Civil Engineering 2 Mathematics Autumn 2011

M. Ottobre

First and Second Order ODEs

Warning: all the handouts that I will provide during the course are in no way exhaustive, they are just short recaps.
Notation used in this handout: $y(x), f(x), a_1(x), a_2(x), a(x), b(x)$ are scalar functions and $x \in \mathbb{R}$. We will often write just y instead of $y(x)$ and y' is the derivative of y with respect to x.

- **Classification.** Consider the following differential equations

$$y' + a(x)y = b(x) \tag{1}$$

and

$$y'' + a_1(x)y' + a_2(x)y = f(x) \tag{2}$$

in the unknown $y(x)$.
Equation (1) is **first order** because the highest derivative that appears in it is a first order derivative. In the same way, equation (2) is **second order** as also y'' appears.
They are both **linear**, because y, y' and y'' are not squared or cubed etc and their product does not appear. In other words we do not have terms like $(y')^2$, $(y'')^5$ or yy'.
If $f(x)$ ($b(x)$, respectively) is zero, then (2) ((1), respectively) is **homogeneous**, otherwise it is **non homogeneous**.
If $a_1(x)$ and $a_2(x)$ are constant, then (2) has **constant coefficients**.

Example 0.1.

$$y'' + 5y = x$$

is second order, linear, non homogeneous and with constant coefficients.

$$y' + x^2 y = e^x$$

is first order, linear, non homogeneous.

$$yy'' + y' = 0$$

is non linear, second order, homogeneous.

Important Remark: The general solution to a first order ODE has one constant, to be determined through an initial condition $y(x_0) = y_0$ e.g $y(0) = 3$. The general solution to a second order ODE contains two constants, to be determined through two initial conditions which can be for example of the form $y(x_0) = y_0, y'(x_0) = y'_0$, e.g. $y(1) = 2, y'(1) = 6$.

We will in general focus on linear equations. The only non linear ones that we will stumble across are **Separable Equations:**

$$\left(\frac{dy}{dx} =\right) y' = g(x)h(y) \quad \Rightarrow \quad \int \frac{dy}{h(y)} = \int g(x)dx$$

(if $h(y)$ is non linear then the equation is non linear).
Now let's go back to the main object of our study.

- **How do we solve equations of type (1)?**

You might be in a very simple case:

$$y' = b(x) \quad \Rightarrow \quad y(x) = \int b(x)$$

However, in general we will use the **INTEGRATING FACTOR METHOD:**
Step 1: Calculate the indefinite integral $A(x) = \int a(x)dx$.
Step 2: Multiply both sides of (1) by the **integrating factor** $e^{A(x)}$. Hence you get

$$e^{A(x)}\left(y' + a(x)y\right) = e^{A(x)}b(x). \tag{3}$$

Now notice that the LHS of (3) can be rewritten as $(e^{A(x)}y)'$, in fact by the rule for the derivative of the product of functions and the chain rule we have

$$(e^{A(x)}y)' = e^{A(x)}a(x)y + e^{A(x)}y'.$$

Step 3: Equation (3) can be rewritten as

$$(e^{A(x)}y)' = e^{A(x)}b(x).$$

Integrate both sides

$$e^{A(x)}y = \int e^{A(x)}b(x)dx + C$$

and obtain

$$y = e^{-A(x)} \int e^{A(x)}b(x)dx + e^{-A(x)}C. \tag{4}$$

The above formula (4) is the general solution. C is a generic constant and it can be calculated by using the initial conditions.

- **How do we solve equations of type (2)?**

Recap of available methods.
Case 1: If the equation is **homogeneous with constant coefficients**, i.e. if it is of the form

$$y'' + by' + cy = 0, \quad b, c \in \mathbb{R} \tag{5}$$

then we write the associated auxiliary polynomial

$$\lambda^2 + b\lambda + c = 0, \quad \Delta = b^2 - 4c.$$

If $\Delta > 0$ the polynomial has two distinct real roots, $\lambda_1, \lambda_2 \in \mathbb{R}$ and the solution to (5) is

$$y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}, \quad A, B \in \mathbb{R}.$$

If $\Delta = 0$ the polynomial has only one root, $\lambda = -\frac{b}{2}$, and the solution to (5) is

$$y(x) = Ae^{\lambda x} + Bxe^{\lambda x} \quad A, B \in \mathbb{R}.$$

If $\Delta < 0$ $(b^2 < 4c)$ the polynomial has two complex conjugate roots, $\lambda_1, \lambda_2 \in \mathbb{C}$, namely

$$\lambda_1 = -\frac{b}{2} + i\omega, \quad \lambda_2 = -\frac{b}{2} - i\omega, \qquad \omega = \sqrt{\frac{4c - b^2}{4}}$$

and the solution to (5) is

$$y(x) = e^{-\frac{b}{2}x} \left(A\cos(\omega x) + B\sin(\omega x)\right) \quad A, B \in \mathbb{R}.$$

Case 2: If the equation is **homogeneous with NON constant coefficients**, i.e. if it is of the form

$$y'' + a_1(x)y' + a_2(x)y = 0. \tag{6}$$

FACT: If $y_1(x)$ and $y_2(x)$ are two (*distinct*) solutions of (6) then the general solution of (6) is

$$y(x) = Ay_1(x) + By_2(x), \quad A, B \in \mathbb{R}.$$

So our aim is finding y_1 and y_2. Let's see what we can do.

METHOD OF ORDER REDUCTION. This method is based on having a certain amount of luck. What do I mean? Well, if for some reason a solution y_1 rains on you, then this method allows you to find y_2 which is the general solution. But you are still left with the problem of finding y_1...However, assume we have a solution y_1 of (6). To find y_2

Step 1: Find $u(x)$ solution to

$$u' + \left(\frac{2y_1'}{y_1} + a_1\right) u = 0 \tag{7}$$

(using the integrating factor method, see (4)).

Step 2: The general solution y_2 is

$$y_2(x) = y_1(x) \int u(x)dx.$$

This is it. But, why is that? We need to show that y_2 solves (6).

$$y_2' = y_1' \int u + y_1 u,$$

$$y_2'' = y_1'' \int u + 2y_1' u + y_1 u'.$$

Putting everything together

$$y_2'' + a_1 y_2' + a_2 y_2$$

$$= \underbrace{(y_1'' + a_1 y_1' + a_2 y_1)}_{\substack{\| \\ 0}} \int u + \underbrace{y_1 u' + (2y_1' + a_1 y_1)\, u}_{\substack{\| \\ 0}} = 0$$

where the first addend vanishes because y_1 is a solution and the terms in the second brace vanish because u solves (7).

Case 3: If the equation is **NON homogeneous with NON constant coefficients**, i.e. if it is of the form

$$y'' + a_1(x) y' + a_2(x) y = f(x). \tag{8}$$

FACT: Let $y_1(x)$ and $y_2(x)$ be two (distinct) solutions of the homogeneous equation associated to (8), that is (6). Let $\bar{y}$ be a *particular integral* of (8), then the general solution of (8) is

$$y(x) = Ay_1(x) + By_2(x) + \bar{y}(x), \quad A, B \in \mathbb{R}. \tag{9}$$

In other words, the general solution to (8) is

$$y(x) = \text{general solution of homog. eqn.} + \text{particular int. of (8)}.$$

Because the method of order reduction worked so well before, let's see if we can employ it again. So, suppose you are given a solution y_1 of the homogenous equation (6). Let us again look for a solution of (8) in the form

$$y = y_1 \int v$$

and try and determine the equation that v has to satisfy in order for y to be a solution of (8). As before

$$y' = y_1' \int v + y_1 v,$$

$$y'' = y_1'' \int v + 2y_1' v + y_1 v'.$$

Now

$$y'' + a_1 y' + a_2 y = f(x) \Leftrightarrow y_1 v' + (2y_1' + a_1 y_1)\, v = f(x)$$

hence v has to solve the equation

$$v' + \left(\frac{2y_1'}{y_1} + a_1 \right) v = \frac{f(x)}{y_1}.$$

v will then be of the form $v(x) = g_1(x) + Cg_2(x)$ (see (4)) so that $\int v$ is of the form $\int v = G_1(x) + CG_2(x) + D$. Hence $y(x)$ is of the form (9) and it is therefore the general solution we were seeking.

Conservative Case: When $a_2 = a_1'$. In this case equation (8) reads

$$y'' + a_1 y' + a_1' y = f(x),$$

which can be rewritten as

$$(y' + a_1 y)' = f(x)$$

so that integrating both sides and naming $F(x) = \int f(x)$ leads to

$$y' + a_1(x) y = F(x).$$

This equation is in the form (1) and we know how to solve it. Notice that the above works also in case $f \equiv 0$.

Example 0.2. Solve

$$y' = -e^x y, \quad y(0) = 1.$$

First solution: separable variables

$$\int_1^y \frac{dy}{y} = -\int_0^x e^x dx \quad \Rightarrow \quad \log(y) = -e^x + 1 \quad \Rightarrow \quad y = e \cdot e^{-e^x}.$$

Second solution: Integrating factor method. The integrating factor is

$$A(x) = \int e^x dx = e^x.$$

Multiply both sides of the equation by e^{e^x} and obtain

$$\left(e^{e^x} y\right)' = 0 \quad \Rightarrow \quad e^{e^x} y = C \quad \Rightarrow \quad y = e^{-e^x} C.$$

Imposing $y(0) = 1$ gives $C = e$.

Example 0.3. Given that $y_1 = x$ is a solution of

$$y'' - \frac{x(x+2)}{x^2} y' + \frac{x+2}{x^2} y = 0,$$

find the general solution of

$$y'' - \frac{x(x+2)}{x^2} y' + \frac{x+2}{x^2} y = x.$$

Sketch of solution: let $y = x \int v$. In order for y to be a solution of the inhomogeneous equation, v has to solve $v' - v = 1$ (during the exam you need to show all the steps that lead to the equation for v). We solve the equation for v by the integrating factor method and we get

$$v = -1 + Ce^x \quad \Rightarrow \quad \int v = -x + Ce^x + D$$

$$\Rightarrow \quad y = -x^2 + Cxe^x + Dx, \quad C, D \in \mathbb{R}$$

and y is the general solution we were seeking.