PARTIAL DIFFERENTIAL EQUATIONS (PDEs)

First a little bit of notation: let $u(x,t)$ be a scalar function of two variables, for example

$$u(x,t) = x^2 t + tx - x$$

then $\frac{\partial u}{\partial x} = u_x = \partial_x u$ are all equivalent notations for the partial derivative with respect to x. In the example

$$u_x = 2xt + t - 1$$

$$u_t = x^2 + x$$

Analogously, $u_{xx} = \frac{\partial^2 u}{\partial x^2} = \partial_{xx} u$ is the second derivative w.r. to x; in the example

$$u_{xx} = 2t \quad , \quad u_{tt} = 0 \quad , \quad u_{xt} = 0$$

where $u_{xt} = \partial_{xt} u = \frac{\partial^2 u}{\partial x \partial t}$ -

An ODE is a differential equation involving functions of one variable -

A PDE is a differential equation for functions of two (or more) variables, involving partial derivatives.

For example, let $u = u(x,t)$, then

$$u_{tt} - u_{xt} + x u_x = 0 \qquad (*)$$

is a PDE and it is a second order PDE because the higher order derivative that appears in $(*)$ is a second derivative.

Ex: $u_{xx} - u_x + 2u_t + 7u = 0$

is a second order PDE with constant coefficients (because the coefficients in front of u and its derivatives do not depend on either x or t)

• Classification of second order PDEs with constant coefficients (for functions of two variables)

• The most general PDE of this type is

$$a\frac{\partial^2 u}{\partial x^2} + h\frac{\partial^2 u}{\partial x \partial t} + b\frac{\partial^2 u}{\partial t^2} + 2f\frac{\partial u}{\partial x} + 2g\frac{\partial u}{\partial t} + cu = 0 \quad (CC)$$

where a, b, c, h, f, g are all constants.

We say that equation (CC) is

ELLIPTIC if $ab - \frac{h^2}{4} > 0$

PARABOLIC if $ab - \frac{h^2}{4} = 0$

HYPERBOLIC if $ab - \frac{h^2}{4} < 0$

EXAMPLES: i) $3u_{xx} - u_{xt} + u = 0$

$a = 3 \quad b = 0 \quad h = -1 \quad \Rightarrow \quad ab - \frac{h^2}{4} = -\frac{1}{4} < 0 \Rightarrow$ hyperbolic

ii) for $u = u(x,y)$: $u_{xx} + \sqrt{\pi}\, u - e^2 u_y + u_{yy} = 0$

$a = 1 \quad b = 1 \quad h = 0 \quad \Rightarrow \quad ab - \frac{h^2}{4} = 1 > 0 \Rightarrow$ elliptic

• REM (for those who want to know): given a function $f(x,y)$ it is not always true that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$. Though, Schwartz's theorem says that if f is a continuous function s.t. its first and second derivatives are continuous functions then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$, i.e. the order of differentiation does not matter. In the cases that we will talk about the function f will always be very regular, so you won't have to worry about this.

The most famous PDES

- $\boxed{\partial_t u = D\, \partial_{xx} u}$ is the HEAT EQN. or DIFFUSION EQN. where $u = u(x,t)$ and $D > 0$ is a constant, called DIFFUSIVITY CONSTANT.

- $\boxed{\partial_{tt} u = c^2\, \partial_{xx} u}$ is the WAVE EQN or the equation of the VIBRATING STRING, where $u = u(x,t)$ and $c^2 > 0$ is a constant, called WAVE SPEED.

- $\boxed{u_{xx} + u_{yy} = 0}$ is LAPLACE'S EQN, where $u = u(x,y)$.

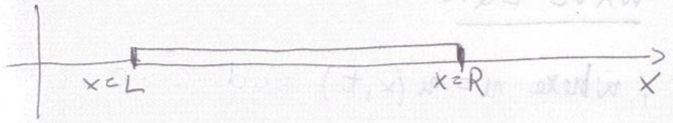
REM: for the first two equations we use the notation $u = u(x,t)$, i.e. u is a function of space x and time t.
In the last eqn, instead, u is a function of two space variables, x and y. We will talk a little about the meaning and physical interpretation of these equations; that should hopefully make things more clear.
Before getting into this matter let us answer a question: why do we need functions of two variables?

EXAMPLE: consider a metallic bar, which is everywhere perfectly isolated from the exterior except at point R.

If we bring a hot body in contact with the bar at the point R the bar starts heating up. Though it is common knowledge that The part of the bar close to R

will warm up before the part of the bar close to the point L
Hence the temperature T depends on time and space, i.e. it depends on when and where we measure it.
Indeed $T = T(x, t)$ and we have for example

$T(R, 0) = 100°$, $T(L, 0) = 10°$. After enough time, we will see that



$T(R, t) \approx T(L, t)$ -

HEAT EQUATION (or DIFFUSION EQN)

Setting for the moment $D = 1$, the heat eqn looks like

(H) $u_t = u_{xx}$ where $u = u(x, t)$ -

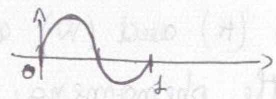
$u(x, t)$ here represents the density of a quantity that diffuses (= "propagates") across a certain body and then it either reaches an equilibrium state or dissipates -
This is typically what happens with heat conduction: if we put a hot body in contact with a cold one, heat diffuses (= spreads) through the cold body - When we remove the hot body, if the cold body was insulated then heat distributes homogeneously across the cold body until a state of equilibrium is reached. If the cold body was not insulated then heat dissipates into the environment. For this reason eqn (H) is said to be a dissipative equation.

So (H) models the phenomenon of heat conduction and similar phenomena; indeed it is also used in population dynamics to describe migratory phenomena -

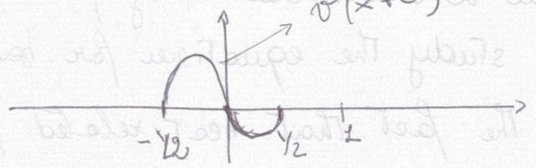
WAVE EQUATION (or VIBRATING STRING EQN)

(W) $u_{tt} = c^2 u_{xx}$

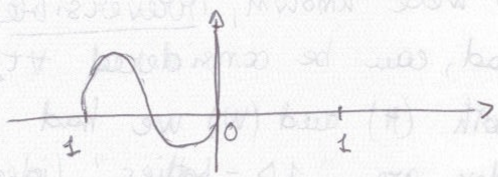
We will see that the general solution of (W) is of the form $u(x,t) = v(x+ct) + f(x-ct)$
where v and f are two arbitrary functions to be determined through initial conditions.

Suppose $0 < x < 1$ and $v(x) =$ 

Then at time $t=1$ we have

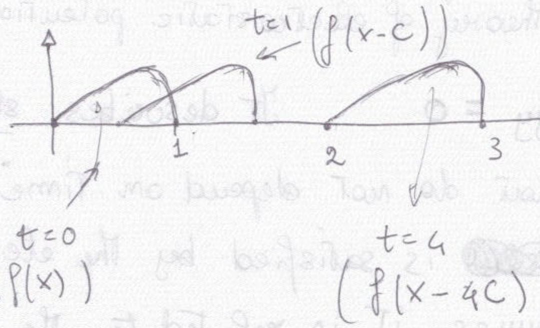
$v(x+c)$ ⇝ 

For this graph we took $c = 1/2$

and at time $t=2$ 

This is to say that $v(x+ct)$ is a wave travelling towards the left with speed c.

In the same way, suppose $f(x) =$ 

then $f(x-ct) =$ 

Then $f(x-ct)$ is a wave travelling towards right with speed c.

In conclusion, this is why (W) is called the wave eqn, indeed its solution is the superimposition of two waves travelling with speed c in opposite directions. It also models the vibration of a string; in this context $u(x,t)$ can be interpreted to be the vertical displacement of the string with respect to the x-axis.

- Equations (H) and (W) are called evolution equations, because the phenomena they describe evolve in time. Nonetheless (H) can be considered only for $t \geq 0$ or, in other words, we cannot study the equation for backward time! This is due to the fact that heat related phenomena are, as it is well known, irreversible!
  (W), instead, can be considered $\forall t$.
  Also in both (H) and (W) we had $x \in \mathbb{R}$, so the bodies we consider are "1D - bodies" (idealization)
  For Laplace's eqn things are different.

LAPLACE'S EQUATION

It has a number of applications, especially in fluid dynamics and in the theory of electrostatic potentials.

(L) $\quad u_{xx} + u_{yy} = 0$ $\quad$ It describes static phenomena, i.e. phenomena that do not depend on time.

Indeed, it is satisfied by the electrostatic potential.
In fluid dynamics it is related to the description of incompressible and irrotational fluids.
The bodies involved are in this case "2D - bodies" as x and y are both space-variables. Solutions of (L) are called HARMONIC functions.