{R})) \to C([0,T];L^2(\mathbb{T};\mathbb{R})).$$

Furthermore, if $z \in C\left([0,T];L^2(\mathbb{T};\mathbb{R})\right)$ then, for all $t \in [0,T]$,

$$\|P\,[z]\,(t)\|_{L^2(\mathbb{T};\mathbb{R})} \le C_2 \int_0^t (t-s)^{-\frac{1}{2}} \|z(s)\|_{L^2(\mathbb{T};\mathbb{R})}\,ds, \tag{4.5}$$

where $C_2 > 0$ is a positive constant.

The above result is well-known in similar settings, see e.g. [14, Lemma 14.2.1] or [12, Lemma 5.2 and Lemma 5.4], though we could not find it for the specific setup in which we work, so we include the proof in Appendix B.

Proposition 4.2. *For any initial datum $u_0 \in L^2(\mathbb{T};\mathbb{R})$ and for a.e. $\omega \in \Omega$, there exists a stopping time $T^* = T^*(\omega)$ such that equation* (4.1) *has a unique local mild solution (in the sense defined above) up to time T^*.*

Proof. We study (4.2), or equivalently (4.4), pathwise for any given $\omega \in \Omega'$, where $\Omega^{'} \subset \Omega$ is the set

$$\Omega^{'} := \left\{\omega \in \Omega \,|\, W_A(\omega) \in C\left([0,T];H^1(\mathbb{T};\mathbb{R})\right)\right\}. \tag{4.6}$$

The set $\Omega^{'}$ is measurable with $\mathbb{P}(\Omega^{'}) = 1$ (as W_A is a continuous $H^1(\mathbb{T};\mathbb{R})$-valued process, see Lemma 5.4).

So, we fix $u_0 \in L^2(\mathbb{T};\mathbb{R})$ and $\omega \in \Omega'$, and choose an $m \in \mathbb{R}_+$ such that $m > \|u_0\|_{L^2(\mathbb{T};\mathbb{R})}$. We want to use a fixed point argument on the space

$$\Sigma(m,T^*) := \left\{v \in C([0,T^*];L^2(\mathbb{T};\mathbb{R})) \;\middle|\; \|v(t)\|_{L^2(\mathbb{T};\mathbb{R})} \le m,\ \forall t \in [0,T^*]\right\}$$

applied to the map $\mathcal{G}$, defined as

$$\mathcal{G}(v)(t) := e^{tA}u_0 + P\left[V^{'}(v+W_A)\right](t) + P\left[(F^{'} * (v+W_A))(v+W_A)\right](t),$$

for any $t \in [0,T^*]$ and $v \in \Sigma(m,T^*)$. We therefore need to show that, provided T^* is small enough, the space $\Sigma(m,T^*)$ is invariant under $\mathcal{G}$ and $\mathcal{G}$ is a contraction on $\Sigma(m,T^*)$.

To show that there exists $T^* > 0$ small enough such that $\Sigma(m, T^*)$ is invariant under $\mathcal{G}$, let $v \in \Sigma(m, T^*)$; then,

$$\begin{aligned}\|\mathcal{G}(v)(t)\|_{L^2(\mathbb{T};\mathbb{R})} \leq \|e^{tA}u_0\|_{L^2(\mathbb{T};\mathbb{R})} + \left\| P\left[V'(v+W_A)\right](t)\right\|_{L^2(\mathbb{T};\mathbb{R})} \\ + \left\| P\left[(F' * (v+W_A))(v+W_A)\right](t)\right\|_{L^2(\mathbb{T};\mathbb{R})},\end{aligned} \tag{4.7}$$

for any $t \in [0, T^*]$. Since e^{tA} is a contraction semigroup, we have

$$\|e^{tA}u_0\|_{L^2(\mathbb{T};\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{T};\mathbb{R})} < m,$$

for any $t > 0$. Moreover, by Lemma 4.1, the second and third addends on the RHS of (4.7) can be bounded by

$$\begin{aligned}&\left\| P\left[V'(v+W_A)\right](t)\right\|_{L^2(\mathbb{T};\mathbb{R})} \\ &\quad\leq C\int_0^t (t-s)^{-\frac{1}{2}} \left\| V'(v(s)+W_A(s))\right\|_{L^2(\mathbb{T};\mathbb{R})} ds \\ &\quad\leq C\|V'\|_{L^\infty(\mathbb{T};\mathbb{R})}\int_0^t (t-s)^{-\frac{1}{2}} \left(\|v(s)\|_{L^2(\mathbb{T};\mathbb{R})} + \|W_A(s)\|_{L^2(\mathbb{T};\mathbb{R})}\right) ds \\ &\quad\leq C\|V'\|_{L^\infty(\mathbb{T};\mathbb{R})}(m+\mu_2)\,(T^*)^{\frac{1}{2}},\end{aligned}$$

and

$$\left\| P\left[(F' * (v+W_A))(v+W_A)\right](t)\right\|_{L^2(\mathbb{T};\mathbb{R})} \leq C\|F'\|_{L^\infty(\mathbb{T};\mathbb{R})}(m^2+\mu_2^2)\,(T^*)^{\frac{1}{2}},$$

respectively, where in the above $\mu_2 := \sup_{t\in[0,T]} \|W_A\|_{L^2(\mathbb{T};\mathbb{R})}$ and C is a generic positive constant (the value of which may change from line to line), independent of m. Therefore, we have the estimate

$$\|\mathcal{G}(v)(t)\|_{L^2(\mathbb{T};\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{T};\mathbb{R})} + C(m+\mu_2+m^2+\mu_2^2)\,(T^*)^{\frac{1}{2}}.$$

If we choose T^* sufficiently small, then the operator $\mathcal{G}$ maps $\Sigma(m, T^*)$ into itself, i.e. the map $\mathcal{G} : \Sigma(m, T^*) \to \Sigma(m, T^*)$ is well-defined. We now want to show that such a map is a contraction. To this end, let $v_1, v_2 \in \Sigma(m, T^*)$; then

$$\begin{aligned}\|\mathcal{G}(v_1)(t) - \mathcal{G}(v_2)(t)\|_{L^2(\mathbb{T};\mathbb{R})} \leq \left\| P\left[V'(v_1-v_2)\right](t)\right\|_{L^2(\mathbb{T};\mathbb{R})} \\ + \left\| P\left[(F' * \tilde{v}_1)\tilde{v}_1 - (F' * \tilde{v}_2)\tilde{v}_2\right](t)\right\|_{L^2(\mathbb{T};\mathbb{R})},\end{aligned}$$

for every $t \in [0, T^*]$, where $\tilde{v}_1 := v_1 + W_A$, $\tilde{v}_2 := v_2 + W_A$. The first addend on the RHS can be bounded using analogous calculations to those we have done in the above. As for the second addend, applying again Lemma 4.1 and using Young's inequality for convolutions, we have

$$\begin{aligned}
&\left\|P\left[(F^{'} * \tilde{v}_1)\tilde{v}_1 - (F^{'} * \tilde{v}_2)\tilde{v}_2\right](t)\right\|_{L^2(\mathbb{T};\mathbb{R})} \\
&\qquad \le C\|F^{'}\|_{L^\infty(\mathbb{T};\mathbb{R})} \int_0^t (t-s)^{-\frac{1}{2}} \|\tilde{v}_1(s)\|_{L^2(\mathbb{T};\mathbb{R})} \|v_1(s)-v_2(s)\|_{L^2(\mathbb{T};\mathbb{R})}\, ds \\
&\qquad\quad + C\|F^{'}\|_{L^\infty(\mathbb{T};\mathbb{R})} \int_0^t (t-s)^{-\frac{1}{2}} \|\tilde{v}_2(s)\|_{L^2(\mathbb{T};\mathbb{R})} \|v_1(s)-v_2(s)\|_{L^2(\mathbb{T};\mathbb{R})}\, ds \\
&\qquad \le C\|F^{'}\|_{L^\infty(\mathbb{T};\mathbb{R})} (T^*)^{\frac{1}{2}} (m+\mu_2) \|v_1 - v_2\|_{C([0,T^*];L^2(\mathbb{T};\mathbb{R}))}.
\end{aligned}$$

Thus, we obtain

$$\|\mathcal{G}(v_1)(t) - \mathcal{G}(v_2)(t)\|_{L^2(\mathbb{T};\mathbb{R})} \le C(1+m+\mu_2)(T^*)^{\frac{1}{2}} \|v_1-v_2\|_{C([0,T^*];L^2(\mathbb{T};\mathbb{R}))},$$

for all $t \in [0,T^*]$ and for any given $v_1, v_2 \in \Sigma(m,T^*)$. Choosing T^* sufficiently small, the conclusion follows from the Banach fixed point theorem. Note that T^* is a stopping time as, by construction, it depends on ω only through μ_2. □

To study a priori estimates for the mild solution to (4.1), we start by recalling the following technical lemma, the proof of which can be found e.g. in [31, Lemma A.2].

Lemma 4.3 (Generalized Gronwall's inequality)**.** *Let $T>0$, $C_1, C_2 \ge 0$ and $q:[0,T]\to\mathbb{R}$ be a non-negative and continuous function and let $r>0$. If*

$$q(t) \le C_1 + C_2 \int_0^t (t-s)^{-1+r} q(s)\, ds, \quad \text{for all } t\in[0,T], \tag{4.8}$$

then there exists a non-negative, increasing and continuous function $\tilde{c}(t) = \tilde{c}(t, C_2, r) \ge 0$, $t \ge 0$ such that

$$q(t) \le C_1 \tilde{c}(t), \quad \text{for all } t \in [0,T].$$

In the next proposition we still work pathwise. After the proof of Proposition 4.4 we explain how to extend the solution up to a time T independent of $\omega\in\Omega$.

Proposition 4.4 (A priori estimates)**.** *Let $\tilde{T} = \tilde{T}(\omega)$ be such that a mild solution v to (4.1) exists up to time $\tilde{T}$ and $v \in C([0,\tilde{T}]; L^2(\mathbb{T};\mathbb{R}))$. Then,*

$$\|v(t)\|^2_{L^2(\mathbb{T};\mathbb{R})} \le a(u_0, W_A)(t)\cdot \exp\left(C\int_0^t b(u_0,W_A)(s)\, ds\right), \quad t\in[0,\tilde{T}],\ \mathbb{P}\text{-a.s.}, \tag{4.9}$$

where the non-negative functions $a(v_0,\varphi)(t)$ and $b(v_0,\varphi)(t)$ are given by

$$\begin{aligned}
a(v_0,\varphi)(t) &:= \|v_0\|^2_{L^2(\mathbb{T};\mathbb{R})} + C\int_0^t \|\varphi(s)\|^4_{L^2(\mathbb{T};\mathbb{R})}\, ds, \\
b(v_0,\varphi)(t) &:= \|\varphi(t)\|^2_{L^2(\mathbb{T};\mathbb{R})} \\
&\qquad + \left(\|v_0\|_{L^1(\mathbb{T};\mathbb{R})} + \int_0^t \left(1 + \|\varphi(s)\|^2_{H^1(\mathbb{T};\mathbb{R})}\right) ds\right) e^{C\int_0^t \|\varphi(s)\|_{H^1(\mathbb{T};\mathbb{R})} ds},
\end{aligned}$$

for any $t\in[0,T]$, $v_0 \in L^2(\mathbb{T};\mathbb{R})$ and $\varphi \in C([0,T]; H^1(\mathbb{T};\mathbb{R}))$, where $C>0$ is a constant depending on V and F.

Proof of Proposition 4.4. In what follows, unless otherwise specified, C denotes a generic deterministic positive constant, which may change from line to line. Note also that the time $\tilde{T}$ in the statement of the proposition does exist (for each ω, just take some $\tilde{T}$ smaller than the time T^* of the local result).

We start with considering a regularised version of the system (4.1), where we replace the stochastic convolution W_A with a $C^\infty([0,T]\times\mathbb{R})$-valued random variable φ. Namely, we consider the following random evolution

$$\begin{cases} \partial_t v = Av + \partial_x\left[V'(v+\varphi) + F' * (v+\varphi)(v+\varphi)\right] \\ v(t,0) = v(t,2\pi), \quad t\in[0,T], \\ v(0,x) = u_0(x), \quad x\in\mathbb{T}, \end{cases} \tag{4.10}$$

with $u_0 \in L^2(\mathbb{T};\mathbb{R})$. We prove in Proposition B.1 that, $\mathbb{P}$-a.s., there exists a unique global solution v to (4.10) and that such a solution satisfies the following a priori estimates for the L^2 norm,

$$\|v(t)\|^2_{L^2(\mathbb{T};\mathbb{R})} \le a(u_0,\varphi)(t)\cdot \exp\left(\int_0^t b(u_0,\varphi)(s)\,ds\right), \quad t\in[0,\tilde{T}],\ \mathbb{P}-\text{a.s.} \tag{4.11}$$

where the functions $a(u_0,\varphi)(t)$ and $b(u_0,\varphi)(t)$ are defined as in the statement of Proposition 4.4. To extend these estimates to the dynamics (4.1), we consider a family $\{\varphi_n\}_{n\in\mathbb{N}}$ of $C^\infty([0,\tilde{T}]\times\mathbb{R})$-valued random variables such that

$$\varphi_n \to W_A \quad \text{in } C([0,\tilde{T}];H^1(\mathbb{T};\mathbb{R})) \qquad \mathbb{P}-\text{a.s., as } n\to\infty. \tag{4.12}$$

For any $n\in\mathbb{N}$, we denote by v_n the unique mild solution to (4.10) with φ replaced by φ_n. Moreover, we denote by $v\in C([0,\tilde{T}];L^2(\mathbb{T};\mathbb{R}))$ the mild solution to (4.1) up to time $\tilde{T}$. We want to show that v_n converges to v in $C([0,\tilde{T}];L^2(\mathbb{T};\mathbb{R}))$, $\mathbb{P}$-a.s.

Recall that v and the sequence $\{v_n\}_{n\in\mathbb{N}}$ satisfy

$$\begin{aligned} v(t) &= e^{tA}u_0 + P\left[V'\tilde{v} + (F' * \tilde{v})\tilde{v}\right](t), \\ v_n(t) &= e^{tA}u_0 + P\left[V'\tilde{v}_n + (F' * \tilde{v}_n)\tilde{v}_n\right](t), \end{aligned}$$

respectively, with P defined in (4.3), having set $\tilde{v} = v + W_A$, $\tilde{v}_n = v_n + \varphi_n$, $n\in\mathbb{N}$. Then, by Lemma 4.1 and with calculations completely analogous to those in the proof of Proposition 4.2, we have

$$\begin{aligned} \|v(t)-v_n(t)\|_{L^2(\mathbb{T};\mathbb{R})} \le{}& C\tilde{T}^{\frac{1}{2}}\|W_A - \varphi_n\|_{C([0,T];L^2(\mathbb{T};\mathbb{R}))} \\ &+ C\int_0^t (t-s)^{-\frac{1}{2}}\|v(s)-v_n(s)\|_{L^2(\mathbb{T};\mathbb{R})}\,ds \\ &+ \int_0^t (t-s)^{-\frac{1}{2}}\left\|\left(F' * \tilde{v}\right)(s)\left(\tilde{v}(s)-\tilde{v}_n(s)\right)\right\|_{L^2(\mathbb{T};\mathbb{R})} ds \\ &+ \int_0^t (t-s)^{-\frac{1}{2}}\left\|\left(F' * (\tilde{v}-\tilde{v}_n)\right)(s)\tilde{v}_n(s)\right\|_{L^2(\mathbb{T};\mathbb{R})} ds\,. \end{aligned}$$

Since $v\in C\left([0,\tilde{T}];L^2(\mathbb{T};\mathbb{R})\right)$ $\mathbb{P}$-a.s., there exists a non-negative random variable $M_{1,\tilde{T}} = M_{1,\tilde{T}}(\omega) < +\infty$ $\mathbb{P}$-a.s. such that $\|v(t)\|_{L^2(\mathbb{T};\mathbb{R})} \le M_{1,\tilde{T}}$, for all $t\in[0,\tilde{T}]$, $\mathbb{P}$-a.s (but we don't know the exact dependence of $M_{1,\tilde{T}}$ on $\tilde{T}$). Hence,

$$\begin{aligned}\left\|\left(F^{'} * \tilde{v}\right)(\tilde{v}-\tilde{v}_n)(s)\right\|_{L^2(\mathbb{T};\mathbb{R})} &\leq C\|\tilde{v}(s)\|_{L^2(\mathbb{T};\mathbb{R})}\|\tilde{v}(s)-\tilde{v}_n(s)\|_{L^2(\mathbb{T};\mathbb{R})}\\ &\leq CM_{1,\tilde{T}}\left(\|\varphi_n(s)-W_A(s)\|_{L^2(\mathbb{T};\mathbb{R})}+\|v(s)-v_n(s)\|_{L^2(\mathbb{T};\mathbb{R})}\right),\end{aligned}$$

and, similarly,

$$\left\|\left(F^{'} * (\tilde{v}-\tilde{v}_n)\right)(s)\tilde{v}_n(s)\right\|_{L^2(\mathbb{T};\mathbb{R})} \leq CM_{2,\tilde{T}}\|v(s)-v_n(s)\|_{L^2(\mathbb{T};\mathbb{R})},$$

for all $s \in [0,\tilde{T}]$, where $M_{2,\tilde{T}} = M_{2,\tilde{T}}(\omega)$ is a non-negative random variable with $M_{2,\tilde{T}} < +\infty$, $\mathbb{P}$-a.s. such that $\|\tilde{v}_n\|_{C([0,\tilde{T}];L^2(\mathbb{T};\mathbb{R}))} \leq M_{2,\tilde{T}}$ uniformly in $n \in \mathbb{N}$, $\mathbb{P}$-a.s.; such a random variable exists by (4.11). From the above, we then have

$$\begin{aligned}\|v(t)-v_n(t)\|_{L^2(\mathbb{T};\mathbb{R})} \leq& C\|W_A-\varphi_n\|_{C([0,T];L^2(\mathbb{T};\mathbb{R}))}\\ &+C(M_{1,\tilde{T}}+M_{2,\tilde{T}})\int_0^t (t-s)^{-\frac{1}{2}}\|v(s)-v_n(s)\|_{L^2(\mathbb{T};\mathbb{R})}\,ds,\end{aligned}$$

so that using Lemma 4.3 finally gives

$$\|v(t)-v_n(t)\|_{L^2(\mathbb{T};\mathbb{R})} \leq \tilde{c}_{\tilde{T}}\left(\|W_A-\varphi_n\|_{C([0,T];L^2(\mathbb{T};\mathbb{R}))}\right),$$

for all $t \in [0,\tilde{T}]$, where $\tilde{c}_{\tilde{T}} > 0$ depends on $\|V^{'}\|_{L^\infty(\mathbb{T};\mathbb{R})}$, $\|F^{'}\|_{L^\infty(\mathbb{T};\mathbb{R})}$, $M_{1,\tilde{T}}$ and $M_{2,\tilde{T}}$. Hence v_n converges to v in $C([0,T];L^2(\mathbb{T};\mathbb{R}))$ $\mathbb{P}$-a.s., as $n \to \infty$. Finally, applying (4.11) to $\{v_n\}_{n\in\mathbb{N}}$ and noting that by definition of the functions a and b, $a(u_0,\varphi_n) \xrightarrow{n\to\infty} a(u_0,W_A)$ and $b(u_0,\varphi_n) \xrightarrow{n\to\infty} b(u_0,W_A)$ $\mathbb{P}$-a.s. uniformly in $[0,\tilde{T}]$ (since $\varphi_n \xrightarrow{n\to\infty} W_A$ in $C([0,T];H^1(\mathbb{T};\mathbb{R}))$ $\mathbb{P}$-a.s.), we obtain the a priori estimate (4.9) for the L^2 norm of the solution v to (4.1). It is important to note that the constants $M_{1,\tilde{T}}$, $M_{2,\tilde{T}}$ do not appear i the definitions of a and b, they only appear in the estimates used to show the convergence of v_n to v. □

Because of the form of the a priori estimate (4.9), from a classical argument (see e.g. [15]) it follows that the solution v to (4.1) can be extended, for almost every $\omega \in \Omega$, up to a time $T = T(\omega)$. [7] To show that such a time can be taken to be independent of ω, referring to the construction in the proof of the above Proposition 4.4 we observe (see [14, Theorem 14.2.4]) the following: first, by Proposition B.1 the solution of (4.10) can be defined up to a time $T > 0$ fixed a priori and independent of ω. Hence all the v_n's exist on an interval $[0,T]$, for any $T > 0$, independent of ω. This, combined with the fact that the sequence φ_n can be taken so that the convergence (4.12) is on $C([0,T];H^1(\mathbb{T};\mathbb{R}))$, allows one to show that the a priori estimates of Proposition 4.4 are in fact valid for any deterministic time interval $[0,T]$.

5. Proof of point i) of Theorem 2.6: irreducibility

In this section we prove irreducibility of the dynamics (2.2) using the methods of [12, Chapter 5], [14, Chapter 14]. We recall that throughout this section Q is assumed to satisfy (2.3) and (2.4).

[7] A difference between [14] and our setting is that in [14, Theorem 14.2.4] estimates independent of the regularity of φ are needed because there the authors consider cylindrical Wiener noise. In our case this further difficulty is not present.

We start by considering the control system associated with (2.2)

$$\begin{cases} \partial_t y = Ay + \partial_x \left[V' y + \left(F' * y \right) y \right] + Q^{\frac{1}{2}} f, \quad t \in [0,T], \\ y(t,0) = y(t,2\pi), \quad t \in [0,T], \\ y(0,x) = y_0(x), \quad x \in \mathbb{T}, \end{cases} \tag{5.1}$$

with initial datum $y_0 \in L^2(\mathbb{T};\mathbb{R})$ and obtained from (2.2) by replacing the stochastic forcing $\partial_t W$ with a *deterministic* control $f \in L^2\left([0,T];L^2(\mathbb{T};\mathbb{R})\right)$. When we wish to emphasize the dependence of the solution of (5.1) on the initial datum y_0 and on the control f we use the notation $y(t;y_0,f)$.

In the same fashion as the proof of Proposition B.1, we can show that system (5.1) admits a unique global mild solution $y = y(t)$ and that the following estimate (analogous to those found in the stochastic case) holds:

$$\|y(t)\|^2_{L^2(\mathbb{T};\mathbb{R})} \leq \ell_2(y_0,f)(t) e^{C\left(t+\int_0^t \ell_1(y_0,f)(s)\,ds\right)}, \tag{5.2}$$

for all $t \geq 0$, for some constant $C > 0$ depending on V and F, and where $t \to \ell_1(y_0,f)(t)$, $t \to \ell_2(y_0,f)(t)$ are time-continuous increasing functions, namely

$$\begin{aligned} \ell_1(y_0,f)(t) &:= \|y_0\|_{L^1(\mathbb{T};\mathbb{R})} + \int_0^t \left\| Q^{\frac{1}{2}} f(s) \right\|_{L^1(\mathbb{T};\mathbb{R})} ds, \\ \ell_2(y_0,f)(t) &:= \|y_0\|_{L^2(\mathbb{T};\mathbb{R})} + \frac{1}{2}\int_0^t \|Q^{\frac{1}{2}} f(s)\|^2_{L^2(\mathbb{T};\mathbb{R})} ds, \end{aligned}$$

for any $t \geq 0$.

We want to show that system (5.1) is approximately controllable. We recall that system (5.1) is *approximately controllable in* $L^2(\mathbb{T};\mathbb{R})$ *at time* $T > 0$ via an $L^2\left([0,T];L^2(\mathbb{T};\mathbb{R})\right)$-control if for any $y_0, y_1 \in L^2(\mathbb{T};\mathbb{R})$ and for all $\varepsilon > 0$ there exists $f \in L^2\left([0,T];L^2(\mathbb{T};\mathbb{R})\right)$ such that the solution $y = y(t;y_0,f)$ of (5.1) satisfies

$$\|y(T;y_0,f) - y_1\|_{L^2(\mathbb{T};\mathbb{R})} \leq \varepsilon.$$

If the dynamics (5.1) is approximately controllable in $L^2(\mathbb{T};\mathbb{R})$ at time T for any $T > 0$ then we simply say that (5.1) is approximately controllable in $L^2(\mathbb{T};\mathbb{R})$.

To show that (5.1) is approximately controllable in $L^2(\mathbb{T};\mathbb{R})$ via an $L^2\left([0,T];L^2(\mathbb{T};\mathbb{R})\right)-$control, we will first show that it is approximately controllable in $C^2(\mathbb{T};\mathbb{R})$ via a $C([0,T];C(\mathbb{T};\mathbb{R}))$-control, see Lemma 5.1. Then the smoothing properties of the deterministic part of the equation allow one to conclude the desired approximate controllability in $L^2(\mathbb{T};\mathbb{R})$ (via an $L^2\left([0,T];L^2(\mathbb{T};\mathbb{R})\right)$-control), see Proposition 5.3. Finally, irreducibility (in $L^2(\mathbb{T};\mathbb{R})$) of the semigroup $\{\mathcal{P}_t\}_{t\geq 0}$ associated with (2.2) is deduced once we show that mild solutions of the SPDE (2.2) can be approximated by solutions of the deterministic problem (5.1). This is the content of Lemma 5.4 and Theorem 5.5, the latter being the main result of this section. We now begin to carry out the programme described above.

Lemma 5.1. *Let Q satisfy* (2.3)-(2.4). *For any $T > 0$, $y_0, y_1 \in C^2(\mathbb{T};\mathbb{R})$ and $\varepsilon > 0$ there exists $f \in C\left([0,T];L^2(\mathbb{T};\mathbb{R})\right)$ such that*

$$\|y(T;y_0,f) - y_1\|_{L^2(\mathbb{T};\mathbb{R})} \leq \varepsilon. \tag{5.3}$$

Proof. Let y_0, y_1 be two arbitrary but fixed points in $C^2(\mathbb{T};\mathbb{R})$ and let $\alpha_{y_0,y_1} = \alpha_{y_0,y_1}(t)$ be a continuous path in $C^2(\mathbb{T};\mathbb{R})$, joining y_0 and y_1, i.e.

$$\alpha_{y_0,y_1}(0) = y_0, \quad \alpha_{y_0,y_1}(T) = y_1. \tag{5.4}$$

To fix ideas, we will take

$$\alpha_{y_0,y_1}(t) := \frac{T-t}{T}y_0 + \frac{t}{T}y_1, \qquad t \in [0,T],$$

and note that α_{y_0,y_1} is a function of time and space, but we omit the dependence on the space variable when not needed. Because $y_0, y_1 \in C^2$, they belong to the domain of A, hence we can define the path $\beta_{y_0,y_1} = \beta_{y_0,y_1}(t) \subset C(\mathbb{T};\mathbb{R})$ as follows:

$$\begin{aligned}\beta_{y_0,y_1}(t) &:= \partial_t \alpha_{y_0,y_1}(t) - A\alpha_{y_0,y_1}(t)\\ &\quad - \partial_x \left[V' \alpha_{y_0,y_1}(t) + \left(F' * \alpha_{y_0,y_1}\right)(t)\alpha_{y_0,y_1}(t)\right], \quad t \in [0,T].\end{aligned}$$

Then, by definition, α_{y_0,y_1} is a classical solution to the PDE

$$\begin{cases}\partial_t \alpha_{y_0,y_1} = A\alpha_{y_0,y_1} + \partial_x \left[V' \alpha_{y_0,y_1} + \left(F' * \alpha_{y_0,y_1}\right)\alpha_{y_0,y_1}\right] + \beta_{y_0,y_1}, \quad t \in [0,T],\\ \alpha_{y_0,y_1}(t,0) = \alpha_{y_0,y_1}(t,2\pi), \quad t \in [0,T],\\ \alpha_{y_0,y_1}(0,x) = y_0(x), \quad x \in \mathbb{T},\end{cases} \tag{5.5}$$

such that (5.4) holds. If we prove that for every $\epsilon > 0$ there exists $f \in C([0,T];L^2(\mathbb{T};\mathbb{R}))$ such that the following holds

$$\|y(t,y_0;f) - \alpha_{y_0,y_1}(t)\|_{L^2(\mathbb{T};\mathbb{R})} \le C\varepsilon, \quad \text{for every } t \in [0,T], \tag{5.6}$$

for some $C > 0$ (and independent of ϵ) then the proof is concluded. Indeed (5.3) readily follows from (5.6) and (5.4).

We will show that if f is any function in $C\left([0,T];L^2(\mathbb{T};\mathbb{R})\right)$ such that

$$\|\beta_{y_0,y_1}(t) - Q^{\frac{1}{2}}f(t)\|_{L^2(\mathbb{T};\mathbb{R})} \le \varepsilon, \quad t \in [0,T], \tag{5.7}$$

then (5.6) holds.[8] So, let f be such that (5.7) holds and set $w(t) := y(t;y_0,f) - \alpha_{y_0,y_1}(t)$. Since α_{y_0,y_1} is a (mild) solution to (5.5) and $y(t;y_0,f)$ is a mild solution to (5.1), we have

$$\begin{aligned}w(t) &= \int_0^t e^{(t-s)A}\partial_x \left[V' w(s)\right] ds\\ &\quad + \int_0^t e^{(t-s)A}\partial_x \left[\left(F' * w\right)(s)\alpha_{y_0,y_1}(s) - \left(F' * y\right)(s)w(s)\right] ds\\ &\quad + \int_0^t e^{(t-s)A}Q^{\frac{1}{2}}f(s)\, ds - \int_0^t e^{(t-s)A}\beta_{y_0,y_1}(s)\, ds.\end{aligned} \tag{5.8}$$

Let now $\kappa > 0$ be such that

$$\sup_{t\in[0,T]} \|y(t;y_0,f)\|_{L^2(\mathbb{T};\mathbb{R})} + \sup_{t\in[0,T]} \|\alpha_{y_0,y_1}(t)\|_{L^2(\mathbb{T};\mathbb{R})} \le \kappa. \tag{5.9}$$

[8]The existence of at least one such function is obvious, as β_{y_0,y_1} is a continuous function.

Such a κ exists by definition of α_{y_0,y_1} and because $y \in C([0,T];L^2)$ if $f \in C([0,T];L^2)$.[9] From (4.5) and the L^2-contraction property of $\{e^{tA}\}_{t\geq 0}$ we obtain

$$\begin{aligned}\|w(t)\|_{L^2(\mathbb{T};\mathbb{R})} \leq C \int_0^t (t-s)^{-\frac{1}{2}} \left\|V' w(s)\right\|_{L^2(\mathbb{T};\mathbb{R})} ds \\ + C \int_0^t (t-s)^{-\frac{1}{2}} \left\|(F' * w)(s)\alpha_{y_0,y_1}(s)\right\|_{L^2(\mathbb{T};\mathbb{R})} ds \\ + C \int_0^t (t-s)^{-\frac{1}{2}} \left\|(F' * y)(s)w(s)\right\|_{L^2(\mathbb{T};\mathbb{R})} ds \\ + \int_0^t \left\|\beta_{y_0,y_1}(s) - Q^{\frac{1}{2}} f(s)\right\|_{L^2(\mathbb{T};\mathbb{R})} ds,\end{aligned}$$

where $C > 0$ is a generic constant (independent of $\varepsilon \in (0,1)$ but possibly dependent on T). Since for every $s \in [0,t]$ we have

$$\begin{aligned}\left\|(F' * w)(s)\alpha_{y_0,y_1}(s)\right\|_{L^2(\mathbb{T};\mathbb{R})} &\leq \left\|(F' * w)(s)\right\|_{L^\infty(\mathbb{T};\mathbb{R})} \|\alpha_{y_0,y_1}(s)\|_{L^2(\mathbb{T};\mathbb{R})} \\ &\leq \left\|F'\right\|_{L^\infty(\mathbb{T};\mathbb{R})} \|w(s)\|_{L^2(\mathbb{T};\mathbb{R})} \|\alpha_{y_0,y_1}(s)\|_{L^2(\mathbb{T};\mathbb{R})}\end{aligned} \tag{5.11}$$

(and acting similarly on the term $(F' * y)w$), from (5.9) we deduce

$$\begin{aligned}\|w(t)\|_{L^2(\mathbb{T};\mathbb{R})} \leq \bar{C}\,(1+\kappa) \int_0^t (t-s)^{-\frac{1}{2}} \|w(s)\|_{L^2(\mathbb{T};\mathbb{R})} ds \\ + \int_0^t \left\|\beta_{y_0,y_1}(s) - Q^{\frac{1}{2}} f(s)\right\|_{L^2(\mathbb{T};\mathbb{R})} ds,\end{aligned}$$

where $\bar{C}$ is a positive constant depending on $\|V'\|_{L^\infty(\mathbb{T};\mathbb{R})}$, $\|F'\|_{L^\infty(\mathbb{T};\mathbb{R})}$ and $T > 0$. The conclusion now follows from (5.7) and from the generalized Gronwall inequality (see Lemma 4.3). □

Note 5.2. By slightly modifying the above proof, it is easy to see that one can always take the control f in $C([0,T];C^\infty(\mathbb{T};\mathbb{R}))$ – just take f smooth such that (5.7) holds, then consider the L^∞ bound in (5.9) and finally adapt the manipulations in (5.11) accordingly. However, we don't need this regularity in our proofs so we simply consider f in $C([0,T];L^2(\mathbb{T};\mathbb{R}))$.

Proposition 5.3. *Let Q satisfy* (2.3) *and* (2.4). *Then the control system* (5.1) *is approximately controllable in $L^2(\mathbb{T};\mathbb{R})$ via an $L^2\left([0,T];L^2(\mathbb{T};\mathbb{R})\right)$-control. That is, for any $T > 0$, $\varepsilon > 0$ and $y_0, y_1 \in L^2(\mathbb{T};\mathbb{R})$ there exists $f \in L^2\left([0,T];L^2(\mathbb{T};\mathbb{R})\right)$ such that*

$$\|y(T;y_0,f) - y_1\|_{L^2(\mathbb{T};\mathbb{R})} \leq \varepsilon.$$

Proof. From the previous lemma we know that (5.1) is approximately controllable provided the initial datum and the endpoint are in $C^2(\mathbb{T};\mathbb{R})$. Moreover we recall that, when $f = 0$, (5.1) has smoothing properties (see e.g. [5, Theorem 2.2]); namely, if $y_0 \in L^2(\mathbb{T};\mathbb{R})$ then $y(t;y_0,0) \in C^1\left((0,+\infty);C^2(\mathbb{T};\mathbb{R})\right)$ for any $t > 0$. With this premise, let $\bar{y}_1$ be any point

[9] To be thorough, note also that κ can be chosen independently of $\varepsilon \in (0,1)$, as

$$\left\|(-B)^{-\frac{\gamma}{2}} f(t)\right\|_{L^2(\mathbb{T};\mathbb{R})} \leq \|\beta_{y_0,y_1}(t)\|_{L^2(\mathbb{T};\mathbb{R})} + 1, \quad \text{for all } t \in [0,T], \tag{5.10}$$

in $C^2(\mathbb{T};\mathbb{R})$ and $\bar{t}>0$ be any positive time. Using the mentioned smoothing properties and Lemma 5.1, there exists a function $\bar{f}\in L^2\left([\bar{t},T];L^2(\mathbb{T};\mathbb{R})\right)$ such that the control

$$f(t):=\begin{cases}0, & t\in[0,\bar{t}),\\ \bar{f}(t), & t\in[\bar{t},T],\end{cases} \tag{5.12}$$

will drive system (5.1) from y_0 to an arbitrarily small neighbourhood of $\bar{y}_1$ in time T (more precisely it will drive (5.1) first from y_0 to a point in C^2 and then from such a point to an arbitrarily small neighbourhood of $\bar{y}_1$). That is, for any $\bar{y}_1\in C^2(\mathbb{T};\mathbb{R})$ and $\varepsilon>0$ there exists $f\in L^2\left([0,T];L^2(\mathbb{T};\mathbb{R})\right)$ such that

$$\|y(T;y_0,f)-\bar{y_1}\|_{L^2(\mathbb{T};\mathbb{R})}\le\varepsilon.$$

Since $C^2(\mathbb{T};\mathbb{R})$ is dense in $L^2(\mathbb{T};\mathbb{R})$, to conclude the argument we choose $\bar{y}_1\in C^2(\mathbb{T};\mathbb{R})$ such that $\|\bar{y}_1-y_1\|_{L^2(\mathbb{T};\mathbb{R})}\le\varepsilon$; then from the triangle inequality we obtain

$$\|y(T;y_0,f)-y_1\|_{L^2(\mathbb{T};\mathbb{R})}\le\|y(T,y_0;f)-\bar{y_1}\|_{L^2(\mathbb{T};\mathbb{R})}+\|\bar{y_1}-y_1\|_{L^2(\mathbb{T};\mathbb{R})}\le 2\varepsilon.$$

□

Before proving the irreducibility of the semigroup $\{\mathcal{P}_t\}_{t\ge0}$ we recall the following lemma, which is relatively elementary, and it is the main place in this paper where assumption (2.4) is needed.

Lemma 5.4. *Let Q satisfy* (2.3) *and* (2.4)*. For $f\in L^2([0,T];L^2(\mathbb{T};\mathbb{R}))$, define*

$$f_A(t):=\int_0^t e^{(t-s)A}Q^{\frac{1}{2}}f(s)\,ds,\quad t\in[0,T]. \tag{5.13}$$

Then $f_A\in C([0,T];H^1(\mathbb{T};\mathbb{R}))$ and its weak derivative is given by

$$\partial_x f_A(t)=\sum_{k\in\mathbb{Z}}|k|\lambda_k\left(\int_0^t e^{-(t-s)k^2}f_k(s)\,ds\right)e_{-k},\quad t\in[0,T]. \tag{5.14}$$

where $f_k(s):=\langle f(s),e_k\rangle_{L^2(\mathbb{T};\mathbb{R})}$, $s\in[0,T]$, $k\in\mathbb{Z}$. As a consequence, the stochastic convolution W_A belongs to $C([0,T];H^1(\mathbb{T};\mathbb{R}))$.
Moreover, for any function f in $L^2\left([0,T];L^2(\mathbb{T};\mathbb{R})\right)$ and for any $\varepsilon>0$ the following holds

$$\mathbb{P}\left(\sup_{t\in[0,T]}\|W_A(t)-f_A(t)\|_{H^1(\mathbb{T};\mathbb{R})}\le\varepsilon\right)>0\,. \tag{5.15}$$

We give a brief proof of the above lemma in Appendix B.

Theorem 5.5. *Let Q satisfy* (2.3) *and* (2.4)*. Then, the semigroup $\{\mathcal{P}_t\}_{t\ge0}$ generated by SPDE* (2.2) *is irreducible in $L^2(\mathbb{T};\mathbb{R})$, that is, for any $u_0\in L^2(\mathbb{T};\mathbb{R})$, $(\mathcal{P}_T 1_G)(u_0)>0$ for all open sets $G\subset L^2(\mathbb{T};\mathbb{R})$ and every $T>0$.*

Proof. Let $u=u(t;u_0)$, $t\ge0$, be the solution to (2.2) with initial condition $u_0\in L^2(\mathbb{T};\mathbb{R})$. Throughout the proof u_0 will be fixed but arbitrary and we don't repeat this in every statement. Proving the assertion is equivalent to showing that for any $T>0$, $u_0\in L^2(\mathbb{T};\mathbb{R})$ and for all y_1 in a dense set of $L^2(\mathbb{T};\mathbb{R})$, the following holds

$$\mathbb{P}\left(\|u(T;u_0)-y_1\|_{L^2(\mathbb{T};\mathbb{R})}\le\varepsilon\right)>0,\quad\text{for every }\varepsilon>0\,. \tag{5.16}$$

To this end, let us fix $T > 0$ for the remaining part of the proof and note that due to Proposition 5.3 we know that the set of reachable points at time T of the control system (5.1) is a dense subset of $L^2(\mathbb{T};\mathbb{R})$. In other words, the set

$$\mathfrak{F} := \left\{ y(T; u_0, f) \,|\, f \in L^2\left([0,T]; L^2(\mathbb{T};\mathbb{R})\right) \right\}, \tag{5.17}$$

is dense in $L^2(\mathbb{T};\mathbb{R})$. Hence, it suffices to prove that (5.16) holds for any $y_1 \in \mathfrak{F}$; in particular we will show that

$$\mathbb{P}\left(\sup_{t\in[0,T]} \|u(t;u_0) - y(t;u_0,f)\|_{L^2(\mathbb{T};\mathbb{R})} \leq \varepsilon \right) > 0, \tag{5.18}$$

for every $\epsilon > 0$ and $f \in L^2\left([0,T]; L^2(\mathbb{T};\mathbb{R})\right)$ as in the above. To do so, we follow the method adopted in Lemma 5.1 and we start by estimating the difference in the $L^2(\mathbb{T};\mathbb{R})$-norm between $u(t;u_0)$ and $y(t;u_0,f)$. In turn, because of the non-linearity, this requires a bound similar to (5.9). Since in this case $u(t;u_0)$ is random (hence, the analogous bound to (5.9) will not hold for every ω) we proceed as follows. Using the estimates (4.9) (this estimate is for v, to get the one for u it suffices to recall that $u = v + W_A$) and (5.2), we can see that there exists $\bar{\kappa}$ such that

$$\sup_{t\in[0,T]} \|u(t)\|_{L^2(\mathbb{T};\mathbb{R})} + \sup_{t\in[0,T]} \|y(t)\|_{L^2(\mathbb{T};\mathbb{R})} \leq \bar{\kappa}, \quad \text{for all } \omega \in S_\varepsilon \text{ and } \varepsilon \in (0,1), \tag{5.19}$$

where the set S_ε is defined for all $\varepsilon \in (0,1)$ as

$$S_\varepsilon := \left\{ \omega \in \Omega \,:\, \sup_{t\in[0,T]} \|W_A(t)\|_{H^1(\mathbb{T};\mathbb{R})} \leq \hat{\kappa}_\varepsilon \right\}, \quad \text{where} \tag{5.20}$$

$$\hat{\kappa}_\varepsilon := \sup_{t\in[0,T]} \|f_A(t)\|_{H^1(\mathbb{T};\mathbb{R})} + \varepsilon. \tag{5.21}$$

Similarly to footnote 9, $\bar{\kappa}$ can be chosen independently of $\varepsilon \in (0,1)$. In what follows, we write in short $u(t)$ in place of $u(t;u_0)$ and $y(t)$ in place of $y(t,u_0;f)$. With this notation in mind, from the mild formulation of u and y we have that

$$\begin{aligned} u(t) - y(t) = &\int_0^t e^{(t-s)A} \partial_x \left[V'(u(s) - y(s)) \right] ds + W_A(t) - f_A(t) \\ &+ \int_0^t e^{(t-s)A} \partial_x \left[\left(F' * u \right)(s) u(s) - \left(F' * y \right)(s) y(s) \right] ds, \end{aligned}$$

where the above equality holds for all $t \in [0,T]$, $\mathbb{P}$-a.s. Now, we first restrict to the realizations $\omega \in S_\varepsilon$ (see (5.20) for its definition). Hence, from (5.19) and (4.5) we obtain

$$\begin{aligned} \|u(t) - y(t)\|_{L^2(\mathbb{T};\mathbb{R})} \leq C \|V'\|_{L^\infty(\mathbb{T};\mathbb{R})} &\int_0^t (t-s)^{-\frac{1}{2}} \|u(s) - y(s)\|_{L^2(\mathbb{T};\mathbb{R})}\, ds \\ &+ C \|F'\|_{L^\infty(\mathbb{T};\mathbb{R})} \bar{\kappa} \int_0^t (t-s)^{-\frac{1}{2}} \|u(s) - y(s)\|_{L^2(\mathbb{T};\mathbb{R})}\, ds \\ &+ \sup_{t\in[0,T]} \|W_A(t) - f_A(t)\|_{L^2(\mathbb{T};\mathbb{R})}, \end{aligned}$$

where the above inequality holds for all $t \in [0,T]$, $\omega \in S_\varepsilon$ and for all $\varepsilon \in (0,1)$. Now, by applying the generalized Gronwall's inequality we obtain that there exists a deterministic constant $C_{\bar{\kappa},T} \in \mathbb{R}_+$ such that

$$\|u(t) - y(t)\|_{L^2(\mathbb{T};\mathbb{R})} \le C_{\bar{\kappa},T} \sup_{t\in[0,T]} \|W_A(t) - f_A(t)\|_{L^2(\mathbb{T};\mathbb{R})}, \tag{5.22}$$

and the above inequality holds for all $t \in [0,T]$, $\omega \in S_\varepsilon$ and for all $\varepsilon \in (0,1)$. The proof is concluded by using (5.22) and noting that, since $\|W_A(t)\|_{H^1(\mathbb{T};\mathbb{R})} - \|f_A(t)\|_{H^1(\mathbb{T};\mathbb{R})} \le \|W_A(t) - f_A(t)\|_{H^1(\mathbb{T};\mathbb{R})}$ for all $t \in [0,T]$, the sets

$$S_\varepsilon^{'} := S_\varepsilon \cap \left\{\omega : \sup_{T\in[0,T]} \|W_A(t) - f_A(t)\|_{H^1(\mathbb{T};\mathbb{R})} < \varepsilon\right\}$$

$$S_\varepsilon^{''} := \left\{\omega : \sup_{T\in[0,T]} \|W_A(t) - f_A(t)\|_{H^1(\mathbb{T};\mathbb{R})} < \varepsilon\right\}$$

have the same probability, which we know to be strictly positive due to (5.15), i.e. $\mathbb{P}\left(S_\varepsilon^{'}\right) = \mathbb{P}\left(S_\varepsilon^{''}\right) > 0$ for all $\varepsilon \in (0,1)$. □

6. Proof of part ii) of Theorem 2.6: Strong Feller Property

From Proposition 4.4, we readily deduce that the semigroup $\{\mathcal{P}_t\}_{t\ge 0}$ associated with (2.2) (defined in (2.7)) is a Feller semigroup on $L^2(\mathbb{T};\mathbb{R})$. The purpose of this section is to prove that the semigroup $\mathcal{P}_t$ is strong Feller as well, i.e. to prove Theorem 6.1 below. We recall that throughout this section we work under Assumption 2.5; we explain where this assumption is used in Note 6.3 below and in the comments before Lemma 6.5.

The proof of Theorem 6.1 requires showing the Strong Feller property for a (class of) SPDE with Lipshitz non-linearity, see (6.3) and (6.4) below. This result is used in the proof of the main theorem and then proved in Subsection 6.1 - more comments on this matter can be found at the beginning of that subsection as well.

Theorem 6.1. *Let Assumption 2.5 hold. Then the semigroup $\{\mathcal{P}_t\}_{t\ge 0}$ generated by SPDE (2.2) is strong Feller in L^2, i.e. $\mathcal{P}_t$ maps $\mathcal{B}_b(L^2(\mathbb{T};\mathbb{R});\mathbb{R})$ into $C_b(L^2(\mathbb{T};\mathbb{R});\mathbb{R})$, for any $t > 0$.*

Proof of Theorem 6.1. The strategy is inspired by [14, Chap.14]. Let $\{\xi_R\}_{R>0} \subset C^\infty(\mathbb{R};\mathbb{R})$ be a family of smooth cutoff functions such that

$$\xi_R(x) = \begin{cases} 1, & |x| \le R, \\ 0, & |x| \ge R+1. \end{cases} \tag{6.1}$$

We consider a 'damped' version of the SPDE (2.2) where we replace the non-linearity $(F' * u)u$ with the truncated operator

$$\mathcal{M}_R(u) := \left(F^{'} * u\right) u\, \xi_R\left(\|u\|^2_{L^2(\mathbb{T};\mathbb{R})}\right), \quad u \in L^2(\mathbb{T};\mathbb{R}). \tag{6.2}$$

Namely, we consider the family of SPDEs

$$\begin{cases} \partial_t u_R = A u_R + \partial_x\left[V^{'} u_R + \mathcal{M}_R(u_R)\right] + Q^{\frac{1}{2}}\partial_t W, & (0,T)\times\mathbb{T}, \\ u_R(t,0) = u_R(t,2\pi), & t\in[0,T], \\ u_R(0,x) = u_0(x), & x\in\mathbb{T}, \end{cases} \tag{6.3}$$

with $R > 0$. The nonlinear term $\mathcal{M}_R(u)$ is globally Lipschitz-continuous, as opposed to the non-linearity in the original system (2.2). Hence a standard application of the Banach fixed point theorem gives the well-posedness in mild sense (in L^2) of (6.3) for any initial datum $u_0 \in L^2$. Therefore we can define the (Feller) semigroup associated to (6.3), and we denote it by $\mathcal{P}_t^R$. Because the non-linearity in (6.3) is globally Lipschitz, it is easier to prove the Strong Feller property for $\mathcal{P}_t^R$ rather than for $\mathcal{P}_t$ directly. In particular, if we prove the following two facts

(i) for each $R > 0$, the semigroup $\{\mathcal{P}_t^R\}_{R>0}$ is Strong Feller in $L^2(\mathbb{T};\mathbb{R})$;
(ii) for any $t > 0$ and any $\psi \in \mathcal{B}_b\left(L^2(\mathbb{T};\mathbb{R});\mathbb{R}\right)$, $\mathcal{P}_t^R\psi$ converges to $\mathcal{P}_t\psi$, as $R \to +\infty$, locally uniformly in $L^2(\mathbb{T};\mathbb{R})$ (i.e. uniformly on any bounded subset $X \subset L^2(\mathbb{T};\mathbb{R})$);

then $\{\mathcal{P}_t\}_{t\geq 0}$ is Strong Feller in $L^2(\mathbb{T};\mathbb{R})$. Indeed, let $\{u_0^n\}_{n\in\mathbb{N}}, u_0 \subset L^2(\mathbb{T};\mathbb{R})$ be such that $u_0^n \xrightarrow{n\to+\infty} u_0$ in $L^2(\mathbb{T};\mathbb{R})$ and let X be a bounded subset of $L^2(\mathbb{T};\mathbb{R})$ such that $\{u_0^n\}_{n\in\mathbb{N}}, u_0 \subset X$. First, we write

$$\begin{aligned}
|(\mathcal{P}_t\psi)(u_0^n) - (\mathcal{P}_t\psi)(u_0)| &\leq \left|(\mathcal{P}_t\psi)(u_0^n) - \left(\mathcal{P}_t^R\psi\right)(u_0^n)\right| \\
&\quad + \left|(\mathcal{P}_t^R\psi)(u_0^n) - \left(\mathcal{P}_t^R\psi\right)(u_0)\right| \\
&\quad + \left|(\mathcal{P}_t^R\psi)(u_0) - (\mathcal{P}_t\psi)(u_0)\right| \\
&\leq 2\sup_{u_0\in X}\left|(\mathcal{P}_t\psi)(u_0) - \left(\mathcal{P}_t^R\psi\right)(u_0)\right| \\
&\quad + \left|(\mathcal{P}_t^R\psi)(u_0^n) - \left(\mathcal{P}_t^R\psi\right)(u_0)\right|.
\end{aligned}$$

Letting $n \to +\infty$, by (i) we obtain

$$\limsup_{n\to+\infty}|(\mathcal{P}_t\psi)(u_0^n) - (\mathcal{P}_t\psi)(u_0)| \leq 2\sup_{u_0\in X}\left|(\mathcal{P}_t\psi)(u_0) - \left(\mathcal{P}_t^R\psi\right)(u_0)\right|.$$

Hence, the conclusion follows by letting $R \to +\infty$ and using (ii).

Statement (i), i.e. the strong Feller property for $\{\mathcal{P}_t^R\}_{t\geq 0}$, is proved in Subsection 6.1 under Assumption 2.5. More precisely, (i) follows directly from the bound

$$\left|\left(\mathcal{P}_t^R\psi\right)(u_0) - \left(\mathcal{P}_t^R\psi\right)(v_0)\right|^2 \leq \frac{(1+t)^\gamma}{t^{1+\gamma}}e^{CL_R^2 t}\|u_0 - v_0\|^2_{L^2(\mathbb{T};\mathbb{R})}, \tag{6.4}$$

where $C > 0$ is some constant (independent of R) and L_R is the Lipschitz constant associated to $V' + \mathcal{M}_R$. The above bound is proved in Proposition 6.8.

To prove Statement (ii), i.e. to show that the following limit holds

$$\sup_{u_0\in X}\left|\mathcal{P}_t\psi(u_0) - \mathcal{P}_t^R\psi(u_0)\right| \to 0, \qquad \text{as } R \to +\infty, \tag{6.5}$$

for any given bounded set $X \subset L^2(\mathbb{T};\mathbb{R})$ and any fixed $t \in [0,T]$, we introduce the family of stopping times $\{\tau_{u_0}^R\}_{R>0}$ given by

$$\tau_{u_0}^R := \inf\{t \geq 0 \,:\, \|u(t,u_0)\|_{L^2(\mathbb{T};\mathbb{R})} \geq R\} \wedge T.$$

Let X be a bounded subset of $L^2(\mathbb{T};\mathbb{R})$ and fix $t \in [0,T]$. To prove (6.5), it is enough to show that

$$\sup_{u_0\in X}\mathbb{P}(\tau_{u_0}^R < t) \to 0, \quad \text{as } R \to +\infty. \tag{6.6}$$

Indeed, since $u(t;u_0)=u^R(t;u_0)$ for $t\in[0,\tau^R_{u_0}]$, we have

$$\begin{aligned}\mathcal{P}_t\psi(u_0)-\mathcal{P}^R_t\psi(u_0)=&\,\mathbb{E}\left(\left(\psi(u(t;u_0))-\psi(u^R(t;u_0)\right)1_{\{\tau^R_{u_0}>t\}}\right)\\&+\mathbb{E}\left(\left(\psi(u(t;u_0))-\psi(u^R(t;u_0)\right)1_{\{\tau^R_{u_0}<t\}}\right)\\=&\,\mathbb{E}\left(\left(\psi(u(t;u_0))-\psi(u^R(t;u_0)\right)1_{\{\tau^R_{u_0}<t\}}\right),\end{aligned}$$

and the RHS of the above can be bounded by

$$\left|\mathbb{E}\left(\left(\psi(u(t;u_0))-\psi(u^R(t;u_0)\right)1_{\{\tau^R_{u_0}<t\}}\right)\right|\le 2\|\psi\|_{L^\infty(L^2(\mathbb{T};\mathbb{R});\mathbb{R}))}\mathbb{P}(\tau^R_{u_0}<t).$$

Hence, to conclude, we need to show (6.6). By Proposition 4.4, we know that there exists an increasing a.s. continuous random function $t\in[0,T]\to C(t,X)\in\mathbb{R}_+$ such that

$$\|u(t;u_0)\|_{L^2(\mathbb{T};\mathbb{R})}\le C(t,X),\text{ for all } t\in[0,T] \text{ and } u_0\in X,\ \mathbb{P}\text{-a.s},$$

uniformly in $R>0$. From this we deduce that (6.6) holds and this concludes the proof. □

Let us now introduce some notation that will be needed in Subsection 6.1. Given two Banach spaces X and Y endowed with the norm $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively, we denote by $\mathfrak{L}(X;Y)$ the Banach space of linear bounded operators from X to Y endowed with the norm

$$\|J\|_{\mathfrak{L}(X;Y)}:=\sup_{\|a\|_X\le 1}\|Ja\|_Y,\quad \text{for any given } J\in\mathfrak{L}(X;Y).$$

We use the shorthand notation $\mathfrak{L}\left(L^2(\mathbb{T};\mathbb{R})\right)=\mathfrak{L}\left(L^2(\mathbb{T};\mathbb{R});L^2(\mathbb{T};\mathbb{R})\right)$ to denote the Banach space of linear bounded operators from $L^2(\mathbb{T};\mathbb{R})$ into itself. We further introduce the Banach spaces $\mathcal{C}^2_T$ defined as the set of adapted square-integrable processes with values in $C\left([0,T];L^2(\mathbb{T};\mathbb{R})\right)$. We recall that a map $J:X\to Y$ is *Fréchet differentiable* at $a_0\in X$ if there exists a bounded linear operator $D_aJ(a_0)\in\mathfrak{L}(X;Y)$ such that

$$\lim_{\|h\|_X\to 0}\frac{\|J(a_0+h)-J(a_0)-D_aJ(a_0)h\|_Y}{\|h\|_X}=0,$$

where $D_aJ(a_0)h$ denotes the operator $D_aJ(a_0)$ applied to h. Furthermore, if J is Fréchet differentiable at every point of X then we simply say that J is Fréchet differentiable in X. Let us clarify that, in what follows, while D_a denotes the Fréchet derivative with respect to an element a in some appropriate infinite dimensional space, ∂_x denotes the derivative with respect to $x\in\mathbb{T}$.

Note 6.2. See also [17, Example 3.2.4]. Consider the special case $X=H$ and $Y=\mathbb{R}$, where $(H,\langle\,,\,\rangle_H)$ is a Hilbert space, and let $J:H\to\mathbb{R}$ be a Fréchet differentiable map, with Fréchet derivative $D_aJ\in\mathfrak{L}(H;\mathbb{R})$. From the Riesz representation theorem $\mathfrak{L}(H;\mathbb{R})\simeq H$, hence there exists a unique point $\nabla_aJ(a)\in H$ such that

$$D_aJ(a)h=\langle\nabla_aJ(a),h\rangle_H,\quad \text{for all } h\in H.$$

Then, for any given $h\in H$, with slight abuse of notation we will write $\langle D_aJ(a),h\rangle_H$ instead of $D_aJ(a)h$.

6.1. **Strong Feller property for gradient form Lipschitz non-linearities.** In this subsection we consider SPDEs of the form

$$\begin{cases} \partial_t u = Au + \partial_x [\mathcal{F}(u)] + Q^{\frac{1}{2}} \partial_t W, \quad (0,T] \times \mathbb{T}, \\ u(t,0) = u(t,2\pi), \quad t \in [0,T], \\ u(0,x) = u_0(x), x \in \mathbb{T}, \end{cases} \tag{6.7}$$

where $\mathcal{F} \in C_b^2 \left(L^2(\mathbb{T};\mathbb{R}); L^2(\mathbb{T};\mathbb{R}) \right)$[10] and $T > 0$, and we show that the semigroup $\mathcal{P}_t^{\mathcal{F}}$ associated with the above evolution is Strong Feller (see Proposition 6.15). Observe that the SPDE (6.3) is a particular case of (6.7), when $\mathcal{F}(u) = V'u + \mathcal{M}_R(u)$, hence the results of this section imply the Strong Feller property for the semigroup $\mathcal{P}_t^R$. Moreover, note that since $\mathcal{F} \in C_b^2 \left(L^2(\mathbb{T};\mathbb{R}); L^2(\mathbb{T};\mathbb{R}) \right)$, the functions $\mathcal{F} : L^2(\mathbb{T};\mathbb{R}) \to L^2(\mathbb{T};\mathbb{R})$ and $D_u\mathcal{F} : L^2(\mathbb{T};\mathbb{R}) \to \mathfrak{L}(L^2(\mathbb{T};\mathbb{R}))$ are both globally Lipschitz continuous (see [17, Proposition 3.2.7]); so the non-linearity in (6.7) is the gradient of a globally Lipschitz continuous functional, hence the name of this subsection. We denote by $L_{\mathcal{F}}$ and $L_{D\mathcal{F}}$ the Lipschitz constants of $\mathcal{F}$ and $D_u\mathcal{F}$, respectively.

To show that the semigroup $\mathcal{P}_t^{\mathcal{F}}$ is strong Feller we adapt the methods in [7, Chap.4], which have been developed to prove smoothing properties of SPDEs with globally Lipschitz non-linearities. In our setting we can't apply the results of [7, Chap.4] directly, as the type of non-linearity in (6.3) is different from the one in [7]. Indeed, in [7, Chapter 4] the non-linearity is allowed to depend on x and u but not on $\partial_x u$, which is the case here. However, the general approach of [7, Chap.4] can still be adapted to our case. We outline the strategy to prove that $\mathcal{P}_t^{\mathcal{F}}$ is strong Feller in Note 6.3 below.

We recall that $u(t)$, $t \in [0,T]$, is called a mild solution to (6.7) if $u(t)$ is a continuous $L^2(\mathbb{T};\mathbb{R})$-valued stochastic process such that

$$u(t) = e^{tA}u_0 + P[\mathcal{F}(u)](t) + W_A(t), \quad \text{for all } t \in [0,T],\ \mathbb{P}\text{-a.s.},$$

where P is the operator defined in (4.3). We emphasize that only throughout this subsection we denote by $u(t)$ (or $u(t;u_0)$ to stress dependence on initial conditions) the solution to (6.7), rather than the solution to (2.2).

Consider the map $\mathcal{I} : L^2(\mathbb{T};\mathbb{R}) \times \mathcal{C}_T^2 \to \mathcal{C}_T^2$ defined as

$$\mathcal{I}(u_0,u)(t) := e^{tA}u_0 + P\left[\mathcal{F}(u)\right](t) + W_A(t), \quad t \in [0,T]. \tag{6.8}$$

Since $\mathcal{F}(u(s)) \in L^2(\mathbb{T};\mathbb{R})$ for all $s \in [0,T]$, $\mathbb{P}$-a.s., we can apply (4.5). Hence, in a similar manner of proof of Proposition 4.2, we obtain

$$\|\mathcal{I}(u_0,u)(t) - \mathcal{I}(u_0,v)(t)\|_{L^2(\mathbb{T};\mathbb{R})}^2 \le C^2 L_{\mathcal{F}}^2 T \sup_{t\in[0,T]} \|u-v\|_{L^2(\mathbb{T};\mathbb{R})}^2, \quad \mathbb{P}\text{-a.s.},$$

for some constant $C > 0$. By taking the supremum over $t \in [0,T]$ and then the expectation on both sides of the above, we have

$$\|\mathcal{I}(u_0,u) - \mathcal{I}(u_0,v)\|_{\mathcal{C}_T^2}^2 \le C^2 L_{\mathcal{F}}^2 T \|u-v\|_{\mathcal{C}_T^2}^2 ,$$

[10] We recall that $C_b^2 \left(L^2(\mathbb{T};\mathbb{R}); L^2(\mathbb{T};\mathbb{R}) \right)$ is the space consisting of twice Fréchet differentiable functions from $L^2(\mathbb{T};\mathbb{R})$ to $L^2(\mathbb{T};\mathbb{R})$ with continuous and bounded first and second Fréchet derivative.

from which (local and then global) in time well-posedness of (6.7) follows.

We can then define the semigroup $\{\mathcal{P}_t^{\mathcal{F}}\}_{t\geq 0}$ associated to SPDE (6.7), namely

$$\left(\mathcal{P}_t^{\mathcal{F}}\psi\right)(u_0) := \mathbb{E}\left(\psi(u(t;u_0))\right), \qquad \psi \in \mathcal{B}_b(L^2(\mathbb{T};\mathbb{R});\mathbb{R}), \tag{6.9}$$

where we recall that throughout this section $u(t)$ denotes the solution to (6.7).

Note 6.3. The strategy to show that $\{\mathcal{P}_t^{\mathcal{F}}\}_{t\geq 0}$ is Strong Feller is as follows. By definition, we want to show that $\mathcal{P}_t^{\mathcal{F}}\psi$ is continuous if ψ is bounded and measurable. We will in fact show that $\mathcal{P}_t^{\mathcal{F}}\psi$ is Lipschitz if ψ is bounded and measurable (see Proposition 6.8). To prove the Lipschitzianity of $\mathcal{P}_t^{\mathcal{F}}\psi$, we will find bounds on the Fréchet derivative $D_{u_0}(\mathcal{P}_t^{\mathcal{F}}\psi)(u_0)$ (see Proposition 6.7). In turn, in order to find such bounds we will use a Bismut-Elworthy-Li type of formula, which is a representation formula for $D_{u_0}(\mathcal{P}_t^{\mathcal{F}}\psi)(u_0)$, see Proposition 6.6. This representation formula is the reason why we impose condition (2.22) in Assumption 2.5. More comments on this before Lemma 6.5.

Lemma 6.4 (Fréchet differentiability of the solution)**.** *Let $u(t;u_0)$ denote the solution of* (6.7) *with initial datum u_0 and suppose Q satisfies* (2.3)-(2.4)*. Then the map $u_0 \in L^2(\mathbb{T};\mathbb{R}) \mapsto u(t;u_0) \in \mathcal{C}_T^2$ is Fréchet differentiable* [11] *and, for any $h \in L^2(\mathbb{T};\mathbb{R})$, the directional derivative $\eta_h := D_{u_0}u(t;u_0)h$ of u in the direction h satisfies the following bounds:*

$$\|\eta_h(t)\|^2_{L^2(\mathbb{T};\mathbb{R})} \leq \|h\|^2_{L^2(\mathbb{T};\mathbb{R})} e^{CL^2_{\mathcal{F}}t}, \tag{6.10}$$

and

$$\int_0^t \|\partial_x \eta_h(s)\|^2_{L^2(\mathbb{T};\mathbb{R})}\, ds \leq \|h\|^2_{L^2(\mathbb{T};\mathbb{R})} e^{CL^2_{\mathcal{F}}t}, \tag{6.11}$$

$\mathbb{P}$-a.s., for $t \geq 0$. As a consequence, the semigroup $\mathcal{P}_t^{\mathcal{F}}$ is Fréchet differentiable with respect to u_0 and the Fréchet derivative $D_{u_0}(\mathcal{P}_t^{\mathcal{F}}\psi)(u_0)$ satisfies the identity

$$\left\langle D_{u_0}\left(\mathcal{P}_t^{\mathcal{F}}\psi\right)(u_0), h\right\rangle_{L^2(\mathbb{T};\mathbb{R})} = \mathbb{E}\left(\left\langle D_{u_0}\psi(u(t;u_0)), \eta_h(t)\right\rangle_{L^2(\mathbb{T};\mathbb{R})}\right),$$

for every $\psi \in \mathcal{B}_b(L^2(\mathbb{T};\mathbb{R});\mathbb{R})$.

Proof. See Appendix B. □

We clarify that in the above statement $D_{u_0}u(t;u_0)h$ denotes the action of $D_{u_0}u(t;u_0) \in \mathfrak{L}(L^2(\mathbb{T};\mathbb{R}))$ on the element h of $L^2(\mathbb{T};\mathbb{R})$ and note that $\eta_h(t) = \eta_h(t,x)$, as $\eta_h(t) \in L^2(\mathbb{T};\mathbb{R})$ for every $t \geq 0$, but we omit the explicit dependence on $x \in \mathbb{T}$ in the notation, as customary. Since $\mathcal{P}_t^{\mathcal{F}}\psi : L^2(\mathbb{T};\mathbb{R}) \to \mathbb{R}$, in the last equality of Lemma 6.4 we have indicated the action of the Fréchet derivative of $\mathcal{P}_t^{\mathcal{F}}$ applied to a vector h with the scalar product of $L^2(\mathbb{T};\mathbb{R})$ - see Note 6.2.

Let us introduce the following stochastic process $Z_h(t)$, $t \geq 0$, defined as

$$Z_h(t) := \int_0^t \left\langle Q^{-\frac{1}{2}}\eta_h(s), dW(s)\right\rangle_{L^2(\mathbb{T};\mathbb{R})}, \; t \geq 0, \tag{6.12}$$

where $\eta_h = D_{u_0}u(t;u_0)h$ is as in the statement of Lemma 6.4. Thanks to Lemma 6.4 and Lemma 6.5 below, $Z_h(t)$, $t \geq 0$, is well-defined as long as (2.22) holds.

[11] Note that one can also prove that the map $u_0 \in L^2(\mathbb{T};\mathbb{R}) \to u(t;u_0) \in \mathcal{C}_T^2$ is twice Fréchet differentiable, but we don't need the second Fréchet derivative in what follows.

Lemma 6.5. *Let Assumption 2.5 hold and let f be any function in $H^1(\mathbb{T};\mathbb{R})$. Then there exists a constant $c>0$ such that*

$$\|Q^{-\frac{1}{2}}f\|_{L^2(\mathbb{T};\mathbb{R})} \le c\|f\|^{1-\gamma}_{L^2(\mathbb{T};\mathbb{R})}\|f\|^{\gamma}_{H^1(\mathbb{T};\mathbb{R})}\,. \tag{6.13}$$

Proof. Let $f_k := \langle f, e_k\rangle_{L^2(\mathbb{T};\mathbb{R})}$, $k\in\mathbb{Z}$, where $\{e_k\}_{k\in\mathbb{Z}}$ is the orthonormal basis defined in (2.1). From the assumption on Q and Parseval's identity, for any $\gamma\in\mathbb{R}$ we have

$$\|Q^{-\frac{1}{2}}f\|^2_{L^2(\mathbb{T};\mathbb{R})} = \sum_{k\in\mathbb{Z}}\lambda_k^{-2}f_k^2 = \sum_{k\in\mathbb{Z}}|f_k|^{2(1-\gamma)}\cdot\lambda_k^{-2}|f_k|^{2\gamma}\,. \tag{6.14}$$

Choosing $\gamma<1$, we can apply Hölder's inequality with $p=\frac{1}{1-\gamma}$ and $q=\frac{1}{\gamma}$, and obtain

$$\|Q^{-\frac{1}{2}}f\|^2_{L^2(\mathbb{T};\mathbb{R})} \le \left(\sum_{k\in\mathbb{Z}}|f_k|^2\right)^{1-\gamma}\left(\sum_{k\in\mathbb{Z}}\lambda_k^{-2/\gamma}|f_k|^2\right)^{\gamma} \le c\|f\|^{2(1-\gamma)}_{L^2(\mathbb{T};\mathbb{R})}\|f\|^{2\gamma}_{H^1(\mathbb{T};\mathbb{R})},$$

where the last inequality follows by the assumption on λ_k, provided $\gamma<1$. □

Proposition 6.6 (Bismut-Elworthy-Li formula)**.** *Let Assumption 2.5 hold, and let $\{\mathcal{P}_t^{\mathcal{F}}\}_{t\ge0}$ be the semigroup generated by the SPDE (6.7). Then, the Fréchet derivative of $\{\mathcal{P}_t^{\mathcal{F}}\}_{t\ge0}$ satisfies*

$$\langle D_{u_0}\left(\mathcal{P}_t\psi\right)(u_0), h\rangle_{L^2(\mathbb{T};\mathbb{R})} = \frac{1}{t}\mathbb{E}\left[\psi\left(u(t;u_0)\right)Z_h(t)\right],$$

for any $t>0$, $\psi\in C_b^2\left(L^2(\mathbb{T};\mathbb{R});\mathbb{R}\right)$, and any given $h\in L^2(\mathbb{T};\mathbb{R})$.

Sketch of the proof. This follows a standard argument (see e.g. [7, Proposition 4.4.3]), which we summarise for the reader's convenience. From Itô's formula, we have

$$\psi(u(t;u_0)) = \mathcal{P}_t^{\mathcal{F}}\psi(u_0) + \int_0^t \left\langle D_{u_0}\left(\mathcal{P}_{t-s}^{\mathcal{F}}\psi\right)(u(s;u_0)), Q^{\frac{1}{2}}dW(s)\right\rangle_{L^2(\mathbb{T};\mathbb{R})},$$

for all $t\ge0$, $\mathbb{P}$-a.s.[12] Multiplying both sides of the above equality by $Z_h(t)$ and taking the expectation, we obtain

$$\begin{aligned}
\mathbb{E}\left[\psi(u(t;u_0))Z_h(t)\right] &= \mathbb{E}\left(\int_0^t \left\langle D_{u_0}\left(\mathcal{P}_{t-s}^{\mathcal{F}}\psi\right)(u(s;u_0)), Q^{\frac{1}{2}}dW(s)\right\rangle_{L^2(\mathbb{T};\mathbb{R})} Z_h(t)\right)\\
&= \mathbb{E}\left(\int_0^t \left\langle Q^{\frac{1}{2}}D_{u_0}\left(\mathcal{P}_{t-s}^{\mathcal{F}}\psi\right)(u(s;u_0)), Q^{-\frac{1}{2}}\eta_h(s)\right\rangle_{L^2(\mathbb{T};\mathbb{R})} ds\right)\\
&= \mathbb{E}\left(\int_0^t \left\langle D_{u_0}\left(\mathcal{P}_{t-s}^{\mathcal{F}}\psi\right)(u(s;u_0)), \eta_h(s)\right\rangle_{L^2(\mathbb{T};\mathbb{R})} ds\right)\\
&= \mathbb{E}\left(\int_0^t \left\langle D_{u_0}\left[\left(\mathcal{P}_{t-s}^{\mathcal{F}}\psi\right)(u(s;u_0))\right], h\right\rangle_{L^2(\mathbb{T};\mathbb{R})} ds\right).
\end{aligned}$$

From Fubini-Tonelli's theorem and using the semigroup property, we then conclude

$$\begin{aligned}
\mathbb{E}\left[\psi(u(t;u_0))Z_h(t)\right] &= \left\langle D_{u_0}\int_0^t \mathbb{E}\left[(P_{t-s}^{\mathcal{F}}\psi)(u(s;u_0))\right]ds, h\right\rangle_{L^2(\mathbb{T};\mathbb{R})}\\
&= t\left\langle D_{u_0}\left(\mathcal{P}_t^{\mathcal{F}}\psi\right)(u_0), h\right\rangle_{L^2(\mathbb{T};\mathbb{R})}.
\end{aligned}$$

□

[12]The proof of the above is straightforward in finite dimension, see e.g. [11], but more delicate in infinite dimension, see [13, Lemma 4.1].

The next step toward the strong Feller property is to obtain an estimate of the $L^2(\mathbb{T};\mathbb{R})$-norm of the Fréchet derivative $D_{u_0}\mathcal{P}_t^{\mathcal{F}}\psi$, for $\psi \in C_b^2(L^2(\mathbb{T};\mathbb{R});\mathbb{R})$.

Proposition 6.7. *Let Assumption 2.5 hold. Then,*

$$\sup_{u_0\in L^2(\mathbb{T};\mathbb{R})} \left\| D_{u_0}\left(\mathcal{P}_t^{\mathcal{F}}\psi\right)(u_0)\right\|^2_{L^2(\mathbb{T};\mathbb{R})} \leq \frac{(1+t)^\gamma}{t^{1+\gamma}} e^{CL_{\mathcal{F}}^2 t}\|\psi\|^2_{L^\infty(L^2(\mathbb{T};\mathbb{R});\mathbb{R})},$$

for any $\psi \in C_b^2(L^2(\mathbb{T};\mathbb{R});\mathbb{R})$ and $t>0$.

Proof. Let $h\in L^2(\mathbb{T};\mathbb{R})$. From Bismut-Elworthy-Li formula (Proposition 6.6) and Itô's isometry we deduce

$$\begin{aligned}\left|\left\langle D_{u_0}\left(\mathcal{P}_t^{\mathcal{F}}\psi\right)(u_0), h\right\rangle_{L^2(\mathbb{T};\mathbb{R})}\right|^2 &\leq \frac{1}{t^2}\|\psi\|^2_{L^\infty(L^2(\mathbb{T};\mathbb{R});\mathbb{R})} \quad \mathbb{E}\,|Z(t)|^2\\ &= \frac{1}{t^2}\|\psi\|^2_{L^\infty(L^2(\mathbb{T};\mathbb{R});\mathbb{R})} \quad \mathbb{E}\int_0^t \left\|Q^{-\frac{1}{2}}\eta_h(s)\right\|^2_{L^2(\mathbb{T};\mathbb{R})} ds.\end{aligned}$$

Applying Lemma 6.5 and then Hölder's inequality with $p=\frac{1}{1-\gamma}$ and $p'=\frac{1}{\gamma}$, we can write

$$\begin{aligned}\int_0^t \left\|Q^{-\frac{1}{2}}\eta_h(s)\right\|^2_{L^2(\mathbb{T};\mathbb{R})} ds &\leq c\int_0^t \|\eta_h(s)\|^{2(1-\gamma)}_{L^2(\mathbb{T};\mathbb{R})}\|\eta_h(s)\|^{2\gamma}_{H^1(\mathbb{T};\mathbb{R})}\, ds\\ &\leq c\left(\int_0^t \|\eta_h(s)\|^2_{L^2(\mathbb{T};\mathbb{R})}\, ds\right)^{1-\gamma}\left(\int_0^t \|\eta_h(s)\|^2_{H^1(\mathbb{T};\mathbb{R})}\, ds\right)^{\gamma}\\ &\leq c\left(\|h\|^{2(1-\gamma)}_{L^2(\mathbb{T};\mathbb{R})} e^{CL_{\mathcal{F}}^2 t(1-\gamma)}\, t^{1-\gamma}\right)\cdot\left(\|h\|^{2\gamma}_{L^2(\mathbb{T};\mathbb{R})}(t+1)^\gamma e^{CL_{\mathcal{F}}^2 t\gamma}\right)\\ &= \|h\|^2_{L^2(\mathbb{T};\mathbb{R})}(1+t)^\gamma e^{CL_{\mathcal{F}}^2 t}\, t^{1-\gamma}, \quad \mathbb{P}\text{-a.s.},\end{aligned}$$

where the last inequality follows from the bounds (6.10)-(6.11). Combining the above bounds,

$$\left|\left\langle D_{u_0}\left(\mathcal{P}_t^{\mathcal{F}}\psi\right)(u_0), h\right\rangle_{L^2(\mathbb{T};\mathbb{R})}\right|^2 \leq \frac{(1+t)^\gamma}{t^{1+\gamma}} e^{CL_{\mathcal{F}}^2 t}\|\psi\|^2_{L^\infty(L^2(\mathbb{T};\mathbb{R});\mathbb{R})}\|h\|^2_{L^2(\mathbb{T};\mathbb{R})}.$$

Choosing $h = D_{u_0}\left(\mathcal{P}_t^{\mathcal{F}}\psi\right)(u_0)$ we obtain the statement. □

The strong Feller property of $\{\mathcal{P}_t^{\mathcal{F}}\}_{t\geq 0}$ is now a straightforward consequence of the mean value theorem.

Proposition 6.8. *Let Assumption 2.5 hold. If $\psi \in \mathcal{B}_b(L^2(\mathbb{T};\mathbb{R});\mathbb{R})$ then $\mathcal{P}_t^{\mathcal{F}}\psi$ is Lipschitz continuous (and hence Strong Feller) for every $t>0$ (as a function from L^2 to $\mathbb{R}$), i.e.*

$$\left|\left(\mathcal{P}_t^{\mathcal{F}}\psi\right)(u_0) - \left(\mathcal{P}_t^{\mathcal{F}}\psi\right)(v_0)\right|^2 \leq \frac{(1+t)^\gamma}{t^{1+\gamma}} e^{CL_{\mathcal{F}}^2 t}\|\psi\|_{L^\infty}\|u_0-v_0\|^2_{L^2(\mathbb{T};\mathbb{R})}, \tag{6.15}$$

for any $\psi\in\mathcal{B}_b(L^2(\mathbb{T};\mathbb{R});\mathbb{R}), u_0, v_0 \in L^2(\mathbb{T};\mathbb{R})$ and $t>0$. In particular, the semigroup $(P_t^R\psi)$ associated to (6.3) *is Lipschitz continuous and satisfies* (6.4)*, for every $R>0$.*

Proof. First, recall that if (6.15) holds for all $\psi \in C_b^2\left(L^2(\mathbb{T};\mathbb{R});\mathbb{R}\right)$, then it also holds for all $\psi\in\mathcal{B}_b\left(L^2(\mathbb{T};\mathbb{R});\mathbb{R}\right)$, see [39, Lemma 2.2]. Hence, it is enough to prove (6.15) for $\psi \in C_b^2\left(L^2(\mathbb{T};\mathbb{R});\mathbb{R}\right)$. From the (infinite dimensional version of the) mean value theorem, see

Proposition [17, Proposition 3.2.7], and from Proposition 6.7, we then have

$$\begin{aligned}\left|\mathcal{P}_t^{\mathcal{F}}\psi(u_0)-\mathcal{P}_t^{\mathcal{F}}\psi(v_0)\right|^2 &\leq \sup_{\bar{u}\in[u_0;v_0]}\left\|D_{u_0}\left(\mathcal{P}_t^{\mathcal{F}}\psi\right)(\bar{u})\right\|^2_{L^2(\mathbb{T};\mathbb{R})}\|u_0-v_0\|^2_{L^2(\mathbb{T};\mathbb{R})}\\ &\leq \frac{(1+t)^\gamma}{t^{1+\gamma}}e^{CL_{\mathcal{F}}^2 t}\|\psi\|^2_{L^\infty(L^2(\mathbb{T};\mathbb{R});\mathbb{R})}\|u_0-v_0\|^2_{L^2(\mathbb{T};\mathbb{R})},\end{aligned}$$

where in the above $[u_0;v_0]:=\{ru_0+(1-r)v_0 \,:\, r\in[0,1]\}$.

Finally, to show that, for any given $R>0$, the semigroup $(P_t^R\psi)$ generated by (6.3) is Lipschitz continuous, it is enough to note that $V'u+\mathcal{M}_R(u)\in C_b^2\left(L^2(\mathbb{T};\mathbb{R});\mathbb{R}\right)$. □

Appendix A. Proofs of Section 3

A.1. Characterization of stationary solutions.

Proof of Proposition 2.1. The approach we use is well established, at least since [18], so we only give a sketch.

We first prove that for any given $\mu\in\mathcal{P}_{ac}(\mathbb{T})$ the solution ρ_μ to the linear equation

$$0=\sigma\partial_{xx}\rho_\mu(x)+\partial_x\left[\left(V'(x)+(F'*\mu)(x)\right)\rho_\mu(x)\right] \tag{A.1}$$

is unique and it is given by

$$\rho_\mu(x)=\frac{1}{C_\sigma}e^{-\frac{1}{\sigma}\left(\int_0^x(V'(y)+F'*\mu(y)\,dy\right)},\quad x\in\mathbb{T}, \tag{A.2}$$

with C_σ normalisation constant. Indeed, since $V'\in C^\infty(\mathbb{T};\mathbb{R})$ and for any fixed $\mu\in\mathcal{P}_{ac}(\mathbb{T})$ the function $F'*\mu$ is smooth, i.e. $F'*\mu\in C^\infty(\mathbb{T};\mathbb{R})$ (see e.g. [41, Lemma 2.3., p.14] for further details), the (linear) operator

$$\mathcal{L}(\rho)(x):=\partial_{xx}\rho(x)+\partial_x\left[\left(V'(x)+(F'*\mu)(x)\right)\rho(x)\right],\quad x\in\mathbb{T},$$

is uniformly elliptic and with smooth coefficients, hence, any weak solution $\rho_\mu\in H^1(\mathbb{T};\mathbb{R})\cap\mathcal{P}_{ac}(\mathbb{T})$ to (A.1) is actually smooth, i.e. $\rho_\mu\in C^\infty(\mathbb{T};\mathbb{R})\cap\mathcal{P}_{ac}(\mathbb{T})$, so the derivatives can be intended in the classical sense. Equation (A.1) can be then solved explicitly:

$$\begin{aligned}&\partial_x[(V'(x)+F'*\mu(x))\rho_\mu(x)]+\sigma\partial_{xx}\rho_\mu(x)=0\\ &\partial_x[(V'(x)+F'*\mu(x))\rho_\mu(x)+\sigma\partial_x\rho_\mu(x)]=0\\ &(V'(x)+F'*\mu(x))\rho_\mu(x)+\sigma\partial_x\rho_\mu(x)=d\\ &\sigma\partial_x\rho_\mu(x)=d-(V'(x)+F'*\mu(x))\rho_\mu(x),\end{aligned}$$

for some constant $d\in\mathbb{R}$. Finally, from the variation of constants formula we deduce that ρ_μ has the following expression

$$\rho_\mu(x)=\left(c+\frac{d}{\sigma}\int_0^x e^{\frac{1}{\sigma}\left(\int_0^y(V'(z)+F'*\mu(z)\,dz\right)}dy\right)e^{-\frac{1}{\sigma}\left(\int_0^x(V'(y)+F'*\mu(y)\,dy\right)},\ x\in\mathbb{T},$$

where c and d are real constants to be determined later. From the periodicity of ρ_μ we know that $\rho_\mu(0) = \rho_\mu(2\pi)$ which gives

$$c = c + \frac{d}{\sigma}\int_0^{2\pi} e^{\frac{1}{\sigma}\left(\int_0^y (V'(z)+F'*\mu(z)\,dz\right)}\,dy,$$

where the above equality is a consequence of the periodicity of V and F; indeed

$$\begin{aligned}\int_0^{2\pi} \left(V'(y) + F' * \mu(y)\right)\,dy &= V(2\pi) - V(0) + F*\mu(2\pi) - F*\mu(0)\\ &= \int_{\mathbb{T}} F(2\pi - x)\mu(x)\,dx - F*\mu(0) = \int_{\mathbb{T}} F(-x)\mu(x)\,dx - F*\mu(0)\\ &= F*\mu(0) - F*\mu(0) = 0,\end{aligned}$$

therefore, we have $e^{-\frac{1}{\sigma}\left(\int_0^{2\pi}(V'(y)+F'*\mu(y)\,dy\right)} = 1$. Hence, $d\int_0^{2\pi} e^{\frac{1}{\sigma}\left(\int_0^y(V'(z)+F'*\mu(z)\,dz\right)}\,dy = 0$, which implies $d = 0$. The constant c is now determined by renormalization. We omit the rest of the argument and just recall that if we consider the map $K : \mathcal{P}_{ac}(\mathbb{T}) \to \mathcal{P}_{ac}(\mathbb{T})$ defined as $K(\mu) := \rho_\mu$ from the above we then have that a solution to the non-linear problem (2.9) must be of the form (2.10). □

A.2. **Step 4 of the proof of Theorem 2.2.** We restrict to the case $\sigma \geq \frac{1}{2}$, the approach adopted is similar to what we have done for $\bar{g}_\sigma$. Namely, let $\xi_\sigma : \mathbb{R} \to \mathbb{R}$ be the map defined as

$$\xi_\sigma(m) := \int_{\mathbb{T}} (\cos x - m)e^{-\frac{1}{\sigma}\cos(2x)+\frac{m}{\sigma}\cos x}\,dx, \quad m \in \mathbb{R}, \tag{A.3}$$

and note that $m \in \mathbb{R}$ is a fixed point of h_σ if and only if m is a zero of ξ_σ, i.e. $\xi_\sigma(m) = 0$. Let us also introduce the sequence $\{c_k\}_{k\in\mathbb{N}}$ defined as

$$c_k := \int_{\mathbb{T}} (\cos x)^k\, e^{-\frac{1}{\sigma}\cos(2x)}\,dx, \quad k \in \mathbb{N}. \tag{A.4}$$

Since $\cos x$, $x \in \mathbb{T}$, is an anti-symmetric function and $\cos(2x)$, $x \in \mathbb{T}$, is a symmetric function with respect to $x = \frac{\pi}{2}$ and $(\cos x)^2 < 1$ for all $x \in (0, 2\pi)$ we have

$$c_{2k+1} = 0, \quad \text{for all } k \in \mathbb{N}, \tag{A.5}$$

$$c_{2k+2} < c_{2k}, \quad \text{for all } k \in \mathbb{N}. \tag{A.6}$$

In Proposition 3.4 we provide a power series expansion of ξ_σ; similarly to what we have done in Section 3.2 this power expansion will then allow us to prove the following result.

Proposition A.1. *[42, cfr. Step 1, Theorem 2.1] The function $\xi_\sigma : \mathbb{R} \to \mathbb{R}$ admits the following series expansion*

$$\xi_\sigma(m) = \sum_{k\in\mathbb{N}} \frac{1}{(2k)!}\left(\frac{m}{\sigma}\right)^{2k+1} c_{2k}\iota_k(\sigma), \tag{A.7}$$

where $\{\iota_k(\sigma)\}_{k\in\mathbb{N}}$ is the sequence defined as

$$\iota_k(\sigma) := \frac{c_{2k+2}}{(2k+1)c_{2k}} - \sigma, \quad k \in \mathbb{N},\ \sigma > 0. \tag{A.8}$$

We omit the proof of the above result as it can be done with calculations similar to Lemma 3.4.

Theorem A.2. *If $\sigma \geq \frac{1}{2}$ then $\xi_\sigma : \mathbb{R} \to \mathbb{R}$ admits a unique zero which is $m = 0$. This implies that $m = 0$ is the unique fixed point of $h_\sigma : \mathbb{R} \to \mathbb{R}$ for $\sigma \geq \frac{1}{2}$.*

Proof. Let us note that $\frac{c_{2k+2}}{(2k+1)c_{2k}} \leq \frac{1}{3}$ for all $k \geq 1$. With this in mind, from (A.7) we can write

$$\xi_\sigma(m) = \left(\frac{m}{\sigma}\right)(c_2 - \sigma c_0) + \sum_{k=1}^{+\infty} \frac{1}{(2k)!}\left(\frac{m}{\sigma}\right)^{2k+1} c_{2k}\iota_k(\sigma).$$

By using the identity $\cos(2x) = 2\cos^2 x - 1$ the factor $c_2 - \sigma c_0$ can be rearranged into the following form:

$$\begin{aligned} c_2 - \sigma c_0 &= \int_{\mathbb{T}} \cos^2 x \, e^{-\frac{1}{\sigma}\cos(2x)}\, dx - \sigma \int_{\mathbb{T}} e^{-\frac{1}{\sigma}\cos(2x)}\, dx \\ &= \left(\frac{1}{2} - \sigma\right) \int_{\mathbb{T}} e^{-\frac{1}{\sigma}\cos(2x)}\, dx + \frac{1}{2}\int_{\mathbb{T}} \cos(2x) e^{-\frac{1}{\sigma}\cos(2x)}\, dx. \end{aligned} \tag{A.9}$$

Using the modified Bessel functions of first kind (defined in (3.2)) we can recast (A.9) into the following form

$$\begin{aligned} c_2 - \sigma c_0 &= \left(\frac{1}{2} - \sigma\right) I_0\left(-\frac{1}{\sigma}\right) + \frac{1}{2} I_1\left(-\frac{1}{\sigma}\right) \\ &= \left(\frac{1}{2} - \sigma\right) I_0\left(\frac{1}{\sigma}\right) - \frac{1}{2} I_1\left(\frac{1}{\sigma}\right) < \left(\frac{1}{2} - \sigma\right) I_0\left(\frac{1}{\sigma}\right). \end{aligned} \tag{A.10}$$

Recalling that the functions $I_0(z) > 0$, for all $z \in \mathbb{R}$ and $I_1(z) > 0$ for all $z > 0$, if $\sigma \geq \frac{1}{2}$ then $c_2 - \sigma c_0 < 0$ and, moreover, $\iota_k(\sigma) < 0$, for all $k \geq 1$. Hence, since all the coefficients of the power series expansion of ξ are strictly negative, we readily obtain that $\xi_\sigma(m) = 0$ if and only if $m = 0$. This concludes the proof. □

Proof of Lemma 3.7. We begin with proving formula (3.25). We want to apply Proposition 3.6 with $U(x) := \cos(2x)$, $x \in \mathbb{T}$, and $G = 0$. To this end, we have to look for the minimum points of the function U on the torus $\mathbb{T}$. Clearly, U admits two global minima $x_1 = \frac{\pi}{2}$ and $x_2 = \frac{3\pi}{2}$. Therefore, in order to apply Lemma 3.6 we split the integral into two parts. Namely,

$$\int_{\mathbb{T}} e^{-\frac{1}{\sigma}\cos(2x)}\, dx = \int_0^{\pi} e^{-\frac{1}{\sigma}\cos(2x)}\, dx + \int_{\pi}^{2\pi} e^{-\frac{1}{\sigma}\cos(2x)}\, dx.$$

Using the π-periodicity of $\cos(2x)$ we reduce to a single integral i.e.

$$\int_{\mathbb{T}} e^{-\frac{1}{\sigma}\cos(2x)}\, dx = 2\int_0^{\pi} e^{-\frac{1}{\sigma}\cos(2x)}\, dx = 2I.$$

On the interval $[0, \pi]$ the function U admits a unique global minimum at $x = \frac{\pi}{2}$. Hence, we can apply Lemma 3.6 to I and obtain the desired result

$$I = \sqrt{\frac{\pi\sigma}{2}} e^{\frac{1}{\sigma}} (1 + o(1)). \tag{A.11}$$

Formula (3.26) is obtained with a similar reasoning. □

Proof of Lemma 3.9. We want to apply Lemma 3.6 with $U(x) := \cos(2x)$ and $G(x) := -\cos x$ so, we consider the function

$$U_m(x) := \cos(2x) - m\cos x, \quad x \in \mathbb{T}, \quad m \in [0,1].$$

By direct calculation one can see $x_{1,m} = \arccos\left(\frac{m}{4}\right)$ and $x_{2,m} = 2\pi - \arccos\left(\frac{m}{4}\right)$ are points of global minimum for U_m as m ranges in $[0,1]$. If we now let $f \in C^3(\mathbb{T};\mathbb{R})$ then by applying Lemma 3.6 we obtain that the following asymptotic expansion holds

$$\int_{\mathbb{T}} f(x) e^{-\frac{1}{\sigma}(\cos(2x) - m\cos x)}\, dx = \sqrt{\frac{2\pi\sigma}{4 - \frac{m^2}{4}}}\, e^{\frac{1}{\sigma}\left(1 + \frac{m^2}{8}\right)} \Big(f(x_{1,m}) + f(x_{2,m}) \\ + \left(\gamma^f_{1,m} + \gamma^f_{2,m}\right)\sigma + o_m(\sigma)\Big),$$

where $\gamma_{1,m}, \gamma_{2,m}$ are constants defined as in (3.24). In our case of interest, since the second, third and fourth derivative of U_m at $x_{1,m}$ and $x_{2,m}$ are respectively

$$U^{(2)}_m(x_{1,m}) = U^{(2)}_m(x_{2,m}) = 4 - \frac{m^2}{4},$$
$$U^{(3)}_m(x_{1,m}) = -U^{(3)}_m(x_{2,m}) = 12m\sqrt{1 - \frac{m^2}{16}},$$
$$U^{(4)}_m(x_{1,m}) = U^{(4)}_m(x_{2,m}) = \frac{7}{4}m^2 - 16,$$

if we set $f = 1$ we have

$$\int_{\mathbb{T}} e^{-\frac{1}{\sigma}(\cos(2x) - m\cos x)}\, dx = \sqrt{\frac{2\pi\sigma}{4 - \frac{m^2}{4}}}\, e^{\frac{1}{\sigma}\left(1 + \frac{m^2}{8}\right)} \left(2 + (\gamma_{1,m} + \gamma_{2,m})\sigma + o_m(\sigma)\right).$$

Moreover, since $f^{'} = f^{''} = 0$ we have $\gamma_{1,m} = \gamma_{2,m}$ and by recalling that $\mathcal{U}_k = \frac{d^k}{dx^k}(\cos(2x) - m\sin x)|_{x = x_{1,m}}$ we obtain

$$\gamma_{1,m} = \frac{5\mathcal{U}_3^2}{24\mathcal{U}_2^3} - \frac{\mathcal{U}_4}{8\mathcal{U}_2^2} = \frac{c(m)}{2} + \frac{2}{(4 - \frac{m^2}{2})^2},$$

where $[0,1] \ni m \to c(m) \in \mathbb{R}$ is a continuous function such that $c(m) \to 0$ as $m \downarrow 0$.
The expansions (3.33) and (3.34) are obtained analogously from (3.23) by using $f(x) = \cos x$ and $f(x) = \cos^2 x$, respectively. For the reader who would like to check the details we point out that if $f(x) = \cos x$ then since $f^{'}(x) = \sin x$ it follows that

$$f^{'}(x_{1,m}) = -f^{'}(x_{2,m}), \;\; f^{'}(x_{1,m})U^{'''}(x_{1,m}) = f^{'}(x_{2,m})U^{'''}(x_{2,m}), \tag{A.12}$$

hence, we obtain $\gamma^{(\cos)}_{1,m} = \gamma^{(\cos)}_{2,m}$ and

$$\gamma^{(\cos)}_{1,m} = f(x_{1,m})\left(\frac{5\mathcal{U}_3^2}{24\mathcal{U}_2^3} - \frac{\mathcal{U}_4}{8\mathcal{U}_2^2}\right) - f^{'}(x_{1,m})\frac{\mathcal{U}_3}{2\mathcal{U}_2^2} + \frac{f^{''}(x_{1,m})}{2\mathcal{U}_2} = \frac{\bar{c}(m)}{2},$$

where $[0,1] \ni m \to \bar{c}(m) \in \mathbb{R}$ is a continuous function such that $\bar{c}(m) \to 0$ as $m \downarrow 0$. If we set $f(x) = \cos^2 x$ then

$$f^{'}(x) = -2\cos x \sin x, \;\; f^{'}(x_{1,m})U^{'''}(x_{1,m}) = f^{'}(x_{2,m})U^{'''}(x_{2,m}), \tag{A.13}$$

hence, we obtain $\gamma_{1,m}^{(\cos^2)} = \gamma_{2,m}^{(\cos^2)}$ and

$$\gamma_{1,m}^{(\cos^2)} = f(x_{1,m})\left(\frac{5\mathcal{U}_3^2}{24\mathcal{U}_2^3} - \frac{\mathcal{U}_4}{8\mathcal{U}_2^2}\right) - f^{'}(x_{1,m})\frac{\mathcal{U}_3}{2\mathcal{U}_2^2} + \frac{f^{''}(x_{1,m})}{2\mathcal{U}_2} = \frac{\hat{c}(m)}{2} + \frac{1}{4-\frac{m^2}{4}}.$$

where $[0,1] \ni m \to \hat{c}(m) \in \mathbb{R}$ is a continuous function such that $\hat{c}(m) \to 0$ as $m \downarrow 0$. □

Appendix B. Proofs of Section 4 to Section 6

We recall that if G_t, $t > 0$, is the heat kernel on $\mathbb{R}$, i.e.

$$G_t(x) := \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}, \qquad x \in \mathbb{R},\, t > 0,$$

the periodic heat kernel G_t^{per} is defined as

$$G_t^{per}(x) := \sum_{k\in\mathbb{Z}} G_t(x + 2k\pi), \qquad x \in \mathbb{R},\, t > 0. \tag{B.1}$$

Clearly, $G_t^{per} \in C^\infty((0,+\infty)\times\mathbb{T})$ and has L^1-norm equal to 1.

We also recall that, since

$$|\partial_x G_t(x)| \le \frac{C}{\sqrt{t}} G_{2t}(x), \tag{B.2}$$

one has

$$|\partial_x G_t^{per}(x)| \le \frac{C}{\sqrt{t}} G_{2t}^{per}(x), \tag{B.3}$$

for all $x \in \mathbb{R}$, $t > 0$, with $C = \sqrt{2}e^{-\frac{1}{2}}$. Hence,

$$\|\partial_x G_t^{per}\|_{L^1(\mathbb{T};\mathbb{R})} \le \frac{C_1}{\sqrt{t}}, \tag{B.4}$$

for some constant $C_1 > 0$. We briefly recall that (B.2) follows from writing

$$|\partial_x G_t(x)| = h_t(x) e^{-\frac{x^2}{8t}},$$

where $x \in \mathbb{R} \mapsto h_t(x) \in \mathbb{R}$ is the even function defined as $h_t(x) := \frac{|x|}{4\sqrt{\pi t^3}} e^{-\frac{x^2}{8t}}, \quad x \in \mathbb{R},\ t > 0.$ The maximum of h_t is attained at $x = \pm 2\sqrt{t}$, from which (B.2) follows. The bounds (B.3) and (B.4) are then obvious.

Proof of Lemma 4.1. We begin with showing that the bounded linear operator $P : C\left([0,T]; H^1(\mathbb{T};\mathbb{R})\right) \to C\left([0,T]; L^2(\mathbb{T};\mathbb{R})\right)$ defined by (4.3) can be extended to a bounded linear operator from $C\left([0,T]; L^2(\mathbb{T};\mathbb{R})\right)$ into $C\left([0,T]; L^2(\mathbb{T};\mathbb{R})\right)$ satisfying (4.5). To this end, first consider $z \in C\left([0,T]; H^1(\mathbb{T};\mathbb{R})\right)$ and $\psi \in L^2(\mathbb{T};\mathbb{R})$. In this case, we obtain

$$\begin{aligned} |\langle P[z](t), \psi\rangle| &= \left| \int_0^t \left\langle e^{(t-s)A}\partial_x z(s), \psi \right\rangle_{L^2(\mathbb{T};\mathbb{R})} ds \right| \\ &\le \int_0^t \left| \left\langle \partial_x z(s), e^{(t-s)A}\psi \right\rangle_{L^2(\mathbb{T};\mathbb{R})} \right| ds \\ &\le \int_0^t \|z(s)\|_{L^2(\mathbb{T};\mathbb{R})} \left\| \partial_x \left(e^{(t-s)A}\psi \right) \right\|_{L^2(\mathbb{T};\mathbb{R})} ds. \end{aligned} \tag{B.5}$$

From Young's inequality for convolutions and (B.4), for any $\psi \in L^2(\mathbb{T};\mathbb{R})$ the heat semigroup satisfies

$$\left\|\partial_x\left(e^{tA}\psi\right)\right\|_{L^2(\mathbb{T};\mathbb{R})} \leq \|\partial_x G_t^{per}\|_{L^1(\mathbb{T};\mathbb{R})}\,\|\psi\|_{L^2(\mathbb{T};\mathbb{R})} \leq \frac{C_1}{t^{\frac{1}{2}}}\|\psi\|_{L^2(\mathbb{T};\mathbb{R})},$$

for all $t>0$. Thus,

$$\int_0^t \|z(s)\|_{L^2(\mathbb{T};\mathbb{R})}\left\|\partial_x\left(e^{(t-s)A}\psi\right)\right\|_{L^2(\mathbb{T};\mathbb{R})} ds \leq C_1\|\psi\|_{L^2(\mathbb{T};\mathbb{R})}\int_0^t (t-s)^{-\frac{1}{2}}\|z(s)\|_{L^2(\mathbb{T};\mathbb{R})}\,ds,$$

for all $\psi \in L^2(\mathbb{T};\mathbb{R})$ and $t\in[0,T]$. Setting $\psi = P[z](t)$, we have

$$\|P[z](t)\|_{L^2(\mathbb{T};\mathbb{R})} \leq C_1\int_0^t (t-s)^{-\frac{1}{2}}\|z(s)\|_{L^2(\mathbb{T};\mathbb{R})}\,ds \tag{B.6}$$

$$\leq C_1 T^{\frac{1}{2}}\,\|z\|_{C([0,T];L^2(\mathbb{T};\mathbb{R}))}, \tag{B.7}$$

for all $t\in[0,T]$. Therefore, inequality (4.5) holds for all $z\in C\left([0,T];H^1(\mathbb{T};\mathbb{R})\right)$.

Now, let $z\in C\left([0,T];L^2(\mathbb{T};\mathbb{R})\right)$ and let $\{z_n\}_{n\in\mathbb{N}}\subset C\left([0,T];H^1(\mathbb{T};\mathbb{R})\right)$ be such that $z_n\to z$ in $C\left([0,T];L^2(\mathbb{T};\mathbb{R})\right)$ as $n\to+\infty$. Then from (B.6) we know that

$$\|P[z_n](t)\|_{L^2(\mathbb{T};\mathbb{R})} \leq C_1\int_0^t (t-s)^{-\frac{1}{2}}\|z_n(s)\|_{L^2(\mathbb{T};\mathbb{R})}\,ds,$$

for all $n\in\mathbb{N}$ and $t\in[0,T]$. Moreover, from the linearity of P and inequality (B.7) we obtain that $\{P[z_n]\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C\left([0,T];L^2(\mathbb{T};\mathbb{R})\right)$. Therefore, if we define $P[z]$ as the limit in $C([0,T];L^2(\mathbb{T};\mathbb{R}))$ of $P[z_n]$ as $n\to+\infty$ (due to (B.7) such limit is independent from the choice of the approximating sequence $\{z_n\}_{n\in\mathbb{N}}$) then we obtain that inequality (4.5) holds for all $z\in C\left([0,T];L^2(\mathbb{T};\mathbb{R})\right)$ and for any $t\in[0,T]$. □

We now prove global well-posedness and L^2 bounds for the solution of the PDE (4.10) used in the proof of Proposition 4.4. We denote by $W^{1,\infty}(\mathbb{T};\mathbb{R})$ the Sobolev space of bounded functions with weak derivative in $L^\infty(\mathbb{T};\mathbb{R})$, endowed with the norm

$$\|f\|_{W^{1,\infty}(\mathbb{T};\mathbb{R})} = \|f\|_{L^\infty(\mathbb{T};\mathbb{R})} + \|\partial_x f\|_{L^\infty(\mathbb{T};\mathbb{R})}, \quad \text{for all } f\in W^{1,\infty}(\mathbb{T};\mathbb{R}).$$

Proposition B.1. *For any $T>0$ (independent of ω) the random PDE* (4.10) *admits a $\mathbb{P}$-a.s. continuous $L^2(\mathbb{T};\mathbb{R})$-valued mild solution v on the interval $[0,T]$; moreover such a solution is a classical solution and satisfies the a priori estimates* (4.11).

Proof. The local existence of a $\mathbb{P}$-a.s. continuous $L^2(\mathbb{T};\mathbb{R})$-valued mild solution can be proven exactly as in Proposition 4.2 with φ in place of W_A. We denote such a solution by v. We also note that, due to the smoothing properties of A, v is a smooth solution $\mathbb{P}$-a.s. as long as v does not blow up, since the coefficients V,F and the external forcing term φ are smooth. In other words, v is a classic solution to (4.10) defined up to a time $T^*=T^*(\omega)>0$ small enough. In the same fashion of proof of Theorem 2.4, to prove the global existence of v, it is enough to show that if v is a solution up to time $\tilde{T}>0$, then the estimate (4.11) is satisfied for all $t\in[0,\tilde{T}]$.

To ease the presentation, we denote $v_\varphi := v+\varphi$ and we omit the dependence on time t for v and φ, i.e. we write v and φ in place of $v(t)$ and $\varphi(t)$, respectively. In what follows, if not further specified, C denotes a generic deterministic positive constant, the value of which may change from line to line. Because of the non-linear term, to estimate the L^2-norm we need to start by estimating the L^1-norm. To do so, we analyse the derivative $\frac{d}{dt}\|v\|_{L^1(\mathbb{T};\mathbb{R})}$. To differentiate the $L^1(\mathbb{T};\mathbb{R})$-norm of v, since the function $x \to |x|$ is not smooth, we need to approximate it with a family of regular functions. To this end, let us consider the following convex $C^2(\mathbb{R};\mathbb{R})$-approximation $\{\chi_\varepsilon\}_{\varepsilon>0}$ of the absolute value

$$\chi_\varepsilon(r) = \begin{cases} |r|, & |r| > \varepsilon, \\ -\frac{r^4}{8\varepsilon^3} + \frac{3r^2}{4\varepsilon} + \frac{3\varepsilon}{8}, & |r| \le \varepsilon. \end{cases} \tag{B.8}$$

Then, we have

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{T}} \chi_\varepsilon(v)\,dx &= \int_{\mathbb{T}} \chi_\varepsilon^{'}(v)\left\{\partial_{xx}v + \partial_x\left[V^{'}v_\varphi + (F^{'} * v_\varphi)v_\varphi\right]\right\}dx \\ &= \int_{\mathbb{T}} \chi_\varepsilon^{'}(v)\partial_x\left[V^{'}\varphi + (F^{'} * v_\varphi)\,\varphi\right]dx \\ &\quad - \int_{\mathbb{T}} \chi_\varepsilon^{''}(v)\partial_x v\left[\partial_x v + V^{'}v + (F^{'} * v_\varphi)\,v\right]dx, \quad \mathbb{P}-a.s. \end{aligned} \tag{B.9}$$

Since $|\chi_\varepsilon^{'}(r)| \le 1$ for all $r > 0$ and all $\varepsilon > 0$, the first addend on the RHS of (B.9) can be estimated by

$$\begin{aligned} &\int_{\mathbb{T}} \chi_\varepsilon^{'}(v)\partial_x\left[V^{'}\varphi + (F^{'} * v_\varphi)\,\varphi\right]dx \\ &\quad \le \int_{\mathbb{T}} |V^{'} + F^{'} * v_\varphi||\partial_x\varphi|\,dx + \int_{\mathbb{T}} |V^{''} + F^{''} * v_\varphi||\varphi|\,dx \\ &\quad \le \sqrt{2\pi}\|V^{'}\|_{W^{1,\infty}(\mathbb{T};\mathbb{R})}\|\varphi\|_{H^1(\mathbb{T};\mathbb{R})} + 2\pi\|F^{'}\|_{W^{1,\infty}(\mathbb{T};\mathbb{R})}\|\varphi\|^2_{H^1(\mathbb{T};\mathbb{R})} \\ &\qquad + \sqrt{2\pi}\|F^{'}\|_{W^{1,\infty}(\mathbb{T};\mathbb{R})}\|v\|_{L^1(\mathbb{T};\mathbb{R})}\|\varphi\|_{H^1(\mathbb{T};\mathbb{R})}, \quad \mathbb{P}-a.s.. \end{aligned} \tag{B.10}$$

As for the last addend on the RHS of (B.9), using the fact that $\chi_\varepsilon^{''} \ge 0$ and applying Young's inequality, we have

$$\begin{aligned} &-\int_{\mathbb{T}} \chi_\varepsilon^{''}(v)\partial_x v\left[\partial_x v + V^{'}v + (F^{'} * v_\varphi)\,v\right]dx \\ &\quad \le -\int_{\mathbb{T}} \chi_\varepsilon^{''}(v)|\partial_x v|^2\,dx + \frac{1}{2}\int_{\mathbb{T}} \chi_\varepsilon^{''}(v)|\partial_x v|^2\,dx + \frac{1}{2}\int_{\mathbb{T}} \chi_\varepsilon^{''}(v)|V^{'}|^2|v|^2\,dx \\ &\qquad + \frac{1}{2}\int_{\mathbb{T}} \chi_\varepsilon^{''}(v)|\partial_x v|^2\,dx + \frac{1}{2}\int_{\mathbb{T}} \chi_\varepsilon^{''}(v)|F^{'} * v_\varphi|^2|v|^2\,dx \\ &\quad \le \frac{\|V^{'}\|^2_{L^\infty(\mathbb{T};\mathbb{R})}}{2}\int_{\mathbb{T}} \chi_\varepsilon^{''}(v)|v|^2\,dx + \frac{\|F^{'} * v_\varphi\|^2_{L^\infty(\mathbb{T};\mathbb{R})}}{2}\int_{\mathbb{T}} \chi_\varepsilon^{''}(v)|v|^2\,dx, \quad \mathbb{P}-a.s. \end{aligned}$$

By (B.8), we note that

$$\chi_\varepsilon^{''}(r) = \begin{cases} 0, & |r| > \varepsilon, \\ -\frac{3r^2}{2\varepsilon^3} + \frac{3}{2\varepsilon}, & |r| \le \varepsilon, \end{cases}$$

is a non-negative continuous function and $r \to \chi_\varepsilon^{''}(r)|r|^2$ is bounded from above by the constant $\frac{3\varepsilon}{2}$. Hence, we deduce

$$-\int_{\mathbb{T}} \chi_\varepsilon^{''}(v)\partial_x v \left[\partial_x v + V^{'} v + (F^{'} * v_\varphi)\, v\right] dx \\ \leq \frac{3\pi\|V^{'}\|^2_{L^\infty(\mathbb{T};\mathbb{R})}\varepsilon}{2} + 3\pi\|F^{'}\|^2_{L^\infty(\mathbb{T};\mathbb{R})}\varepsilon\left(\|v\|^2_{L^1(\mathbb{T};\mathbb{R})} + \|\varphi\|^2_{L^1(\mathbb{T};\mathbb{R})}\right), \quad \mathbb{P}-a.s. \tag{B.11}$$

Putting together (B.10), (B.11) and (B.9), we have

$$\frac{d}{dt}\int_{\mathbb{T}} \chi_\varepsilon(v)dx \leq \sqrt{2\pi}\|V^{'}\|_{W^{1,\infty}(\mathbb{T};\mathbb{R})}\|\varphi\|_{H^1(\mathbb{T};\mathbb{R})} + 2\pi\|F^{'}\|_{W^{1,\infty}(\mathbb{T};\mathbb{R})}\|\varphi\|^2_{H^1(\mathbb{T};\mathbb{R})} \\ + \sqrt{2\pi}\|F^{'}\|_{W^{1,\infty}(\mathbb{T};\mathbb{R})}\|v\|_{L^1(\mathbb{T};\mathbb{R})}\|\varphi\|_{H^1(\mathbb{T};\mathbb{R})} + \frac{3\pi\|V^{'}\|^2_{L^\infty(\mathbb{T};\mathbb{R})}\varepsilon}{2} \\ + 3\pi\|F^{'}\|^2_{L^\infty(\mathbb{T};\mathbb{R})}\varepsilon\left(\|v\|^2_{L^1(\mathbb{T};\mathbb{R})} + \|\varphi\|^2_{L^1(\mathbb{T};\mathbb{R})}\right), \quad \mathbb{P}-a.s.$$

Finally, integrating with respect to time, letting $\varepsilon \to 0$ and applying Gronwall's lemma, we obtain

$$\|v(t)\|_{L^1(\mathbb{T};\mathbb{R})} \leq \left(\|u_0\|_{L^1(\mathbb{T};\mathbb{R})} + C\int_0^t \left(1 + \|\varphi(s)\|^2_{H^1(\mathbb{T};\mathbb{R})}\right) ds\right) e^{C\int_0^t \|\varphi(s)\|_{H^1(\mathbb{T};\mathbb{R})}\, ds}, \tag{B.12}$$

for any $t \in [0,\tilde{T}]$, $\mathbb{P}$-a.s.

We can now estimate the $L^2(\mathbb{T};\mathbb{R})$-norm of v. To this end, by differentiating the $L^2(\mathbb{T};\mathbb{R})$-norm with respect to time and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2(\mathbb{T};\mathbb{R})} = -\int_{\mathbb{T}} |\partial_x v|^2\, dx - \int_{\mathbb{T}} V^{'} v_\varphi \partial_x v\, dx - \int_{\mathbb{T}} (F^{'} * v_\varphi)v_\varphi \partial_x v\, dx, \quad \mathbb{P}-a.s. \tag{B.13}$$

By Young's inequality and integration by parts, the second addend on the RHS of (B.13) can be estimated as

$$-\int_{\mathbb{T}} V^{'} v_\varphi \partial_x v\, dx = -\int_{\mathbb{T}} V^{'}(v+\varphi)\partial_x v = \frac{1}{2}\int_{\mathbb{T}} V^{''} v^2\, dx - \int_{\mathbb{T}} V^{'}\varphi\partial_x v\, dx \\ \leq \frac{\|V^{''}\|_{L^\infty(\mathbb{T};\mathbb{R})}}{2}\|v\|^2_{L^2(\mathbb{T};\mathbb{R})} + \|V^{'}\|^2_{L^\infty(\mathbb{T};\mathbb{R})}\|\varphi\|^2_{L^2(\mathbb{T};\mathbb{R})} + \frac{1}{4}\|\partial_x v\|^2_{L^2(\mathbb{T};\mathbb{R})}, \quad \mathbb{P}-a.s.$$

As for the third addend on the RHS of (B.13), we proceed similarly and we are going to use the $L^1(\mathbb{T};\mathbb{R})$-norm estimate of v obtained beforehand:

$$-\int_{\mathbb{T}} (F^{'} * v_\varphi)v_\varphi\partial_x v\, dx = -\int_{\mathbb{T}} (F^{'} * v)v\partial_x v\, dx - \int_{\mathbb{T}} (F^{'} * v)\varphi\partial_x v\, dx \\ -\int_{\mathbb{T}} (F^{'} * \varphi)v\partial_x v\, dx - \int_{\mathbb{T}} (F^{'} * \varphi)\varphi\partial_x v\, dx, \quad \mathbb{P}-a.s. \tag{B.14}$$

Since $F^{''} \in L^\infty(\mathbb{T};\mathbb{R})$, by Young's inequality for convolutions, the first two terms in (B.14) are bounded respectively by

$$-\int_{\mathbb{T}} (F^{'} * v)v\partial_x v\, dx = \frac{1}{2}\int_{\mathbb{T}} (F^{''} * v)|v|^2\, dx \leq \frac{1}{2}\|F^{''}\|_{L^\infty(\mathbb{T};\mathbb{R})}\|v\|_{L^1(\mathbb{T};\mathbb{R})}\|v\|^2_{L^2(\mathbb{T};\mathbb{R})}, \quad \mathbb{P}-a.s.,$$

and

$$-\int_{\mathbb{T}} (F^{'} * v)\varphi\partial_x v\, dx \leq \frac{1}{4}\|\partial_x v\|^2_{L^2(\mathbb{T};\mathbb{R})} + 2\pi\|F^{'}\|^2_{L^\infty(\mathbb{T};\mathbb{R})}\|v\|^2_{L^2(\mathbb{T};\mathbb{R})}\|\varphi\|^2_{L^2(\mathbb{T};\mathbb{R})}, \quad \mathbb{P}-a.s.$$

As for the latter two addends in (B.14), by similar arguments we have

$$\begin{aligned}\left|\int_{\mathbb{T}}(F^{'} * \varphi)v\partial_x v\,dx\right| &\le \sqrt{2\pi}\|F^{'}\|_{L^\infty(\mathbb{T};\mathbb{R})}\|\varphi\|_{L^2(\mathbb{T};\mathbb{R})}\int_{\mathbb{T}}|v||\partial_x v|\,dx\\ &\le 2\pi\|F^{'}\|^2_{L^\infty(\mathbb{T};\mathbb{R})}\|v\|^2_{L^2(\mathbb{T};\mathbb{R})}\|\varphi\|^2_{L^2(\mathbb{T};\mathbb{R})} + \frac{1}{4}\|\partial_x v\|^2_{L^2(\mathbb{T};\mathbb{R})}, \quad \mathbb{P}-a.s.,\end{aligned}$$

and

$$\left|\int_{\mathbb{T}}(F^{'} * \varphi)\varphi\partial_x v\,dx\right| \le \frac{1}{4}\|\partial_x v\|^2_{L^2(\mathbb{T};\mathbb{R})} + 2\pi\|F^{'}\|^2_{L^\infty(\mathbb{T};\mathbb{R})}\|\varphi\|^4_{L^2(\mathbb{T};\mathbb{R})}, \quad \mathbb{P}-a.s.$$

Thus, by Gronwall's lemma and using (B.12) to estimate $\|v\|_{L^1(\mathbb{T};\mathbb{R})}$, we obtain the estimate (4.11) for any $t \in [0,\tilde{T}]$. In particular, from these estimates it follows that the solution v to (4.10) does not blow-up in $L^2(\mathbb{T};\mathbb{R})$ and, therefore, it can be extended up to time T $\mathbb{P}$-a.s. □

Proof of Lemma 5.4. Let $f \in L^2([0,T];L^2(\mathbb{T};\mathbb{R}))$. Writing the deterministic convolution f_A (5.13) in Fourier basis, i.e.

$$f_A(t) = \sum_{k\in\mathbb{Z}}\lambda_k\left(\int_0^t e^{-(t-s)k^2} f_k(s)\,ds\right)e_k, \quad t \in [0,T],$$

we can see that, for any $\psi \in C^\infty(\mathbb{T};\mathbb{R})$, the following holds

$$\begin{aligned}\langle f_A, \partial_x\psi\rangle_{L^2(\mathbb{T};\mathbb{R})} &= \sum_{k\in\mathbb{Z}}\lambda_k\int_0^t e^{-(t-s)k^2} f_k(s)\,ds\int_{\mathbb{T}} e_k\partial_x\psi\,dx\\ &= -\sum_{k\in\mathbb{Z}}|k|\lambda_k\int_0^t e^{-(t-s)k^2} f_k(s)\,ds\int_{\mathbb{T}} e_{-k}\psi\,dx,\end{aligned}$$

where we have used the identity $\partial_x e_k(x) = |k|\,e_{-k}(x)$, $k \in \mathbb{Z}$. From this, we can see that the weak derivative of f_A is given by (5.14). From (5.14), since Q is trace-class, it is easy to see that $\partial_x f_A(t)$ belongs to $L^2(\mathbb{T};\mathbb{R})$, for every $t > 0$.

In a similar fashion as for what we have done for f_A one can prove that the weak derivative of W_A (at this stage on a formal level only) is given by

$$\partial_x W_A(t,x) = \sum_{k\in\mathbb{Z}}|k|\lambda_k\left(\int_0^t e^{-(t-s)k^2}\,d\beta_s^k\right)e_{-k}(x), \tag{B.15}$$

dx-a.e. in $\mathbb{T}$ for all $t \in [0,T]$, $\mathbb{P}$-a.s. To prove that the above object is well-defined we intend to apply [12, Theorem 2.13]. As one can see, [12, Theorem 2.13] hinges on [12, Hypothesis 2.10]. In our context, the operators A and C of [12, Hypothesis 2.10] correspond to $A = \partial_{xx}$ and $C = (-A)^{\frac{1}{2}}\,Q$, respectively. It is clear that the heat semi-group $\left\{e^{tA}\right\}_{t\ge0}$ satisfies conditions (i) and (ii) in [12, Hypothesis 2.10] (see [12, Example 2.11]). Conditions (iii) and (iv) in [12, Hypothesis 2.10] are clearly satisfied since $Ae_k = -k^2e_k$, for every $k \in \mathbb{Z}$, and $|e_k(x)| \le \frac{1}{\sqrt{\pi}}$, for every $x \in \mathbb{T}$ and $k \in \mathbb{Z}$. As for condition (v) in [12, Hypothesis 2.10], from our choice of A and C it follows that condition (v) is equivalent to requiring (2.4); which is satisfied by assumption. Therefore, one can now apply [12, Theorem 2.13] and obtain $\partial_x W_A \in L^{2m}\left(\Omega; C([0,T];L^2(\mathbb{T};\mathbb{R}))\right)$ for $m > \frac{1}{2\delta}$ where δ coincides with the one in (2.4).

From the above and the fact $W_A \in C\left([0,T]; L^2(\mathbb{T};\mathbb{R})\right)$, $\mathbb{P}$-a.s. (see [12, Theorem 2.9] applied with $A = \partial_{xx}$ and $C = Q$) we readily obtain that W_A belongs to $C\left([0,T]; H^1(\mathbb{T};\mathbb{R})\right)$, $\mathbb{P}$-a.s. □

Proof of Lemma 6.4. The part of the statement which is lengthiest to prove is the differentiability of the solution $u(t;u_0)$ of (6.7) with respect to the initial datum u_0. To do so one starts by considering the so-called *first variation* equation, namely the equation

$$\begin{cases} \partial_t z(t) = Az(t) + \partial_x[(D\mathcal{F})\,(u(t;u_0))\,z(t)], & t \in (0,T], \\ z(0) = h, \end{cases} \tag{B.16}$$

for the unknown $z(t) \in L^2(\mathbb{T};\mathbb{R})$. We clarify that in the above $(D\mathcal{F})\,(u(t;u_0))$ denotes the Fréchet derivative of $\mathcal{F} = \mathcal{F}(u)$ (with respect to u), calculated at the point $u(t;u_0)$. At this point there are (at least) two possible approaches. One approach, which is the one taken in [7, Chapter 4], is to observe that the solution $u(t;u_0)$ is a fixed point of the map $\mathcal{I}$ defined in (6.8) and then apply standard results that allow one to deduce differentiability of the fixed point from the regularity properties of the fixed point map ($\mathcal{I}$, in our case), see [7, Appendix C]. Once the desired differentiability of u is obtained, one observes that $\eta_h = D_{u_0}u(t;u_0)h$ needs to satisfy the first variation equation; from this observation, the estimates (6.10) and (6.11) are easily obtained (as we will explain below). This approach is lengthy but it works in general circumstances. Applied to our case, it allows one to obtain that u is once Fréchet differentiable and (at least) twice Gateaux differentiable. We don't take this approach here to contain the length of the paper and because, strictly speaking, we only need one Fréchet derivative of the solution, but [30] will contain the details of how to use this approach in our case. The approach we take here is the one of [36, Theorem 2], namely: one first observes that the first variation equation admits a mild solution (by standard contraction mapping arguments). Using this fact, it is easy to show that the following inequality holds:

$$\|z(t)\|^2_{L^2(\mathbb{T};\mathbb{R})} + \int_0^t \|\partial_x z(s)\|^2_{L^2(\mathbb{T};\mathbb{R})}\,ds \le \|h\|^2_{L^2(\mathbb{T};\mathbb{R})} + CL^2_{\mathcal{F}} \int_0^t \|z(s)\|^2_{L^2(\mathbb{T};\mathbb{R})}\,ds, \quad \mathbb{P}-a.s.\,, \tag{B.17}$$

where $L_{\mathcal{F}}$ is the Lipschitz continuity constant of $\mathcal{F}$ and C is a positive (deterministic) constant, see e.g. [12, Lemma 5.8 and Prop. 5.9], from which (6.10) and (6.11) are then easily deduced. At this point one shows that there exist a constant $C > 0$ and a function $\nu_T(h) : L^2 \to \mathbb{R}_+$ such that $\nu_T(h) \to 0$ as $\|h\|^2_L \to 0$ and

$$\|u(t;u_0+h) - u(t;u_0) - z(t)\|^2_L \le C\nu_T(h)\|h\|_{L^2}, \quad \mathbb{P}\ a.s.$$

Hence $z(t)$ coincides with the Fréchet derivative $D_{u_0}u(t;u_0)h$. The proof of the above follows the lines of [36, Theorem 2], with calculations similar to those we have shown so far, so we don't repeat it here.

The differentiability of the semigroup $\mathcal{P}^{\mathcal{F}}_t$ now follows from the differentiability of $u(t;u_0)$. Indeed, as a result of the Banach fixed point theorem we know that the continuous dependence with respect to the initial datum holds, i.e. if $\{u_0^n\}_{n\in\mathbb{N}} \subset L^2(\mathbb{T};\mathbb{R})$ such that $u_0^n \to u_0$ as $n \to +\infty$ in $L^2(\mathbb{T};\mathbb{R})$ then $u(t;u_0^n) \to u(t;u_0)$ as $n \to +\infty$ in $L^2(\mathbb{T};\mathbb{R})$ for all fixed

$t \geq 0$. Consequently, if $\psi \in C_b(L^2(\mathbb{T};\mathbb{R});\mathbb{R})$ then from the dominated convergence theorem it follows that $\mathcal{P}_t^{\mathcal{F}}\psi \in C_b(L^2(\mathbb{T};\mathbb{R});\mathbb{R})$ for all $t > 0$. Hence, $\{\mathcal{P}_t^{\mathcal{F}}\}_{t\geq 0}$ is a Feller semigroup. Furthermore, if $\psi \in C_b^2\left(L^2(\mathbb{T};\mathbb{R});\mathbb{R}\right)$ then from the differentiation under the integral sign and the fact that $u(t;u_0)$, $t \geq 0$, is Fréchet differentiable in $L^2(\mathbb{T};\mathbb{R})$ we deduce that $\mathcal{P}_t^{\mathcal{F}}\psi$ is Fréchet differentiable in $L^2(\mathbb{T};\mathbb{R})$ as well (to be precise, since $u(t;u_0)$, $t \geq 0$ is Fréchet differentiable and twice Gâteaux differentiable in $L^2(\mathbb{T};\mathbb{R})$ we obtain that $\{\mathcal{P}_t^{\mathcal{F}}\}_{t\geq 0}$ is Fréchet differentiable and twice Gâteaux differentiable in $L^2(\mathbb{T};\mathbb{R})$).

□

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