• HOW TO SOLVE eqns (H), (W) and (L)? by the

METHOD of SEPARATION of VARIABLES

Before getting to this method, jus a quick reminder on ODEs

$y'' = -\omega^2 y \quad \Rightarrow \quad y(x) = A\cos(\omega x) + B\sin(\omega x)$

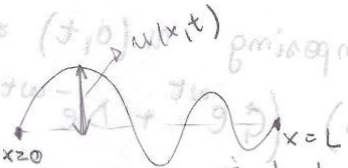
$y'' = \omega^2 y \quad \Rightarrow \quad y(x) = Ae^{\omega x} + Be^{-\omega x}$

$\omega^2 > 0$ is a constant

Also $y' = -\omega^2 y \quad \Rightarrow \quad y(x) = e^{-\omega^2 x} C \quad$ (R)

so $y(x) \to 0$ as $x \to \infty$.

• WAVE EQUATION



$$\begin{cases} \partial_{tt} u = c^2 \partial_{xx} u & (W) \\ u(0,t) = u(L,t) = 0 & t \geq 0 \\ u(x,0) = f(x) & 0 \leq x \leq L \\ \left.\dfrac{\partial u}{\partial t}(x,t)\right|_{t=0} = g(x) & 0 \leq x \leq L \end{cases}$$

B.C.

$u(0,t) = u(L,t) = 0$ → the string is tied down at the extrema $x=0$ and $x=L$

$u(x,0) = f(x)$ → initial configuration of the string

• remember that $u(x,t)$ is the vertical displacement of the string at point x and time t.

• $f(x)$ and $g(x)$ are given functions and L is known.

• We exclude the case $u(x,t)=0 \ \forall x,t$.

• We seek for a solution in the form

$u(x,t) = X(x)\,T(t) \quad (S) \quad \Rightarrow \partial_t u = T'(t)X(x), \ \partial_{tt} u = T''(t)X(x)$

Substituting (S) in (W) we have

$X(x)\,T''(t) = c^2 X''(x)\,T(t)$

$$\Rightarrow \underbrace{\frac{c^2 X''(x)}{X(x)}}_{\text{function of } x \text{ only}} = \frac{T''(t)}{T(t)} \rightarrow \text{function of } t \text{ only}$$

$\Rightarrow c^2 \frac{X''(x)}{X(x)} = \text{const}$ and $\frac{T''(t)}{T(t)} = \text{const}$ (EE)

- suppose that const is positive, so I can call it ω^2

$\Rightarrow \frac{X''(x)}{X(x)} = \frac{\omega^2}{c^2}$ and $\frac{T''(t)}{T(t)} = \omega^2 \qquad \omega^2 > 0$

$\Rightarrow X''(x) = \frac{\omega^2}{c^2} X(x) \Rightarrow X(x) = A e^{\omega/c\, x} + B e^{-\omega/c\, x}$

and $T''(t) = \omega^2 T(t) \Rightarrow T(t) = C e^{\omega t} + D e^{-\omega t}$

$\Rightarrow u(x,t) = \left(A e^{\omega/c\, x} + B e^{-\omega/c\, x}\right)\left(C e^{\omega t} + D e^{-\omega t}\right)$

- Imposing $u(0,t) = 0$ gives

$(A+B)\left(C e^{\omega t} + D e^{-\omega t}\right) = 0 \Rightarrow A = -B$

because it cannot be $C e^{\omega t} + D e^{-\omega t} = 0$, otherwise I would have $u(x,t) = 0 \ \forall x, t$.

Imposing $u(L,t) = 0$ we have

$\left(-B e^{\omega/c\, L} + B e^{-\omega/c\, L}\right) = 0$

$\Rightarrow B e^{-\omega/c\, L} = B e^{\omega/c\, L} \Rightarrow e^{-\omega/c\, L} = e^{+\omega/c\, L}$ impossible!

So it cannot be $\omega^2 > 0$.

Try with $\omega^2 = 0$ i.e. $\omega = 0$

$\Rightarrow X''(x) = 0$ and $T''(t) = 0$

$\Rightarrow X(x) = Ax + B \ , \ T(t) = Ct + D$

now $u(0,t) = 0 \Rightarrow B = 0$

and $u(L,t) = 0 \Rightarrow AL = 0 \Rightarrow A = 0$

- But $A = B = 0 \Rightarrow X(x) = 0 \Rightarrow u(x,t) = 0$ not acceptable.

So the last chance is that our constant is negative i.e.

$$c^2 \frac{X''(x)}{X(x)} = -\omega^2 \quad \text{and} \quad \frac{T''(t)}{T(t)} = -\omega^2$$

$$\Rightarrow X''(x) = -\frac{\omega^2}{c^2} X(x) \quad \text{and} \quad T''(t) = -\omega^2 T(t)$$

$$\Rightarrow X(x) = A\cos\left(\frac{\omega}{c}x\right) + B\sin\left(\frac{\omega}{c}x\right)$$

$$T(t) = C\cos(\omega t) + D\sin(\omega t)$$

$\Rightarrow$ oscillatory solution

$$\Rightarrow u(x,t) = \left[A\cos\left(\frac{\omega}{c}x\right) + B\sin\left(\frac{\omega}{c}x\right)\right]\left[C\cos(\omega t) + D\sin(\omega t)\right]$$

Impose $u(0,t) = 0$ and get $A = 0$

$$\Rightarrow u(x,t) = B\sin\left(\frac{\omega}{c}x\right)\left[C\cos\omega t + D\sin\omega t\right]$$

$$= \sin\left(\frac{\omega}{c}x\right)\left[BC\cos\omega t + BD\sin\omega t\right]$$

still a generic constant call $BC = E$, $BD = F$

$$= \sin\left(\frac{\omega}{c}x\right)\left[E\cos\omega t + F\sin\omega t\right]$$

Impose $u(L,t) = 0$ and get

$$\sin\left(\frac{\omega}{c}L\right) = 0 \Rightarrow \frac{\omega}{c}L = n\pi \quad n = 0, 1, 2, \ldots$$

$$\Rightarrow \omega = \frac{n\pi c}{L}$$. What does this mean?

It means that

for $n=1$ $\sin\left(\frac{\pi x}{L}\right)\left[E_1\cos\left(\frac{\pi c t}{L}\right) + F_1\sin\left(\frac{\pi c t}{L}\right)\right]$ is a sol of (W)

for $n=2$ $\sin\left(\frac{2\pi x}{L}\right)\left[E_2\cos\left(\frac{2\pi c t}{L}\right) + F_2\sin\left(\frac{2\pi c t}{L}\right)\right]$ is a sol (W)

and so on and so forth for any n.

Hence we have found ∞ many solutions.

Now notice that if $h(x)$ and $\ell(x)$ are solutions of (W) then also $h(x) + \ell(x)$ is a solution of (W). So in our case

$$u(x,t) = \sum_{m=0}^{\infty} \sin\left(\frac{m\pi x}{L}\right)\left[E_m \cos\left(\frac{m\pi c}{L} t\right) + F_m \sin\left(\frac{m\pi c}{L} t\right)\right]$$

is a solution of (W).

We still need to satisfy $u(x,0) = f(x) \quad 0 \le x \le L$

and $\frac{\partial u}{\partial t}(x,t)\Big|_{t=0} = g(x) \quad 0 \le x \le L$

So impose $u(x,0) = f(x)$, $0 \le x \le L$:

$$u(x,0) = \sum_{m=1}^{\infty} \sin\left(\frac{m\pi x}{L}\right) \cdot E_m = f(x) \quad (1)$$

and impose also $\frac{\partial u}{\partial t}(x,0) = g(x)$, $0 \le x \le L$.

$$\frac{\partial u}{\partial t}(x,t) = \sum_{m=0}^{\infty} \sin\left(\frac{m\pi x}{L}\right)\left[-E_m \sin\left(\frac{m\pi c}{L} t\right) \cdot \frac{m\pi c}{L} + F_m \frac{m\pi c}{L} \cos\left(\frac{m\pi c}{L} t\right)\right]$$

$$\Rightarrow \frac{\partial u}{\partial t}(x,0) = \sum_{m=1}^{\infty} \sin\left(\frac{m\pi x}{L}\right) \cdot F_m \frac{m\pi c}{L} = g(x) \quad (2)$$

If $g(x)$ and $f(x)$ are continuous I can represent them through their Fourier half range series for $0 < x < L$

then (3) $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$, $b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

(4) $g(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$, $B_n = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$

So from (1) and (3) we have $E_m = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx$

and from (2) and (4) we have $F_m \cdot \frac{m\pi c}{L} = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{m\pi x}{L}\right) dx$

$$\Rightarrow u(x,t) = \sum_{m=0}^{\infty} \sin\left(\frac{m\pi x}{L}\right)\left[\left[\frac{2}{L}\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx\right] \cos\left(\frac{m\pi c}{L} t\right) + \left(\frac{2}{m\pi c}\int_0^L g(x) \sin\left(\frac{m\pi x}{L}\right) dx\right) \sin\left(\frac{m\pi c}{L} t\right)\right].$$

Notice that the solution is indeed an oscillatory function, as it is a combination of sines and cosines, hence it mirrors the intuition that the solution of the wave equation should be something that vibrates (or oscillates).
We obtained an oscillatory solution because we took the constant on the right hand side of (EE) to be negative.

REM: when you are asked to find an oscillatory solution for the wave equation, you need to choose the const in (EE) to be negative, i.e. const $= -\omega^2$.

• HEAT EQUATION

$$\begin{cases} \dfrac{\partial u}{\partial t} = D \dfrac{\partial^2 u}{\partial x^2} \quad (H) & D>0. \\ u(0,t) = u(L,t) = 0 \quad \forall t \geq 0 \\ u(x,0) = f(x) \quad 0 \leq x \leq L \end{cases}$$

boundary conditions

again we look for a solution in the form

$u(x,t) = X(x)\,T(t)$ so

$\partial_t u = X(x)\,T'(t)$ and $\partial_{xx} u = X''(x)\,T(t)$; plug this in (H)

$$X(x)\,T'(t) = D\,X''(x)\,T(t)$$

$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{1}{D}\frac{T'(t)}{T(t)}$$

$$\Rightarrow \frac{X''(x)}{X(x)} = \text{const} \quad \text{and} \quad \frac{1}{D}\frac{T'(t)}{T(t)} = \text{const} \quad (FF)$$

you can check that the choices const $= \omega^2 > 0$ and const $= 0$ are not admissible. Also, we expect to get a solution that decays exponentially in time. So we choose const $= -\omega^2 < 0$

$$\Rightarrow \quad X''(x) = -\omega^2 X(x) \quad \text{and} \quad T'(t) = -\omega^2 D\, T(t)$$

$$\Rightarrow \quad X(x) = A\cos(\omega x) + B\sin(\omega x)$$

$$T(t) = e^{-\omega^2 D t}\, G$$

$$\Rightarrow \quad u(x,t) = G\, e^{-\omega^2 D t}\left(A\cos(\omega x) + B\sin(\omega x)\right) =$$

$$= e^{-\omega^2 D t}\left(\underbrace{AG}_{=E}\cos(\omega x) + \underbrace{BG}_{=F}\sin(\omega x)\right) =$$

$$= e^{-\omega^2 D t}\left(E\cos(\omega x) + F\sin(\omega x)\right)$$

Impose $u(0,t) = 0 \quad \forall t$

$$\Rightarrow \quad u(0,t) = e^{-\omega^2 D t} \cdot E = 0 \quad \Rightarrow \quad E = 0$$

$$\Rightarrow \quad u(x,t) = F\, e^{-\omega^2 D t}\sin(\omega x)$$

Impose $u(L,t) = 0 \quad \forall t$

$$\Rightarrow \quad u(L,t) = F\, e^{-\omega^2 D t}\sin(\omega L) = 0$$

because F cannot be equal to 0, otherwise $u \equiv 0$, then

$\sin(\omega L) = 0 \quad \Rightarrow \quad \omega L = m\pi$. So as before

$$\omega_m = \frac{m\pi}{L} \quad \text{and} \quad u(x,t) = \sum_{m=1}^{\infty} F_m\, e^{-\omega_m^2 D t}\sin(\omega_m x)$$

$$= \sum_{m=1}^{\infty} F_m\, e^{-\frac{m^2\pi^2}{L^2} D t}\sin\left(\frac{m\pi x}{L}\right)$$

Impose $u(x,0) = f(x)$: if $f(x)$ is continuous then we can represent it through its half range sine series on the interval $0 < x < L$ so

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{with} \quad b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

so

$$u(x,0) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

implies $F_n = b_n \ \forall n \geq 1$. Hence

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right)\right] e^{-(n^2\pi^2/L^2)Dt} \sin\left(\frac{n\pi x}{L}\right)$$

is the solution of (H) with the given boundary conditions.

Notice that $u(x,t)$ decays in time due to the factor

$$e^{-\left(\frac{n^2\pi^2 D}{L^2}\right)t} \to 0 \quad \text{as} \quad t \to \infty.$$

- **LAPLACE EQUATION** (for $u(x,y)$)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (L)$$

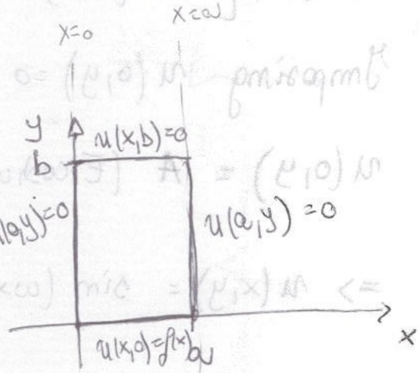
$u(0,y) = 0 \quad 0 \leq y \leq b$

$u(a,y) = 0 \quad 0 \leq y \leq b$

$u(x,b) = 0 \quad 0 \leq x \leq a$

$u(x,0) = f(x) \quad 0 < x < a$

boundary conditions



$u(x,y) = X(x)\,Y(y)$. $\quad \partial_{xx} u = X''(x)\,Y(y)$, $\partial_{yy} u = X(x)\,Y''(y)$

so from (L) we have $\quad X''(x)\,Y(y) = -X(x)\,Y''(y)$

$$\Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} \Rightarrow \frac{X''(x)}{X(x)} = \text{const} \text{ and } \frac{Y''(y)}{Y(y)} = -\text{const}$$

you can check again that $\text{const} = \omega^2$ and $\text{const} = 0$ are not admissible choices. So try with $\text{const} = -\omega^2$. Notice that in this way we will obtain solutions which are periodic in x.

$$\Rightarrow X''(x) = -\omega^2 X(x) \quad \text{and} \quad Y''(y) = \omega^2 Y(y)$$

$$\Rightarrow X(x) = A\cos(\omega x) + B\sin(\omega x)$$

$$Y(y) = C e^{\omega y} + D e^{-\omega y}.$$ Recall now from the revision sheet that $Y(y)$ can be rewritten as

$$Y(y) = E\cosh \omega y + F\sinh(\omega y)$$ for some generic constants E and F, so

$$u(x,y) = [A\cos(\omega x) + B\sin(\omega x)][E\cosh(\omega y) + F\sinh(\omega y)]$$

Imposing $u(0,y) = 0$ for $0 \le y \le b$ we have

$$u(0,y) = A(E\cosh \omega y + F\sinh \omega y) = 0 \Rightarrow A = 0$$

$$\Rightarrow u(x,y) = \sin(\omega x)[\overbrace{BE}^{G}\cosh(\omega y) + \overbrace{BF}^{H}\sinh(\omega y)]$$

$$= \sin(\omega x)[G\cosh(\omega y) + H\sinh(\omega y)],$$

G and H generic constants.

Impose $u(a,y) = 0$ for $0 \le y \le b$:

$$u(a,y) = \sin(\omega a)[G\cosh(\omega y) + H\sinh(\omega y)] = 0$$

$\Rightarrow \sin(\omega a) = 0 \quad \Rightarrow \quad \omega a = m\pi$

so $\omega_m = \frac{m\pi}{a}$ and we have

$$u(x,y) = \sum_{m=1}^{\infty} \sin(\omega_m x) \left[G_m \cosh(\omega_m y) + H_m \sinh(\omega_m y) \right] =$$

$$= \sum_{m=1}^{\infty} \sin\left(\frac{m\pi x}{a}\right) \left(G_m \cosh\left(\frac{m\pi y}{a}\right) + H_m \sinh\left(\frac{m\pi y}{a}\right) \right)$$

Impose $u(x,b) = 0$ for $0 \le x \le a$:

$$u(x,b) = \sum_{m=1}^{\infty} \sin\left(\frac{m\pi x}{a}\right) \left(G_m \cosh\left(\frac{m\pi b}{a}\right) + H_m \sinh\left(\frac{m\pi b}{a}\right) \right) = 0$$

$$\Rightarrow \quad G_m \cosh\left(\frac{m\pi b}{a}\right) = - H_m \sinh\left(\frac{m\pi b}{a}\right)$$

$$\Rightarrow \quad G_m = - H_m \tanh\left(\frac{m\pi b}{a}\right)$$

$$\Rightarrow u(x,y) = \sum_{m=1}^{\infty} \sin\left(\frac{m\pi x}{a}\right) \left[- H_m \tanh\left(\frac{m\pi b}{a}\right) \cosh\left(\frac{m\pi y}{a}\right) + H_m \sinh\left(\frac{m\pi y}{a}\right) \right]$$

$$= \sum_{m=1}^{\infty} H_m \sin\left(\frac{m\pi x}{a}\right) \left[- \tanh\left(\frac{m\pi b}{a}\right) \cosh\left(\frac{m\pi y}{a}\right) + \sinh\left(\frac{m\pi y}{a}\right) \right]$$

Impose $u(x,0) = f(x)$ for $0 < x < a$

If $f(x)$ is continuous then $f(x) = \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{a}\right)$

with $b_m = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{m\pi x}{a}\right) dx$. So

$$u(x,0) = \sum_{m=1}^{\infty} H_m \sin\left(\frac{m\pi x}{a}\right) \left[- \tanh\left(\frac{m\pi b}{a}\right) \right] = f(x)$$

implies $- H_m \tanh\left(\frac{m\pi b}{a}\right) = b_m \quad \forall m$ and hence

$$u(x,y) = \sum_{m=1}^{\infty} \frac{-b_m}{\tanh\left(\frac{m\pi b}{a}\right)} \sin\left(\frac{m\pi x}{a}\right) \left[- \tanh\left(\frac{m\pi b}{a}\right) \cosh\left(\frac{m\pi y}{a}\right) + \sinh\left(\frac{m\pi y}{a}\right) \right]$$

where $b_m = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{m\pi x}{a}\right) dx$.

So far we have found solutions of PDEs when imposing boundary conditions. How do we find general solutions? Before getting into this matter,

CHAIN RULE FOR FUNCTIONS OF TWO VARIABLES

given the function $u(x,y)$, consider the two new variables $\begin{cases} \xi = x + \alpha y \\ \eta = x + \beta y \end{cases}$

so $u(x,y) \longleftrightarrow u(\xi,\eta)$

(I can write u either as a function of x,y or as a function of ξ,η)

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} \qquad (4)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} \qquad (5)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial \xi}\cdot 1 + \frac{\partial u}{\partial \eta}\, 1\right] =$$

notice that $\frac{\partial u}{\partial \xi}(\xi,\eta)$ and $\frac{\partial u}{\partial \eta}(\xi,\eta)$

$$= \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + \frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x} =$$

$$= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \qquad (1)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u(\xi,\eta)}{\partial \xi}\,\alpha + \frac{\partial u(\xi,\eta)}{\partial \eta}\,\beta\right)$$

$$= \alpha\left(\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial y}\right) + \beta\left(\frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial y}\right) =$$

$= \alpha^2 u_{\xi\xi} + 2\alpha\beta\, u_{\xi\eta} + \beta^2 u_{\eta\eta}$ (2)

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(u_\xi \alpha + u_\eta \beta\right) = \alpha\left(u_{\xi\xi}\frac{\partial \xi}{\partial x} + u_{\xi\eta}\frac{\partial \eta}{\partial x}\right) + \beta\left(u_{\xi\eta}\frac{\partial \xi}{\partial x} + u_{\eta\eta}\frac{\partial \eta}{\partial x}\right) =$$

$$= \alpha u_{\xi\xi} + (\alpha+\beta) u_{\xi\eta} + \beta u_{\eta\eta} \quad - \quad (3)$$

GENERAL SOLUTION OF II ORDER PDEs WITH CONSTANT COEFFICIENTS

EXAMPLE: consider the equation

$$2\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (E)$$

$a = 2$, $b = 1$, $h = 3$ so $ab - \frac{h^2}{4} = 2 - \frac{9}{4} < 0 \Rightarrow$ hyperbolic

Consider the change of variable $\begin{cases} \xi = x + \alpha y \\ \eta = x + \beta y \end{cases}$ α and β to be chosen

Using the chain rule, namely (1), (2) and (3), we can rewrite (E) as

$$2\left(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\right) + 3\left(\alpha u_{\xi\xi} + (\alpha+\beta) u_{\xi\eta} + \beta u_{\eta\eta}\right) + \alpha^2 u_{\xi\xi} + 2\alpha\beta\, u_{\xi\eta} + \beta^2 u_{\eta\eta} = 0$$

$$\Rightarrow \left(2 + 3\alpha + \alpha^2\right) u_{\xi\xi} + \left(4 + 3(\alpha+\beta) + 2\alpha\beta\right) u_{\xi\eta} + \left(2 + 3\beta + \beta^2\right) u_{\eta\eta} = 0$$

→ it is the same polynomial!!

So choose α and β to be the roots of the polynomial

$$\lambda^2 + 3\lambda + 2 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \frac{-3 \pm \sqrt{9-8}}{2} = \begin{cases} -2 \\ -1 \end{cases}$$

So choose $\alpha = -1$ and $\beta = -2$.

Then the previous equation becomes

$$\left(4 + 3(-2) + 2 \cdot 2\right) u_{\xi\eta} = 0 \quad \Rightarrow \quad -u_{\xi\eta} = 0$$

$\Rightarrow u_{\xi\eta} = 0$. What is the solution of $\frac{\partial^2 u}{\partial\xi\partial\eta} = 0$?

Well $\boxed{\frac{\partial^2 u}{\partial\xi\partial\eta}(\xi,\eta) = 0}$ (R) $\Rightarrow u(\xi,\eta) = F(\xi) + G(\eta)$

for two arbitrary functions F and G.

<u>Motivation</u>: recall that if I have a function of one variable, say $h(x)$, then $\int dx \left(\frac{d}{dx} h(x)\right) = h(x) + \text{const}$.

So if we integrate (R) in $d\xi$ and regard η as a constant (or better, as a parameter) we have

$$\int d\xi \, \frac{\partial}{\partial\xi}\left(\frac{\partial u}{\partial\eta}(\xi,\eta)\right) = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial\eta}(\xi,\eta) + g(\eta) = 0$$

Integrating again in $d\eta$:

$$\int d\eta \left(\frac{\partial u}{\partial\eta} + g(\eta)\right) = u(\xi,\eta) + \bar{G}(\eta) + \bar{F}(\xi) = 0$$

$\Rightarrow u(\xi,\eta) = -\bar{G}(\eta) - \bar{F}(\xi)$ with $\bar{G}, \bar{F}$ generic functions

$\Rightarrow u(\xi,\eta) = G(\eta) + F(\xi)$ having called $-\bar{F} = F$ and $-\bar{G} = G$.

The expression $u_{\xi\eta} = 0$ is called the <u>CANONICAL FORM</u> of equation (E).

$$u(\xi, \eta) = F(\xi) + G(\eta) \Rightarrow \boxed{u(x,y) = F(x-y) + G(x-2y)}$$

general solution of equation (E)

Imposing boundary conditions gives F and G.
Consider for example the bound. cond.

$$u(x, 0) = 0$$

$$\frac{\partial u}{\partial y}(x, 0) = \frac{x}{2}$$

$u(x, 0) = 0$ gives $F(x) + G(x) = 0 \Rightarrow -F(x) = +G(x)$

$$\Rightarrow u(x, y) = F(x-y) - F(x-2y)$$

$$\frac{\partial u}{\partial y}(x, y) = -F'(x-y) + 2F'(x-2y)$$

$$\frac{\partial u}{\partial y}(x, 0) = -F'(x) + 2F'(x) = F'(x) = \frac{x}{2}$$

$$\Rightarrow F(x) = \int \frac{x}{2} = \frac{x^2}{4} + C$$

$$\Rightarrow u(x, y) = \frac{(x-y)^2}{4} + \cancel{C} - \frac{(x-2y)^2}{4} - \cancel{C} =$$

$$= \frac{(x-y)^2}{4} - \frac{(x-2y)^2}{4}$$