SYSTEMS of ODEs

Why do we want/need to calculate exponentials of matrices?

• Consider the ODE $\begin{cases} y' = a\,y \\ y(0) = 2 \end{cases}$ $a \in \mathbb{R}\setminus\{0\}$

where $y(x)$ is just a scalar function.

$y' = ay \implies y(x) = C_1 e^{ax}$ (check: $y' = a C_1 e^{ax} = a\,y(x)$)

$y(0) = 2 \implies C_1 = 2 \implies y(x) = 2e^{ax}$.

• In general $\begin{cases} y' = a\,y \\ y(t_0) = y_0 \end{cases}$ (From now on we call t the independent variable)

$y(t) = C_1 e^{at}$ $y(t_0) = C_1 e^{at_0} = y_0 \implies C_1 = e^{-at_0} y_0$

$\implies y(t) = e^{a(t-t_0)} y_0$.

• Suppose I want to solve the system of ODEs

$$(*)\begin{cases} y_1'(t) = 2y_1(t) + y_2(t) \\ y_2'(t) = y_1(t) + 2y_2(t) \end{cases}$$

I can rewrite this system in the following way:

Let $y = \begin{vmatrix} y_1(t) \\ y_2(t) \end{vmatrix}$ and denote $y' = \begin{vmatrix} y_1'(t) \\ y_2'(t) \end{vmatrix}$.

Let A be the matrix $A = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$.

REM: notice that $y = y(t)$, I mean that y is a vector but the value of its components is not constant, it depends on t. In other words $y(t)$ is a vector of functions.

We can write system (*) as

$$y' = Ay.$$

Giving initial conditions means, in this case, giving conditions like $y_1(0) = 2$ and $y_2(0) = 3$.

In other words I give the vector $y(0) = \begin{vmatrix} y_1(0) \\ y_2(0) \end{vmatrix} = \begin{vmatrix} 2 \\ 3 \end{vmatrix}$, i.e. a vector of initial conditions.

<u>Thm</u>: the solution of $\begin{cases} y' = Ay \\ y(t_0) = y_0 \end{cases}$, $y \in \mathbb{R}^n$, $y_0 \in \mathbb{R}^n$, A $n \times n$ matrix with constant coeff.

is $y(t) = e^{A(t-t_0)} y_0$.

This can be rephrased as: $\boxed{\text{The general sol is } y = e^{At} D \text{ where } D = \begin{vmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{vmatrix} \text{ is a generic vector of constants to be determined through initial conditions}}$

<u>Example</u>: Solve the system (find general sol)

$$\begin{cases} y_1' = 2y_1 \\ y_2' = 2y_2 \\ y_3' = -y_1 + 3y_3 \end{cases}$$

The system can be rewritten as $y' = Ay$ where $A = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix}$ and $y = \begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix}$, $y \in \mathbb{R}^3$.

The gen sol is $y = e^{At} D$, $D = \begin{vmatrix} d_1 \\ d_2 \\ d_3 \end{vmatrix}$.

We need to calculate e^{At}.

A is not diagonal, let's see if it is diagonalizable

$$\det(A - \lambda I) = (2-\lambda)^2 (3-\lambda) = 0 \iff \lambda = 2 \text{ or } \lambda = 3$$

$$\lambda = 3: \quad (A - 3I)v = 0 \implies \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{vmatrix} \begin{vmatrix} v_1 \\ v_2 \\ v_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

$$\Rightarrow \begin{cases} -v_1 = 0 \\ -v_2 = 0 \\ -v_1 = 0 \end{cases} \Rightarrow v = (0, 0, 1) \quad (\text{when you choose } v_3 = 1)$$

$$\lambda = 2: \quad (A - 2I)\,w = 0 \Rightarrow \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{vmatrix} \begin{vmatrix} w_1 \\ w_2 \\ w_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix} \Rightarrow \begin{cases} 0w_1 + 0w_2 + 0w_3 = 0 \\ -w_1 + w_3 = 0 \end{cases}$$

$\Rightarrow$ $w_1 = w_3$ and w_2 can be anything.

So choose $\begin{cases} w_2 = 0 \\ w_1 = w_3 = 1 \end{cases} \Rightarrow w = (1, 0, 1)$

choose $\begin{cases} w_2 = 1 \\ w_1 = w_3 = 1 \end{cases} \Rightarrow u = (1, 1, 1)$

notice that if you first choose $\begin{matrix} w_2 = 0 \\ w_1 = w_3 = 1 \end{matrix} \Rightarrow w = (1, 0, 1)$

if then you choose $w_2 = 0$, $w_1 = w_3 = 3 \Rightarrow \tilde{w} = (3, 0, 3)$

$\tilde{w} = 3w$ so they are "the same eigenvector" - which is why the next try is with $w_2 = 1$

However, we have three eigenv. so we can construct

$$C = \begin{vmatrix} | & | & | \\ v & w & u \\ | & | & | \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} \quad \det C = 1 \neq 0 \Rightarrow C \text{ invertible}$$

$$C^{-1} = \begin{vmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{vmatrix} ; \quad C^{-1} A C = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = \Delta$$

A is diagonalizable.

$$A = C \Delta C^{-1} \qquad At = C(\Delta t)C^{-1}, \text{ indeed}$$

$$At = \begin{vmatrix} 2t & 0 & 0 \\ 0 & 2t & 0 \\ -t & 0 & 3t \end{vmatrix} = C \begin{vmatrix} 3t & 0 & 0 \\ 0 & 2t & 0 \\ 0 & 0 & 2t \end{vmatrix} C^{-1} -$$

so $e^{At} = C\, e^{\Delta t}\, C^{-1}$.

Δt is clearly still diagonal so $e^{\Delta t} = \begin{vmatrix} e^{3t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{vmatrix}$ and

$$e^{At} = C \begin{vmatrix} e^{3t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{vmatrix} \begin{vmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} \begin{vmatrix} -e^{3t} & 0 & e^{3t} \\ e^{2t} & -e^{2t} & 0 \\ 0 & e^{2t} & 0 \end{vmatrix}$$

$$\Rightarrow e^{At} = \begin{vmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ e^{2t}-e^{3t} & 0 & e^{3t} \end{vmatrix}$$

So general solution is

$$y = e^{At} D = \begin{vmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ e^{2t}-e^{3t} & 0 & e^{3t} \end{vmatrix} \begin{vmatrix} d_1 \\ d_2 \\ d_3 \end{vmatrix} = \begin{vmatrix} e^{2t} d_1 \\ e^{2t} d_2 \\ (e^{2t}-e^{3t}) d_1 + e^{3t} d_3 \end{vmatrix} =$$

$$= \begin{vmatrix} e^{2t} d_1 \\ e^{2t} d_2 \\ e^{2t} d_1 + \tilde{d}_1 e^{3t} \end{vmatrix}$$ where I called $d_3 - d_1 = \tilde{d}_1$, still a generic constant

in other words

$$\begin{cases} y_1 = e^{2t} d_1 \\ y_2 = e^{2t} d_2 \\ y_3 = e^{2t} d_1 + \tilde{d}_1 e^{3t} \end{cases} \leftarrow \text{general solution (with three generic constants)}$$

If I want the solution with $y(1) = \underbrace{\overset{(y_1(1),\, y_2(1),\, y_3(1))}{(e^2, 1, 0)}}_{\text{"}y_0\text{"}}$

~~then I know that the solution is~~

$$\cancel{y(t) = e^{A(t-1)} \cdot \begin{vmatrix} e^2 \\ 1 \\ 0 \end{vmatrix} = e^{At} \cdot e^{-A} \cdot \begin{vmatrix} e^2 \\ 1 \\ 0 \end{vmatrix}}$$

~~we know who this is~~

~~it is the inverse of e^A, but we don't really~~ need to calculate it.

$y(1) = \begin{vmatrix} e^2 \\ 1 \\ 0 \end{vmatrix}$ means $y_1(1) = e^2$, $y_2(1) = 1$, $y_3(1) = 0$

$y_1 = e^{2t} d_1$ so $y_1(1) = e^2 \Rightarrow e^2 d_1 = e^2 \Rightarrow d_1 = 1$

$y_2 = e^{2t} d_2$ so $y_2(1) = 1 \Rightarrow e^2 d_2 = 1 \Rightarrow d_2 = e^{-2}$

$y_3 = e^{2t} d_1 + \tilde{d}_1 e^{3t}$ so $y_3(1) = 0 \Rightarrow e^2 \cdot d_1 + \tilde{d}_1 e^3 = 0$ but $c_1 = 1$

so $e^2 + \tilde{d}_1 e^3 = 0 \Rightarrow \tilde{d}_1 = -\frac{e^2}{e^3} = -e^{-1}$

so solution with given initial conditions is

$$\begin{cases} y_1 = e^{2t} \\ y_2 = e^{-2} \cdot e^{2t} \\ y_3 = e^{2t} - \frac{1}{e} e^{3t} \end{cases}$$

EXAMPLE: solve the system

$$\begin{cases} y_1' = y_1 \\ y_2' = -2y_1 + 3y_2 \\ y_3' = -2y_1 + 3y_3 \end{cases} \qquad y' = By \qquad B = \begin{vmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ -2 & 0 & 3 \end{vmatrix}$$

gen sol is $y(t) = e^{Bt} C$

$\det(B - \lambda I) = 0 \iff (1-\lambda)(3-\lambda)^2 = 0$

eigenvalues $\lambda = 1$, $\lambda = 3$

$\underline{\lambda = 1}$: $(B - I)v = 0 \Rightarrow \begin{vmatrix} 0 & 0 & 0 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{vmatrix} \begin{vmatrix} v_1 \\ v_2 \\ v_3 \end{vmatrix} = 0 \Rightarrow \begin{cases} 0v_1 + 0v_2 + 0v_3 = 0 \\ -2v_1 + 2v_2 = 0 \\ -2v_1 + 2v_3 = 0 \end{cases}$

so $\begin{cases} v_1 = v_2 \\ v_1 = v_3 \end{cases}$ can choose $v_3 = 1$ so you have $v_1 = v_2 = v_3 = 1 \Rightarrow v = (1, 1, 1)$

$\underline{\lambda = 3}$: $(B - 3I)w = 0 \quad \begin{vmatrix} -2 & 0 & 0 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{vmatrix} \begin{vmatrix} w_1 \\ w_2 \\ w_3 \end{vmatrix} = 0 \Rightarrow \begin{cases} w_1 = 0 \\ w_2, w_3 \in \mathbb{R}^2 \setminus \{(0,0)\} \end{cases}$

so $\begin{matrix} w = (0, 1, 0) \\ u = (0, 0, 1) \end{matrix}$ are eigenvect.

$$C = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \quad \det C = 1 \neq 0 \Rightarrow C \text{ invert.} \quad C^{-1} = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

$$C^{-1} B C = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{vmatrix} = \Delta \Rightarrow B = C \Delta C^{-1}$$

$$Bt = C(\Delta t) C^{-1}$$

$$e^{Bt} = C e^{\Delta t} C^{-1} = C \begin{vmatrix} e^t & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{3t} \end{vmatrix} C^{-1} = \begin{vmatrix} e^t & 0 & 0 \\ e^t - e^{3t} & e^{3t} & 0 \\ e^t - e^{3t} & 0 & e^{3t} \end{vmatrix}$$

so general solution is

$$y(t) = e^{Bt} \cdot \begin{vmatrix} c_1 \\ c_2 \\ c_3 \end{vmatrix}$$ which is

$$\begin{cases} y_1(t) = c_1 e^t \\ y_2(t) = c_1 (e^t - e^{3t}) + c_2 e^{3t} \\ y_3(t) = c_1 (e^t - e^{3t}) + c_3 e^{3t} \end{cases}$$

RECAP

- A diagonalizable means $\exists\, G$ invertible matrix s.t. $G^{-1} A G = \Delta$, Δ diagonal
- A diagonalizable $\Rightarrow$ $A = G \Delta G^{-1}$
- If A diagonalizable then $A^k = G \Delta^k G^{-1}$
  and $\boxed{e^A = G e^{\Delta} G^{-1}}$ (*)

- Given the system of ODEs $y' = Ay$
  where $y = y(t) = \begin{vmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{vmatrix}$, A is an $n \times n$ matrix,
  the general solution is $y(t) = e^{At} D$ where
  $D = \begin{vmatrix} d_1 \\ \vdots \\ d_n \end{vmatrix}$ is a vector of generic constants, to be determined through initial conditions.

- <u>Def</u>: $\mathbb{R}^N$ = space of vectors with N components, each component takes values in $\mathbb{R}$.