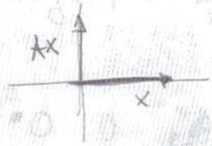
EIGENVALUES and EIGENFUNCTIONS

what do matrices do? They move vectors around

<u>Ex</u>: $\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$ rotates everything counterclockwise by $90°$

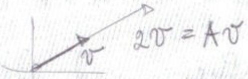
$$\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$$



$$\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 1 \end{vmatrix} = \begin{vmatrix} -1 \\ 1 \end{vmatrix}$$

I am interested in finding out what directions are left ~~of~~ invariant under the action of a matrix A, i.e. given A, can I find λ and $v \neq 0$ s.t. $Av = \lambda v$

Look graphycally



I don't mind wether $Av = v$, I just want to stay on the same straight line where v belongs to.

<u>Def</u>: given an $n \times n$ matrix A, we say that $v \neq 0$ is an eigenvector corresponding to the eigenv. λ if $Av = \lambda v$.

<u>REM</u> $v \neq 0$ but λ can be zero

<u>REM</u> we ask for $v \neq 0$ because $A \cdot 0 = 0$ is always trivially true.

...w do we find eigvalues and eigenfunctions?

We need to solve the equation

$Av = \lambda v$, which is a bit complicated because

we don't know neither λ nor v.

$Av = \lambda v \Rightarrow (A - \lambda I)v = 0$

If $(A - \lambda I)$ is invertible then $v = (A - \lambda I)^{-1} \cdot 0 = 0$

which we no not want. So we look for the values

of λ s.t. $(A - \lambda I)$ is not invert. Indeed

the eigenvalues of the matrix A are the values of λ

s.t. $\underbrace{\det(A - \lambda I)}$ $= 0$.

is a polyn in λ, of degree n. It is called

the characteristic polyn.

Example

$A = \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix}$ $\quad \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix}$

$\det(A - \lambda I) = 0 \Rightarrow (4-\lambda)(3-\lambda) - 2 = 0$ $\quad \frac{7 + \sqrt{49-40}}{2} = 5$

$12 - 7\lambda + \lambda^2 - 2 = 0 \Rightarrow \lambda^2 - 7\lambda + 10 = 0$ $\quad \frac{7-3}{2} = 2$

what are the eigenvectors relative to 5?

$\underline{\lambda = 5}$: $A v = 5\lambda$

$(A - 5I)v = 0 \qquad \begin{vmatrix} -1 & 1 \\ 2 & -2 \end{vmatrix} \begin{vmatrix} v_1 \\ v_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$

This is a system of equations

$\begin{cases} -v_1 + v_2 = 0 \Rightarrow v_1 = v_2 \\ 2v_1 - 2v_2 = 0 \end{cases}$ so $v = \begin{vmatrix} a \\ a \end{vmatrix}$ $a \in \mathbb{R} \setminus \{0\}$

In fact, we found ∞ many eigenvectors corresponding to $\lambda = 5$, all of them lay on the straight line where, for example, $(1,1)$ belongs to.

So $\lambda = 5$ has (for $a=1$) $(1,1)$ as one of the eigenvalues.

REMARK: Why did we find ∞-many eigenvectors?

Because $\det(A-5I)=0$! so $(A-5I)v=0$ has ∞ many solutions. Though, all the eigvect we found belong to the same straight line, i.e. we have found the invariant direction we were seeking.

$\lambda = 2$: $\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} \begin{vmatrix} v_1 \\ v_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \Rightarrow 2v_1 + v_2 = 0$

$\Rightarrow v_1 = -\dfrac{v_2}{2}$

so the invariant direction is $\begin{vmatrix} -b/2 \\ b \end{vmatrix}$ $b \in \mathbb{R} \setminus \{0\}$

One eigenvector is ($b=1$) $v = (-1/2, 1)$

Summary: "the" eigvect. corresponding to the eigenvalue $\lambda = 5$ is $(1,1)$

and the one corresp to $\lambda = 2$ is $(-1/2, 1)$.

EXAMPLE: $\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$ eigv. 1 e 3.

DIAGONALIZATION OF MATRICES

Let us compare the following two examples:

EXAMPLE 1

$A = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{vmatrix}$ Calculate eigenvalues and eigenvectors:

$$\det(A-\lambda I) = \det \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & -1 & -1-\lambda \end{vmatrix} = (2-\lambda)\left[(2-\lambda)(-1-\lambda)+2\right] =$$

$$= (2-\lambda)(-\lambda+\lambda^2) = \lambda(\lambda-1)(2-\lambda)$$

$\det(A-\lambda I) = 0$ when $\lambda=0$, $\lambda=1$ or $\lambda=2$.

So $0, 1, 2$ are the eigenvalues of A.

The eigenvectors satisfy $Av = \lambda v$.

$\lambda=0$: $\quad Av = 0 \quad \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{vmatrix} \begin{vmatrix} v_1 \\ v_2 \\ v_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix} \Rightarrow \begin{cases} 2v_1 = 0 \\ 2v_2 + 2v_3 = 0 \\ -v_2 - v_3 = 0 \end{cases}$

$\Rightarrow \begin{cases} v_1 = 0 \\ v_2 = -v_3 \end{cases}$ so one possible eigenvalue corresponding to $\lambda=0$ is $v = (0, -1, 1)$ (choose $v_3 = 1$)

$\lambda = 1$: $\quad Aw = 1 \cdot w \Rightarrow (A - 1\cdot I)w = 0$

$\Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{vmatrix} \begin{vmatrix} w_1 \\ w_2 \\ w_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix} \Rightarrow \begin{cases} w_1 = 0 \\ w_2 + 2w_3 = 0 \\ -w_2 - 2w_3 = 0 \end{cases} \Rightarrow \begin{cases} w_1 = 0 \\ w_2 = -2w_3 \end{cases}$

so eigenv. corresponding to $\lambda=1$ is $w = (0, -2, 1)$ (choose $w_3 = 1$)

$\lambda = 2$: $\quad Au = 2u \Rightarrow (A - 2I)u = 0$

$\Rightarrow \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -1 & -3 \end{vmatrix} \begin{vmatrix} u_1 \\ u_2 \\ u_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix} \Rightarrow \begin{cases} 0u_1 = 0 \\ 2u_2 = 0 \\ -u_2 - 3u_3 = 0 \end{cases} \Rightarrow \begin{cases} u_1 \text{ can be anything} \\ 0 = u_2 \\ 0 = u_3 \end{cases}$

$\Rightarrow$ eigenvect. corresp. to $\lambda = 2$ is $(1, 0, 0)$ (choose $u_1 = 1$).

Take the matrix $C = \begin{vmatrix} 0 & 0 & 1 \\ -1 & -2 & 0 \\ 1 & 1 & 0 \end{vmatrix}$ which is constructed

by putting v, w, u on the first, second and third column, respectively. You can easily check that C is invertible, indeed $\det C = 1 \neq 0$.

Also, you can check that $C^{-1} = \begin{vmatrix} 0 & 1 & 2 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \end{vmatrix}$.

And now look at the magic:

$$C^{-1} A C = C^{-1} \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{vmatrix} \begin{vmatrix} 0 & 0 & 1 \\ -1 & -2 & 0 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \end{vmatrix} \begin{vmatrix} 0 & 0 & 2 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{vmatrix}$$

$C^{-1} A C = \Delta = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix}$ Δ is a diagonal matrix and it has the eigenvalues of A on the diagonal!

<u>Def</u>: Given an $n \times n$ matrix A, we say that A is diagonalizable if there exists an invertible matrix C s.t. $C^{-1} A C = \Delta$, where Δ is a diagonal matrix.

- Can we always find C? In other words, is any matrix diagonalizable? I am afraid no...

<u>EXAMPLE 2</u>

$B = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{vmatrix}$ Let's calculate eigenval. & eigenvect. of B.

$\det(B - \lambda I) = 0 \iff (1-\lambda)(3-\lambda)^2 = 0$

$\Rightarrow \lambda = 1$ and $\lambda = 3$ are the eigenvalues of B.

Notice that $\lambda = 3$ is repeated twice as root of the characterisitic polynomial, so we say that 3 has ALGEBRAIC MULTIPLICITY 2.

detour: the polynomial $(\lambda-2)(\lambda-5)^3 = 0$ has 2 and 5 as roots, but 5 is repeated three times, indeed $(\lambda-2)(\lambda-5)^3 = (\lambda-2)(\lambda-5)(\lambda-5)(\lambda-5) = 0$, so $\lambda = 2$ has algebraic multiplicity 1 and $\lambda = 5$ has alg. mult 3.
For the polym. $(\lambda-7)(\lambda-9)^{18} = 0$, 7 has alg. mult 1 and 9 has alg. mult 18

However, let's look for our eigenvectors:

$\lambda = 1$: $\quad B v = v \iff (B - I) v = 0 \iff \begin{vmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{vmatrix} \begin{vmatrix} v_1 \\ v_2 \\ v_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$

$\iff \begin{cases} 0 \cdot v_1 = 0 \\ 2v_2 + v_3 = 0 \\ 2v_3 = 0 \end{cases} \implies \begin{cases} v_1 \text{ can be anything (except zero, for our purposes)} \\ v_2 = v_3 = 0 \end{cases}$

$\implies \lambda = 1$ has $v = (1, 0, 0)$ as eigenvect.

$\lambda = 3$: $\quad B w = 3 w \iff (B - 3I) w = 0 \iff \begin{vmatrix} -2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} w_1 \\ w_2 \\ w_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$

$\iff \begin{cases} -2 w_1 = 0 \\ w_3 = 0 \\ 0w_1 + 0w_2 + 0w_3 = 0 \end{cases} \iff \begin{cases} w_1 = w_3 = 0 \\ w_2 \text{ can be any non zero number} \end{cases}$

$\lambda = 3$ has $w = (0, 1, 0)$ as eigenvector.

Now let's try and construct C.

v is the first column, w is the second but...

problem: I don't have a third eigenvector for the third column of C!

Houghston we have a problem! We can't construct C! -
So I am afraid B is not diagonalizable.

EXAMPLE 3

$$M = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{vmatrix} \qquad \det(M-\lambda I) = 0 \iff (1-\lambda)(5-\lambda)^2 = 0$$

so $\lambda = 1$ eigenval with algebr. mult 1
" $\lambda = 5$ " " 2

$$\lambda = 1: \quad \begin{vmatrix} 0 & 2 & -1 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{vmatrix} \begin{vmatrix} v_1 \\ v_2 \\ v_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix} \iff \begin{cases} 2v_2 - v_3 + 0 \cdot v_1 = 0 \\ v_2 = v_3 = 0 \end{cases}$$

$\Rightarrow v_1 \in \mathbb{R} \setminus \{0\}, \ v_2, v_3 = 0 \Rightarrow$ eigenvect $v = (1, 0, 0)$

$$\lambda = 5: \quad \begin{vmatrix} -4 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} w_1 \\ w_2 \\ w_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix} \iff \begin{cases} -4w_1 + 2w_2 - w_3 = 0 \\ 0 \cdot w_2 = 0 \\ 0 \cdot w_3 = 0 \end{cases}$$

$\Rightarrow (w_2, w_3) \in (\mathbb{R} \times \mathbb{R}) \setminus \{(0,0)\} \rightarrow$ it means that one of the two can be zero but not both, otherwise you get $w = 0$

If we choose $\begin{cases} w_2 = 0 \\ w_3 = 4 \end{cases} \Rightarrow -4w_1 = w_3 = 4 \Rightarrow w_1 = -1$

and we get the eigenvector $w = (-1, 0, 4)$

If we choose $\begin{cases} w_3 = 0 \\ w_2 = 2 \end{cases} \Rightarrow +4w_1 = +2w_2 = 4 \Rightarrow w_1 = 1$

we get another eigenvector $u = (1, 2, 0)$.

So $\lambda = 1$ has only one associated eigenvector, whereas $\lambda = 5$ has two associated eigenvectors.

Construct $C = \begin{vmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 4 & 0 \end{vmatrix}$, check that it is invertible

$\det C = -8 \neq 0 \Rightarrow C$ is invertible

Now calculate $C^{-1} = \begin{vmatrix} 1 & -1/2 & 1/4 \\ 0 & 0 & 1/4 \\ 0 & 1/2 & 0 \end{vmatrix}$

and verify that $C^{-1}AC = \Delta = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{vmatrix}$ -

What do we deduce from these examples?

In general: given an $n \times n$ matrix, call it A, if this matrix admits n eigenvectors, $v_1, \dots, v_n$, and these eigenvectors are such that $C = \begin{vmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{vmatrix}$ is invertible, then A is diagonalizable and $C^{-1}AC$ is a diagonal matrix having the eigenvalues of A on the diagonal.

C is called the DIAGONALIZING MATRIX.

REM: if A ($n \times n$ matrix) has n distinct eigenvalues then it is diagonalizable