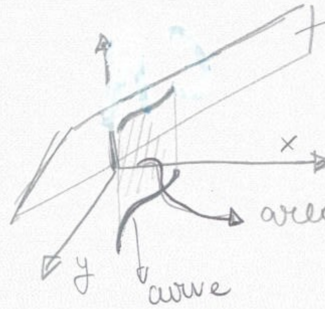
LINE INTEGRAL

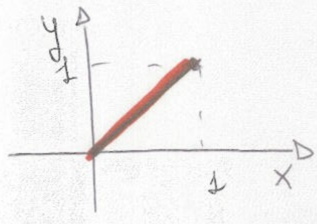
- The double integral is needed when we want to calculate the volume of the solid enclosed by a surface and a certain domain on the (x,y) plane. Suppose instead we want to calculate the area between the surface and a curve on the (x,y) plane; to fix ideas:



- Suppose the curve is described by a function on the (x,y) plane, of the form $y = h(x)$ (in other words the curve can be expressed as a function of x). If we want to integrate the function $f(x,y)$ along this curve, we will have

$$\int_C f(x,y)\,dx = \int_C f(x, y(x))\,dx$$

Ex: Integrate the function $f(x,y) = x+y$ along the curve $y = x$ from $(0,0)$ to $(1,1)$:



This time the domain of integration is a segment, the one in red.

Note: as usual I am sketching the domain of integration, not the function!

Going back to the example, we want to calculate

$$\int_C (x+y)\,dx = \int_0^1 (x+x)\,dx = x^2\Big|_0^1 = 1$$

EX: calculate $\int_C xy\,dy$ where C is the curve $y = x^2$ joining the points $(0,0)$ and $(2,4)$ -

This time the curve is expressed as a function of x, again, but we are integrating in dy, so

$$\int_C xy\,dy = \int_0^2 x \cdot x^2 \cdot 2x\,dx = 2\frac{x^5}{5}\Big|_0^2 = \frac{2^6}{5}.$$

$dy = 2x\,dx$ $\rightarrow$ x goes from 0 to 2.

EX: If the curve we are integrating along has equation $x = \text{const}$ (i.e. it is a line parallel to the y axis) then $\int_C f(x,y)\,dx = 0$ no matter what the function is, because $x = \text{const} \Rightarrow dx = 0$!

Analogous thing holds if the curve is $y = \text{const}$ and we want to integrate in dy.

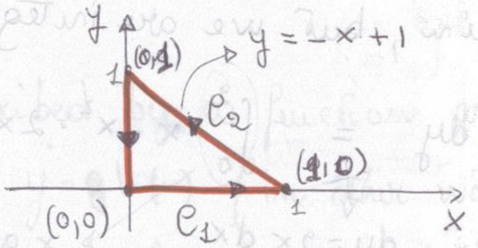
EX: Calculate $\int_{C} e^{x+y}\, dy$ where C is the curve $x = \log y$ (assuming $y>0$) going from $(x_1, y_1) = (0, 1)$ to $(x_2, y_2) = (\log 2, 2)$.

$$\int_C e^{x+y}\, dy = \int_1^2 e^{(\log y)+y}\, dy = \int_1^2 y \cdot e^{y}\, dy$$

y goes from 1 to 2

EX: Calculate $\int_C \left(e^{x+y}\, dx + xy\, dy \right)$

where C is the curve in red, run in the counterclockwise way.



C can be split in three parts,

$C = C_1 \cup C_2 \cup C_3$ where

C_1 is $y = 0$ run from $(0,0)$ to $(1,0)$

C_2 is $y = -x+1$ " $(1,0)$ to $(0,1)$

C_3 is $x = 0$ " $(0,1)$ to $(0,0)$

$$\int_C \left(e^{x+y}\, dx + xy\, dy \right) = \int_{C_1} \left(e^{x+y}\, dx + xy\, dy \right)$$

$$+ \int_{C_2} \left(e^{x+y}\, dx + xy\, dy \right) + \int_{C_3} \left(e^{x+y}\, dx + xy\, dy \right)$$

$$\int_{C_1}\left(e^{x+y}\,dx + xy\,dy\right) = \int_{C_1} e^{x+y}\,dx = \int_{C_1} e^{x}\,dx = \int_0^1 e^x\,dx$$

$y=0$ so $dy=0$ using $y=0$ because x goes from 0 to 1

$$\int_{C_2}\left(e^{x+y}dx + xy\,dy\right) = \int_{C_2}\left(e^{x-x+1}\,dx + x(-x+1)(-dx)\right) =$$

(*)

$$= \int_1^0 e\,dx - (x-x^2)\,dx = \int_1^0 (e-x+x^2)\,dx$$

x goes from 1 to 0

$$\int_{C_3}\left(e^{x+y}\,dx + xy\,dy\right) = \int_1^0 0\,dy$$

$x=0$ so $dx=0$ y goes from 1 to 0.

Notice that in (*) the extrema are $\int_1^0$ and not $\int_0^1$; in the latter case we would be integrating along the same curve, but we would be running the curve in the opposite direction

<u>Rem</u>: because $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$$\int_{C_{A\to B}} = -\int_{C_{B\to A}}$$

integral along the curve running from point A to point B