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Long time asymptotics of a Brownian particle coupled with a random environment with non-diffusive feedback force

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Abstract

We study the long time behavior of a Brownian particle moving in an anomalously diffusing field, the evolution of which depends on the particle position. We prove that the process describing the asymptotic behavior of the Brownian particle has bounded (in time) variance when the particle interacts with a subdiffusive field; when the interaction is with a superdiffusive field the variance of the limiting process grows in time as $t^{2\gamma-1}$, $1/2 < \gamma < 1$. Two different kinds of superdiffusing (random) environments are considered: one is described through the use of the fractional Laplacian; the other via the Riemann–Liouville fractional integral. The subdiffusive field is modeled through the Riemann–Liouville fractional derivative.

Keywords: Anomalous diffusion; Riemann–Liouville fractional derivative (integral); Fractional Laplacian; Continuous time random walk; Lévy flight; Scaling limit; Interface fluctuations

1. Introduction

In [2], Bertini et al. considered the following system of Itô-SDEs, describing the evolution of a one-dimensional interface:

$$\begin{cases} dX(t) = \lambda dw(t) + \alpha \left\langle \varphi_{X(t)}, h(t) \right\rangle dt \\ dh(t) = \frac{1}{2} \Delta h(t) dt - \varphi_{X(t)} dX(t), \end{cases} \tag{1}$$

with initial conditions $X(0) = h(0) = 0$. In the above system $w(t)$ is a one-dimensional Brownian motion (BM) on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ (E is going to denote

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expectation with respect to P) and $\langle \cdot, \cdot \rangle$ is the scalar product of $L^2(\mathbb{R}, dx)$ (for background material on probability theory and stochastic differential equations we refer the reader to [3,8,19]). More precisely, in [2] the authors consider a system thermally isolated from the exterior, in a state in which two phases coexist. Under the assumption of planar symmetry for the system, the interface position is represented by the point $X(t) \in \mathcal{C}(\mathbb{R}_+)$ separating the two phases. In Eq. $(1)_1$ the interface displacements are described as the sum of two contributions: the first is a Brownian fluctuation, related to the macroscopic fluctuations of the system; the second is the interaction with a diffusive field, $h(t) = h(t, x) \in \mathcal{C}(\mathbb{R}_+; \mathcal{C}(\mathbb{R}))$. Also,

$$\left\langle \varphi_{X(t)}, h(t) \right\rangle = \int_{\mathbb{R}} dx \varphi(x - X(t)) h(t, x),$$

where $\varphi(x)$ is a probability density in the Schwartz class (regions of the field far from the interface do not significantly affect the interface evolution) and $\varphi_{X(t)} = \varphi(x - X(t))$.

On the other hand, Eq. $(1)_2$ describes the field variation as the sum of a diffusive term plus a “feedback term” taking into account the latent heat effect.

The parameters $\lambda > 0$ and $\alpha > 0$ determine the intensity of the Brownian noise and of the coupling with the field, respectively. In [2] the authors study a scaling limit of $X(t)$ as $\lambda \to 0$ under the hypothesis $\alpha = \lambda$ of weak coupling.

Notice that the system (1) can also be interpreted as describing a Brownian motion weakly coupled with a (diffusive) random environment, the evolution of which depends on the position of the Brownian motion itself. For further details about the model we refer the reader to [2,1].

Let $\xi(t)$ be the solution of the following integral equation:

$$\xi(t) = \bar{b}(t) - \int_o^t ds \rho_{t-s}(0) \xi(s), \tag{2}$$

where $\bar{b}(t)$ is the scaled BM $\bar{b}(t) = \lambda w(t\lambda^{-2})$ and $\rho_t(x) = \rho(t, x)$ is the density of a centered Gaussian with variance t. In [2] the following asymptotics (3) and (4) are obtained: upon rescaling the interface position, i.e. considering the process $X_\lambda(t) = X(t\lambda^{-2})$, we have that $\forall N \in [1, \infty) \, \exists \, \tau = \tau(N) > 0$ s.t.

$$\lim_{\lambda \to 0} E \sup_{t \le \tau |\log \lambda|} |X_\lambda(t) - \xi(t)|^N = 0. \tag{3}$$

As noticed in [2], this implies that X_λ converges weakly to ξ as $\lambda \to 0$ in $\mathcal{C}(\mathbb{R}_+)$ endowed with the topology of uniform convergence on compacts. Furthermore, $\xi(t)$ is a centered Gaussian process such that

$$\lim_{t \to \infty} \frac{1}{\log(t)} E\,[\xi(t)]^2 = \frac{2}{\pi}; \tag{4}$$

that is, the width of the interface fluctuations increases in time as $\log(t)$.

However, a number of natural phenomena cannot be described by means of simple diffusion; e.g., the way some proteins diffuse across cell membranes or the motion of a particle in systems with geometric constraints, for example on the surface of a perfect crystal. Therefore, it can be of interest to consider systems of SDEs analogous to (1) and in which the Brownian particle interacts with anomalously diffusing fields. The present paper is devoted to extending the results obtained in [2] for system (1) to the case in which the interface fluctuations are due to interactions with anomalously diffusing fields. In other words, we will study the long time behavior of a Brownian particle coupled with an anomalously diffusing environment (see systems (10)–(12)).

Anomalous diffusion processes are characterized by a mean square displacement which, instead of growing linearly in time, grows like $t^{2\gamma}$, $\gamma > 0, \gamma \neq \frac{1}{2}$. When $0 < \gamma < \frac{1}{2}$ the process is subdiffusive; when $\gamma > \frac{1}{2}$ it is superdiffusive.

Diffusion phenomena can be described at the microscopic level by means of BM and macroscopically by means of the heat equation, i.e. the parabolic problem associated with the Laplacian operator; the link between the two descriptions is, roughly speaking, the fact that the fundamental solution to the diffusion equation is the probability density associated with BM.

A similar picture can be obtained for anomalous diffusion. The main difference is that in nature a variety of anomalous diffusion phenomena can be observed and the question is how to characterize them from both the analytical and the statistical points of view. It has been shown that the microscopical (probabilistic) approach can be understood in the context of continuous time random walks (CTRWs) and, in this framework, a process is uniquely determined once the probability density for moving at distance r in time t is known ([4,10,14,9,5,6,13] and references therein). The analytical approach is based on the theory of fractional differentiation operators, where the derivative can be fractional either in time or in space (see [12,11,15,17] and references therein).

For $f(s)$ regular enough (e.g. $f \in \mathcal{C}(0,t]$ with an integrable singularity at $s = 0$), let us introduce the Riemann–Liouville fractional derivative,

$$D_t^{\gamma}(f) := \frac{1}{\Gamma(2\gamma)}\frac{d}{dt}\int_0^t ds \frac{f(s)}{(t-s)^{1-2\gamma}}, \quad 0 < \gamma < \frac{1}{2}, \tag{5}$$

and the Riemann–Liouville fractional integral,

$$I_t^{\gamma}(f) := \frac{1}{\Gamma(2\gamma-1)}\int_0^t ds \frac{f(s)}{(t-s)^{2-2\gamma}}, \quad \frac{1}{2} < \gamma < 1, \tag{6}$$

where Γ is the Euler Gamma function [15]. Appendix B contains a motivation for introducing such operators. For $\frac{1}{2} < \gamma < 1$ let us also introduce the fractional Laplacian $\Delta^{(\gamma)}$, defined through its Fourier transform: if the Laplacian corresponds, in Fourier space, to a multiplication by $-k^2$, the fractional Laplacian corresponds to a multiplication by $-|k|^{\frac{1}{\gamma}}$. (5) and (6) can be defined in a more general way (see [15]), but for our purposes the above definition is sufficient. Furthermore, notice that the operators in (5) and (6) are fractional in time, whereas the fractional Laplacian is fractional in space.

Let us now consider the function $\rho^{\gamma}(t,x)$, the solution to

$$\partial_t \rho^{\gamma}(t,x) = \frac{1}{\Gamma(2\gamma)}\frac{d}{dt}\int_0^t ds\, \frac{\partial_x^2 \rho^{\gamma}(s,x)}{(t-s)^{1-2\gamma}} \quad \text{when } 0 < \gamma < \frac{1}{2}, \tag{7}$$

$$\partial_t \rho^{\gamma}(t,x) = \frac{1}{\Gamma(2\gamma-1)}\int_0^t ds\, \frac{\partial_x^2 \rho^{\gamma}(s,x)}{(t-s)^{2-2\gamma}} \quad \text{when } \frac{1}{2} < \gamma < 1. \tag{8}$$

It has been shown (see [10,6] and references therein) that such a kernel is the asymptotic of the probability density of a CTRW run by a particle either moving at constant velocity between stopping points or instantaneously jumping between halt points, where it waits a random time before jumping again. On the other hand, a classic result states that the Fourier transform of the solution $\rho^\gamma(t,x)$ to

$$\partial_t \rho^\gamma(t,x) = \frac{1}{2}\Delta^{(\gamma)}\rho^\gamma(t,x), \quad \frac{1}{2} < \gamma < 1, \tag{9}$$

is, for $\gamma \geq \frac{1}{2}$, the characteristic function of a (stable) process whose first moment is divergent when $\gamma \geq 1$ (see [17]); this justifies the choice $\frac{1}{2} < \gamma < 1$ in Eq. (9). Processes of this kind are particular CTRWs, the well known Lévy flights; in this case large jumps are allowed with non-negligible probability and this results in the process having divergent second moment.

We will use the notation $\rho^\gamma(t,x) = \rho_t^\gamma(x)$ to indicate the solution to either (7), (8) or (9), as in the proofs we use only the properties that these kernels have in common.

The above described framework is analogous to that of Einstein diffusion: for subdiffusion and Riemann-type superdiffusion the statistical description is given by CTRWs, whose (asymptotical) density is the fundamental solution of the evolution equation associated with the operators of fractional differentiation and integration, i.e. (7) and (8), respectively (see Appendix B). For the Lévy-type superdiffusion, the statistical point of view is given by Lévy flights, whose probability density evolves in time according to the evolution equation associated with the fractional Laplacian, i.e. (9) (see [17]).

In view of the above considerations, we introduce the following three systems of Itô-SDEs:

$$\begin{cases} dX^\gamma(t) = \lambda^{\frac{1}{2\gamma}} dw(t) + \lambda^{\frac{1}{\gamma}-1}\left\langle \varphi_{X^\gamma(t)}, h^\gamma(t)\right\rangle dt \\ dh^\gamma(t) = \dfrac{1}{\Gamma(2\gamma)}\dfrac{d}{dt}\displaystyle\int_0^t ds \frac{\partial_x^2 h^\gamma(s,x)}{(t-s)^{1-2\gamma}} dt - \varphi_{X^\gamma(t)} dX^\gamma(t), \end{cases} \tag{10}$$

$$\begin{cases} dX^\gamma(t) = \lambda^{\frac{1}{2\gamma}} dw(t) + \lambda^{\frac{1}{\gamma}-1}\left\langle \varphi_{X^\gamma(t)}, h^\gamma(t)\right\rangle dt \\ dh^\gamma(t) = \dfrac{1}{\Gamma(2\gamma-1)}\displaystyle\int_0^t ds \frac{\partial_x^2 h^\gamma(s,x)}{(t-s)^{2-2\gamma}} dt - \varphi_{X^\gamma(t)} dX^\gamma(t), \end{cases} \tag{11}$$

and

$$\begin{cases} dX^\gamma(t) = \lambda^{\frac{1}{2\gamma}} dw(t) + \lambda^{\frac{1}{\gamma}-1}\left\langle \varphi_{X^\gamma(t)}, h^\gamma(t)\right\rangle dt \\ dh^\gamma(t) = \dfrac{1}{2}\Delta^{(\gamma)} h^\gamma(t) dt - \varphi_{X^\gamma(t)} dX^\gamma(t). \end{cases} \tag{12}$$

Roughly speaking, the first two systems are obtained from (1), by replacing the Laplacian of the field $h(t,x)$ in Eq. $(1)_2$ with the fractional derivative and fractional integral of $\Delta h(t,x)$, respectively (see (7) and (8)). The last system is obtained by replacing the Laplacian with the fractional Laplacian (see (9)). In this way we model our anomalously diffusing fields.

Again, $w(t)$ is a one-dimensional BM, $\varphi(x)$ is a function in the Schwartz class and $\varphi_{X^\gamma(t)} = \varphi(x - X^\gamma(t))$. A more detailed motivation for introducing the above systems can be found in Appendix B.

We shall denote by $X^\gamma(t)$ the solution to either of the three above systems (the reason for adopting this notation, which might at first seem confusing, will be apparent in a few lines). For $\lambda \in (0,1)$, let us introduce the scaled variables

$$X^{(\lambda,\gamma)}(t) := X^\gamma\left(t\lambda^{-\frac{1}{\gamma}}\right),$$

$$h^{(\lambda,\gamma)}(t,x) := \frac{1}{\lambda} h^\gamma\left(x\lambda^{-1}, t\lambda^{-\frac{1}{\gamma}}\right),$$

$$\varphi^{(\lambda)}(x) := \frac{1}{\lambda}\varphi\left(x\lambda^{-1}\right).$$

For the function φ only, we use the convention $\varphi_a(x) := \varphi(x-a)$, $a \in \mathbb{R}$ and we set

$$\varphi_t^{(\lambda)}(x) := \varphi^{(\lambda)}_{\lambda X^{(\lambda,\gamma)}} = \frac{1}{\lambda}\varphi\left(x\lambda^{-1} - X^{(\lambda,\gamma)}(t)\right); \tag{13}$$

the notation for $\varphi_t^{(\lambda)}$ should include the superscript $^\gamma$, which we omit.

Let also $\xi^\gamma(t)$ be the solution to the integral equation

$$\xi^\gamma(t) = b(t) - \int_0^t ds \rho^\gamma_{t-s}(0)\xi^\gamma(s), \quad \xi^\gamma(0) = 0, 0 < \gamma < 1, \tag{14}$$

where $b(t) = \lambda^{\frac{1}{2\gamma}} w(t\lambda^{-\frac{1}{\gamma}})$. Notice that, by virtue of the scaling property of Brownian motion, the dependence of $\xi^\gamma(t)$ on λ through $b(t)$ is only formal. The main result presented in this paper is a scaling limit (in fact, three scaling limits) of $X^{(\lambda,\gamma)}(t)$ going to $\xi^\gamma(t)$. Also, the solution to (14) is unique by the basic facts of the theory of Volterra integral equations, which we shall recall at the beginning of Section 3.

Theorem 1 (*First Version*)**.** *With the notation introduced above, we have that* $\forall\gamma \in (0,1)$ *and* $\forall N \in [1,\infty)$ *there exists* $\tau = \tau(N,\gamma) > 0$ *such that*

$$\lim_{\lambda\to 0} E \sup_{t \le \tau |\log\lambda|^{\frac{1}{C(\gamma)}}} |X^{(\lambda,\gamma)}(t) - \xi^\gamma(t)|^N = 0,$$

where $C(\gamma)$ *is a positive constant, with* $C(1/2) = 1$.

The fact that $C(1/2) = 1$ is consistent with (3). In Section 4 we prove an equivalent version of Theorem 1, namely Theorem 3, where the constant $C(\gamma)$ is made explicit. Theorem 1 says that the asymptotic behavior of $X^{(\lambda,\gamma)}(t)$, the rescaled solution to either one of the systems (10)–(12), is described by the function $\xi^\gamma(t)$. Hence, we need to determine the behavior of $\xi^\gamma(t)$ for large t, and this is the content of the following Theorem 2.

Theorem 2. *For* $\gamma \in (0, \frac{1}{2})$, $\xi^\gamma(t)$ *is a centered Gaussian process s.t.*

$$\lim_{t\to\infty} E\left[\xi^\gamma(t)\right]^2 = const. \tag{15}$$

For $\gamma \in \left(\frac{1}{2}, 1\right)$, *we prove an invariance principle for* $\xi^\gamma(t)$. *Let* $\xi^\gamma_\epsilon(t) = \epsilon^{\gamma-\frac{1}{2}}\xi^\gamma(\epsilon^{-1}t)$; *then, as* $\epsilon \to 0$, ξ^γ_ϵ *converges weakly in* $\mathcal{C}(\mathbb{R}_+)$ *to a mean-zero Gaussian process,* $Z(t)$, *whose covariance function is*

$$E(Z(s)Z(t)) = \frac{\sin^2(\pi\gamma)}{\pi^2 c(\gamma)^2}\int_0^{t\wedge s} du \frac{1}{(t-u)^{1-\gamma}(s-u)^{1-\gamma}}.$$

Intuitively, this means that in the case in which the particle interacts with a subdiffusive field, the feedback force exerted by the field keeps the process localized. On the other hand, the superdiffusive field (no matter which one of the two we consider) is not strong enough to overcome the effect of the Brownian nature of the particle, and the width of the fluctuation increases in time as $t^{2\gamma-1}$.

Notice also that the CTRWs associated with the operators D_t^γ and I_t^γ are non-Markovian whereas Lévy processes are Markovian processes; nevertheless the limiting process (14) is non-Markovian for any $\gamma \in (0,1)$: in the case of Lévy-type superdiffusion there is loss of Markovianity.

The paper is organized as follows. In Section 2, after establishing the notation, we state a second (equivalent) version of Theorem 1. This version is the one that we shall actually prove in Section 4. Section 3 contains all the technical estimates used in Section 4. This proof is a generalization of the one used in [2] in order to prove (3). Section 5 is devoted to the proof of Theorem 2, which relies on the use of Tauberian theorems. Finally, Appendix A provides a sketch of the proof of the existence, uniqueness and continuity of the paths of the solution to (10)–(12). Appendix B contains a more detailed motivation for the introduction of the operators of fractional differentiation and integration.

2. Notation and results

The kernels in (7) and (8) can be explicitly written both in integral form (see Appendix B):

$$\rho^\gamma(t,x) = \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} dz\, e^{zt}\,\frac{e^{-|x|z^\gamma}}{z^{1-\gamma}} \quad \forall c>0 \text{ and } 0<\gamma<1 \tag{16}$$

and as a series:

$$\rho^\gamma(t,x) = \frac{1}{2t^\gamma} M\left(\frac{|x|}{t^\gamma},\gamma\right), \quad 0<\gamma<1, \text{ where}$$

$$M(z,\gamma) := \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!\,\Gamma(-\gamma(k+1)+1)}. \tag{17}$$

The asymptotic behavior of the *Mainardi function* $M(z,\gamma)$ as $z \to +\infty$ is known:

$$M(z,\gamma) \simeq A(\gamma)\, z^{\frac{2\gamma-1}{2-2\gamma}}\, e^{-B(\gamma)\, z^{\frac{1}{1-\gamma}}},$$

with A and B constants depending on γ [11]; hence $\rho^\gamma(t,x)$ has finite moments of any orders, given by

$$\int_{\mathbb{R}} dx\, \rho^\gamma(t,x)|x|^n < \infty, \quad \forall n \in N.$$

We remark that this property holds when $\rho^\gamma(t,x)$ is the fundamental solution of either (7) or (8). On the other hand, the fundamental solution of (9), namely

$$\rho^\gamma(t,x) = \int_{\mathbb{R}} e^{-\frac{1}{2}t|k|^{\frac{1}{\gamma}}} e^{ikx} dk, \quad \gamma \in (1/2,1), \tag{18}$$

has finite first moment but divergent second moment.

We want to stress that in order to prove Theorem 1 (i.e. Theorem 3), we basically use only the following elementary properties enjoyed by both (16) and (18):

- the scaling property:

$$\rho^\gamma(t,x) = \frac{1}{t^\gamma}\rho^\gamma\left(1,\frac{x}{t^\gamma}\right), \tag{19}$$

from which, setting

$$c(\gamma) := \rho_1^\gamma(0), \tag{20}$$

$$\rho_{t-s}^\gamma(0) = \frac{\rho_1^\gamma(0)}{(t-s)^\gamma} = \frac{c(\gamma)}{(t-s)^\gamma}; \tag{21}$$

- there exists a generic constant $C = C(\gamma) > 0$ such that

$$\left|\frac{\rho_1^\gamma(z)}{c(\gamma)} - 1\right| \le C|z|^\beta, \text{[1]} \quad \forall \beta \in (0,1], \tag{22}$$

and

$$\left|\frac{\rho_1^\gamma(z)}{c(\gamma)} - 1\right| \le C. \text{[2]} \tag{23}$$

For $f, g \in L^2([0,t])$, $f * g$ denotes the *Volterra convolution*, namely

$$(f * g)(t) := \int_0^t ds f(t-s) g(s).$$

For $m \in \mathbb{N}, m \ge 2$, $f^{*(m)} = f * f^{(m-1)}$ is the convolution of f with itself $(m-1)$ times, where we define $f^{*(1)}(t) := f(t)$. Set $\mathbb{K}_\gamma(t) := \rho_t^\gamma(0)$ and notice that

$$\begin{aligned}\mathbb{K}_\gamma^{*(2)}(t-s) = \int_0^{t-s} ds' \rho_{t-s-s'}^\gamma(0)\rho_{s'}^\gamma(0) &= \int_s^t ds' \rho_{t-s'}^\gamma(0)\rho_{s'-s}^\gamma(0) \\ &= k_{(1)}(\gamma)(t-s)^{1-2\gamma}.\end{aligned} \tag{24}$$

If we iterate n times, we end up with

$$\begin{aligned}\mathbb{K}_\gamma^{*(n+1)}(t-s) &:= \int_s^t ds' \rho_{t-s'}^\gamma(0)\mathbb{K}_\gamma^{*(n)}(s'-s) \\ &= k_{(n)}(\gamma)(t-s)^{n-(n+1)\gamma}, \quad n \ge 1,\end{aligned} \tag{25}$$

where

$$k_{(n)}(\gamma) := c(\gamma)^{n+1} \frac{\Gamma(1-\gamma)^{n+1}}{\Gamma((n+1)(1-\gamma))}. \tag{26}$$

To obtain the previous equality we used the fact that the Beta function $B(z,w)$ can be expressed in terms of the Euler Gamma function in the following way:

$$B(z,w) \stackrel{def}{=} \int_0^1 ds\, s^{z-1}(1-s)^{w-1} = \frac{\Gamma(z)\,\Gamma(w)}{\Gamma(z+w)} \quad Re(z), Re(w) > 0.$$

[1] This inequality can be deduced by using (17) when referring to Riemann-type anomalous diffusion; see also footnote 3. For when ρ_t^γ is the kernel in (18), see footnote 3.

[2] The constant that appears in this inequality is equal to 1 when ρ^γ is either the Lévy-type kernel or the subdiffusive kernel and it depends on γ otherwise; see again footnote 3.

In the same way, on setting

$$P_{t,s}^{(\lambda,\gamma)} := \left\langle \varphi_t^{(\lambda)}, \rho_{t-s}^{\gamma} \varphi_s^{(\lambda)} \right\rangle = P_{t,s}^{*(1)}, \tag{27}$$

(on the RHS we drop the superscript $^{(\lambda,\gamma)}$ for notational convenience) we have

$$P_{t,s}^{*(2)} = \int_s^t ds' P_{t,s'}^{(\lambda,\gamma)} P_{s',s}^{(\lambda,\gamma)},$$

and for $n \geq 1$

$$P_{t,s}^{*(n+1)} := \int_s^t ds' P_{t,s'}^{(\lambda,\gamma)} P_{s',s}^{*(n)}. \tag{28}$$

We further introduce

$$\begin{aligned}
K_{t,s}^{(\lambda,\gamma)} &:= \left\langle \varphi_t^{(\lambda)}, \int_s^t db(s') \rho_{t-s'}^{\gamma} \varphi_{s'}^{(\lambda)} \right\rangle, \\
F_0^{(\lambda,\gamma)}(t) &:= -\int_0^t ds K_{s,0}^{(\lambda,\gamma)}, \\
F_1^{(\lambda,\gamma)}(t) &:= \int_0^t ds P_{t,s}^{(\lambda,\gamma)} K_{s,0}^{(\lambda,\gamma)}, \\
F_2^{(\lambda,\gamma)}(t) &:= -\int_0^t ds \int_0^s ds' P_{t,s}^{(\lambda,\gamma)} P_{s,s'}^{(\lambda,\gamma)} K_{s',0}^{(\lambda,\gamma)} \\
&= -\int_0^t ds K_{s,0}^{(\lambda,\gamma)} P_{t,s}^{*(2)},
\end{aligned} \tag{29}$$

and in general

$$F_n^{(\lambda,\gamma)}(t) := (-1)^{n+1} \int_0^t ds K_{s,0}^{(\lambda,\gamma)} P_{t,s}^{*(n)}, \quad n \geq 1 \tag{30}$$

$$= -\int_0^t ds P_{t,s}^{(\lambda,\gamma)} F_{n-1}^{(\lambda,\gamma)}(s) \quad n \geq 2. \tag{31}$$

Via the Duhamel principle (see Lemma 2), systems (10)–(12) can be expressed in integral form through a unique system, that is,

$$\begin{cases} X^{(\lambda,\gamma)}(t) = b(t) + \int_0^t ds \left\langle \varphi_s^{(\lambda)}, h^{(\lambda,\gamma)}(s) \right\rangle \\ h^{(\lambda,\gamma)}(t) = -\int_0^t db(s) \rho_{t-s}^{\gamma} \varphi_s^{(\lambda)} - \int_0^t ds \left\langle \varphi_s^{(\lambda)}, h^{(\lambda,\gamma)}(s) \right\rangle \rho_{t-s}^{\gamma} \varphi_s^{(\lambda)}, \end{cases} \tag{32}$$

where $\gamma \in (0,1)$; in the above system $\rho_t^{\gamma}(x) = \rho^{\gamma}(t,x)$ is either (16) for $\gamma \in (0,1)$ or (18) for $\gamma \in (1/2, 1)$. For any f in the Schwartz class, $\left(\rho_t^{\gamma} f\right)(x)$ is a convolution in the space variable. Namely,

$$\rho_{t-s}^{\gamma} \varphi_s^{(\lambda)} = \left(\rho_{t-s}^{\gamma} \varphi_s^{(\lambda)}\right)(x) = \int_{\mathbb{R}} dy \rho_{t-s}^{\gamma}(x-y) \varphi_s^{(\lambda)}(y).$$

The initial conditions for (32) are $X^{(\lambda,\gamma)}(0) = h^{(\lambda,\gamma)}(0) = 0$. In Appendix A we prove that (32) admits a unique solution in $\mathcal{C}(\mathbb{R}_+; \mathbb{R} \times L^2(\mathbb{R}))$. Notice as well that from (32) one has

$$h^{(\lambda,\gamma)}(t) = -\int_0^t dX^{(\lambda,\gamma)}(s)\rho^{\gamma}_{t-s}\varphi^{(\lambda)}_s. \tag{33}$$

Following [2] page 10, we formally iterate once both the equation for $X^{(\lambda,\gamma)}$ and the one for ξ^{γ}, $(32)_1$ and (14), respectively. Plugging $(32)_2$ into $(32)_1$ and using (33), we get

$$\begin{aligned} X^{(\lambda,\gamma)}_{(1)}(t) &= b(t) - \int_0^t ds K^{(\lambda,\gamma)}_{s,0} \\ &\quad + \int_0^t ds \left\langle \varphi^{(\lambda)}_s, \int_0^s ds' \left\langle \varphi^{(\lambda)}_{s'}, \int_0^{s'} dX^{(\lambda,\gamma)}_{(1)}(s'')\rho^{\gamma}_{s'-s''}\varphi^{(\lambda)}_{s''} \right\rangle \rho^{\gamma}_{s-s'}\varphi^{(\lambda)}_{s'} \right\rangle \\ &= b(t) + F^{(\lambda,\gamma)}_0(t) + \int_0^t ds \int_0^s ds' P^{(\lambda,\gamma)}_{s,s'} \left\langle \varphi^{(\lambda)}_{s'}, \int_0^{s'} dX^{(\lambda,\gamma)}_{(1)}(s'')\rho^{\gamma}_{s'-s''}\varphi^{(\lambda)}_{s''} \right\rangle, \end{aligned} \tag{34}$$

where the subscript $_{(1)}$ of X is to recall that we are considering the first iteration of $(32)_1$. Setting $Y^{(\lambda,\gamma)}_{(1)}(t) = X^{(\lambda,\gamma)}_{(1)}(t) - b(t) - F^{(\lambda,\gamma)}_0(t)$, $Y^{(\lambda,\gamma)}_{(1)}(t)$ solves

$$\begin{aligned} Y^{(\lambda,\gamma)}_{(1)}(t) &= \int_0^t ds F^{(\lambda,\gamma)}_1(s) + \int_0^t ds F^{(\lambda,\gamma)}_2(s) \\ &\quad + \int_0^t ds \int_0^s ds' P^{(\lambda,\gamma)}_{s,s'} \left\langle \varphi^{(\lambda)}_{s'}, \int_0^{s'} dY^{(\lambda,\gamma)}_{(1)}(s'')\rho^{\gamma}_{s'-s''}\varphi^{(\lambda)}_{s''} \right\rangle; \end{aligned} \tag{35}$$

observing that $Y^{(\lambda,\gamma)}_{(1)}(t)$ is a.s. in $\mathcal{C}^1(\mathbb{R})$, we can rewrite the previous expression for $Y^{(\lambda,\gamma)}_{(1)}(t)$ as

$$\dot{Y}^{(\lambda,\gamma)}_{(1)}(t) = F^{(\lambda,\gamma)}_1(t) + F^{(\lambda,\gamma)}_2(t) + \int_0^t ds \dot{Y}^{(\lambda,\gamma)}_{(1)}(s) \int_s^t ds' P^{(\lambda,\gamma)}_{t,s'} P^{(\lambda,\gamma)}_{s',s}, \tag{36}$$

and hence

$$X^{(\lambda,\gamma)}_{(1)}(t) := \int_0^t ds \dot{Y}^{(\lambda,\gamma)}_{(1)}(s) + b(t) + F^{(\lambda,\gamma)}_0(t). \tag{37}$$

On the other hand, iterating the equation for ξ^{γ} and using (24), we get

$$\xi^{\gamma}_{(1)}(t) = b(t) - \int_0^t ds \rho^{\gamma}_{t-s}(0) b(s) + k_{(1)}(\gamma) \int_0^t ds (t-s)^{1-2\gamma} \xi^{\gamma}_{(1)}(s). \tag{38}$$

We can repeat the same procedure n times; for $n \geq 2$ we then have

$$X^{(\lambda,\gamma)}_{(n)}(t) := b(t) + F^{(\lambda,\gamma)}_0(t) + \int_0^t ds [F^{(\lambda,\gamma)}_1 + \cdots + F^{(\lambda,\gamma)}_n](s) + Y^{(\lambda,\gamma)}_{(n)}(t), \tag{39}$$

where

$$Y^{(\lambda,\gamma)}_{(n)}(t) := (-1)^{n+1} \int_0^t ds \left\langle \varphi^{(\lambda)}_s, \int_0^s dX^{(\lambda,\gamma)}_{(n)}(u)\rho^{\gamma}_{s-u}\varphi^{(\lambda)}_u \right\rangle \int_s^t ds' P^{*(n)}_{s',s}. \tag{40}$$

$Y_{(n)}^{(\lambda,\gamma)}(t)$ solves the equation

$$Y_{(n)}^{(\lambda,\gamma)}(t) = \int_0^t ds \left[F_n^{(\lambda,\gamma)} + \cdots + F_{2n}^{(\lambda,\gamma)} \right](s) + (-1)^{n+1} \int_0^t ds \left\langle \varphi_s^{(\lambda)}, \int_0^s dY_{(n)}^{(\lambda,\gamma)}(u) \rho_{s-u}^{\gamma} \varphi_u^{(\lambda)} \right\rangle \int_s^t ds' P_{s',s}^{*(n)}, \tag{41}$$

so by differentiating, using the definition of $P_{t,s}^{(\lambda,\gamma)}$ and (28), we get

$$\dot{Y}_{(n)}^{(\lambda,\gamma)}(t) = \left[F_n^{(\lambda,\gamma)} + \cdots + F_{2n}^{(\lambda,\gamma)} \right](t) + (-1)^{n+1} \int_0^t ds \dot{Y}_{(n)}^{(\lambda,\gamma)}(s) P_{t,s}^{*(n+1)}. \tag{42}$$

Define $A_{(n)}^{\gamma}(t)$ as

$$A_{(n)}^{\gamma}(t) := \sum_{\nu=0}^{n} (-1)^{\nu} \left(\mathbb{K}_{\gamma}^{*(\nu)} * b \right)(t) \quad n \geq 1, 0 < \gamma < \frac{n}{n+1}, \tag{43}$$

where $\left(\mathbb{K}_{\gamma}^{*(0)} * b \right)(t)$ is only formal and we set it to be equal to $b(t)$. Then, at the nth iteration of the equation for the limiting process $\xi^{\gamma}(t)$, we find that $\forall n \geq 1$,

$$\xi_{(n)}^{\gamma}(t) = A_{(n)}^{\gamma}(t) + (-1)^{n+1} \int_0^t ds \xi_{(n)}^{\gamma}(s) \mathbb{K}^{*(n+1)}(t-s). \tag{44}$$

When we write $X_{(n)}^{(\lambda,\gamma)}$, we refer to the expression (39) if $n \geq 2$ and to (37) if $n = 1$. As for $Y_{(n)}^{(\lambda,\gamma)}$ and $\dot{Y}_{(n)}^{(\lambda,\gamma)}$, expressions (41) and (42) coincide with (35) and (36) respectively, when $n = 1$. So $Y_{(n)}^{(\lambda,\gamma)}$ and $\dot{Y}_{(n)}^{(\lambda,\gamma)}$ are defined by (41) and (42) $\forall n \geq 1$.

To prove convergence of $X^{(\lambda,\gamma)}$ to ξ^{γ} we prove convergence of the nth iterates. More precisely, we prove that $\forall n \geq 1$, $X_{(n)}^{(\lambda,\gamma)}$ converges to $\xi_{(n)}^{\gamma}$ (in the sense of Theorem 3 below) when $\gamma \in \left(0, \frac{n}{n+1}\right)$.

The reason why we consider successive iterates of the equation for $X^{(\lambda,\gamma)}$ (and hence for ξ^{γ}) is to gain integrability and some sort of regularity. Notice indeed that $\int_0^t db(s) \rho_{t-s}^{\gamma}(0)$ is not well defined for $\gamma \geq 1/2$, whereas $\forall n \geq 1$

$$\int_0^t db(s) \mathbb{K}_{\gamma}^{*(n+1)}(t-s) \quad \text{is well defined for } \gamma \in \left(0, \frac{2n+1}{2(n+1)}\right). \tag{45}$$

$\forall n \geq 1$, we further restrict the range of γ to $\gamma \in \left(0, \frac{n}{n+1}\right)$ in view of (25) (see Remark 4.2 and (105), as well).

Theorem 3 (*That is, Second Version of* Theorem 1)**.** *With the notation introduced above, we have that for any integer $n \geq 1$, for any $\gamma \in \left(0, \frac{n}{n+1}\right)$ and $\forall N \in [1, \infty)$, there exists $\tau = \tau(N, \gamma) > 0$ s.t.*

$$\lim_{\lambda \to 0} E \sup_{t \leq \tau |\log \lambda|^{\frac{1}{(n+1)(1-\gamma)}}} |X_{(n)}^{(\lambda,\gamma)}(t) - \xi_{(n)}^{\gamma}(t)|^N = 0. \tag{46}$$

$\|\cdot\|_p$, $p \geq 1$, indicates the usual $L^p(\mathbb{R}, dx)$ norm and $(\rho_t^\gamma f)(x) = \int dy \rho_t^\gamma (x-y) f(y)$ is a convolution in space. Now a few observations: $\forall t > 0$ and $\forall n \geq 1$,

$$\varphi_t^{(\lambda)} = \frac{1}{\lambda}\varphi\left(x\lambda^{-1} - X^{(\lambda,\gamma)}(t)\right) = \frac{1}{\lambda}\varphi\left(x\lambda^{-1} - X_{(n)}^{(\lambda,\gamma)}(t)\right), \quad \gamma \in (0,1); \tag{47}$$

so actually the notation for $\varphi_t^{(\lambda)}$, like that for $K_{t,s}^{(\lambda,\gamma)}$ and $\Gamma_s^{(\lambda,\delta,\gamma)}$, the latter defined in (79), should explicitly show the "dependence" on n, but we omit it. This also explains why in some estimates (for example (78)), n appears on the right hand side but not on the left hand side.

$\forall p \geq 1$ there exists a positive constant $C = C(p)$ s.t.

$$\|\varphi_t^{(\lambda)}\|_p \leq C\lambda^{\frac{1}{p}-1}. \tag{48}$$

Moreover, $\forall t > 0$,

$$\rho^\gamma(t,x) \leq B(\gamma)\rho^\gamma(t,0), \tag{49}$$

where $B(\gamma) = 1$ if ρ^γ is either the subdiffusive kernel or (18), and it is a positive constant actually depending on γ in the case of Riemann superdiffusion.[3] (49) implies that

$$P_{t,s}^{(\lambda,\gamma)} \leq B(\gamma)\rho_{t-s}^\gamma(0), \quad \forall\, 0 < s < t, \tag{50}$$

and

$$\left\langle \varphi^{(\lambda)}, \rho_t^\gamma \varphi^{(\lambda)} \right\rangle \leq B(\gamma)\rho_t^\gamma(0), \quad \forall t > 0. \tag{51}$$

From (50), we also have

$$P_{t,s}^{*(n)} \leq C\mathbb{K}_\gamma^{*(n)}(t-s), \tag{52}$$

where $C > 0$ is a generic constant depending on γ.

3. Technical lemmata

Throughout the following lemma we will make extensive use of the Volterra convolution introduced in Section 2. Notice that this convolution is commutative and that it enjoys the property

$$\left[\left(\int_0^\cdot du f(u)\right) * g\right](t) = \int_0^t du (f*g)(u), \tag{53}$$

which easily follows after a change of variable. Indeed

$$\begin{aligned}\left[\left(\int_0^\cdot du f(u)\right) * g\right](t) &= \int_0^t ds \left(\int_0^{t-s} du f(u)\right) g(s) \\ &= \int_0^t ds g(s) \int_s^t dv f(v-s) = \int_0^t dv \int_0^v ds f(v-s) g(s) \\ &= \int_0^t dv (f*g)(v).\end{aligned}$$

[3] A more detailed account and helpful plots of the kernels (16) can be found in [11] on page 1473; see also [15]. As for the kernel in (18), we recall that both ρ_t^γ and its first derivative in space belong to $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\forall t > 0$ and we refer the reader to [17].

Lemma 1. *For $n \in \mathbb{N}, n \geq 1$, consider the integral equation*

$$h_{(n)}(t) - (\mathbb{K}_\gamma^{*(n+1)} * h_{(n)})(t) = g(t), \quad g \in L^2([0,t]), \gamma \in \left(0, \frac{n}{n+1}\right). \tag{54}$$

Name as $\varkappa^\gamma_{(n)}(t)$ the solution to (54) when the forcing $g(t)$ is taken to be equal to $\mathcal{A}^\gamma_{(n)}(t) \in L^2([0,t])$ and as $\varsigma^\gamma_{(n)}(t)$ the solution to the same equation with a different forcing, say $\mathcal{G}^\gamma_{(n)}(t)$. Namely,

$$\varkappa^\gamma_{(n)}(t) + (-1)^n \int_0^t ds \varkappa^\gamma_{(n)}(s) k_{(n)}(\gamma)(t-s)^{n-(n+1)\gamma} = \mathcal{A}^\gamma_{(n)}(t) \tag{55}$$

and

$$\varsigma^\gamma_{(n)}(t) + (-1)^n \int_0^t ds \varsigma^\gamma_{(n)}(s) k_{(n)}(\gamma)(t-s)^{n-(n+1)\gamma} = \mathcal{G}^\gamma_{(n)}(t). \tag{56}$$

If the two forcings $\mathcal{A}^\gamma_{(n)}(t)$ and $\mathcal{G}^\gamma_{(n)}(t)$ are related through

$$(\mathcal{A}^\gamma_{(n)} * \mathbb{K}_\gamma^{*(n+1)})(t) = (-1)^{n+1} \int_0^t ds \mathcal{G}^\gamma_{(n)}(s), \tag{57}$$

then the same relation holds true between the corresponding solutions, i.e.,

$$(\varkappa^\gamma_{(n)} * \mathbb{K}_\gamma^{*(n+1)})(t) = (-1)^{n+1} \int_0^t ds \varsigma^\gamma_{(n)}(s). \tag{58}$$

The proof of this lemma is an immediate consequence of some basic facts in the theory of Volterra integral equations, which we recall here. For more details on this theory we refer the reader to [18]. For some $T > 0$, let $g(t), \mathcal{K}(t) \in L^2([0,T])$. Then the solution $h(t)$ to the equation

$$h(t) - \int_0^t ds \mathcal{K}(t-s) h(s) = g(s)$$

exists and is unique and can be expressed as

$$h(t) = g(t) - \int_0^t ds H(t-s) g(s), \tag{59}$$

where

$$H(t-s) = -\sum_{\nu=0}^{\infty} \mathcal{K}^{*(\nu+1)}(t-s).$$

When the kernel $\mathcal{K}(t)$ is not in L^2, the solution to (59) still exist and is unique provided that for some $n \in \mathbb{N}$ the iterated kernel $\mathcal{K}^{*(n)}$ is bounded on $[0,T]$. The proof of this fact can be found in [18], Section 1.12, where kernels of the form $\mathcal{K}(t) = t^\alpha$, with $\alpha \in (0,1)$ are considered.

Proof of Lemma 1. For $\gamma \in (0, n/n+1)$, the kernel of Eqs. (55) and (56) is a bounded continuous function, so the standard theory for kernels in L^2 applies. Thanks to (59), together with (55)–(57), proving (58) boils down to proving

$$(-1)^{n+1} \int_0^t ds \left(H * \mathcal{G}^\gamma_{(n)}\right)(s) = \left(\mathbb{K}_\gamma^{*(n+1)} * H * \mathcal{A}^\gamma_{(n)}\right)(t).$$

Such an equality holds true because, by the commutativity of the Volterra convolution, the right hand side is equal to

$$\left[H * \left(\mathbb{K}_\gamma^{*(n+1)} * \mathcal{A}_{(n)}^\gamma\right)\right](t) = \int_0^t H(t-s)\left(\mathbb{K}_\gamma^{*(n+1)} * \mathcal{A}_{(n)}^\gamma\right)(s)$$
$$= (-1)^{n+1}\int_0^t ds H(t-s)\int_0^s \mathcal{G}_{(n)}^\gamma(s')ds';$$

now the conclusion follows from property (53). □

In the following lemma and throughout the paper we will be using the notation $\mathcal{F}\{f(x)\}(k) = \hat{f}(k)$ and $\mathcal{L}\{g(t)\}(\mu) = g^{\#}(\mu)$ for the Fourier transform and the Laplace transform respectively and we will use the superscript ˜ for the Fourier–Laplace transform.

Lemma 2. *For $0 < \gamma < \frac{1}{2}$, let $v^\gamma(t,x)$ be a solution to*

$$\begin{cases} \partial_t v^\gamma(t,x) = \dfrac{1}{\Gamma(2\gamma)}\dfrac{d}{dt}\displaystyle\int_0^t ds\,\dfrac{\Delta_x v^\gamma(s,x)}{(t-s)^{1-2\gamma}} + F(t,x) & (0,\infty)\times\mathbb{R} \\ v^\gamma(0,x) = v_0^\gamma(x) & \{0\}\times\mathbb{R} \end{cases}$$

and, for $\frac{1}{2} < \gamma < 1$, let it be a solution to

$$\begin{cases} \partial_t v^\gamma(t,x) = \dfrac{1}{\Gamma(2\gamma-1)}\displaystyle\int_0^t ds\,\dfrac{\Delta_x v^\gamma(s,x)}{(t-s)^{2-2\gamma}} + F(t,x) & (0,\infty)\times\mathbb{R} \\ v^\gamma(0,x) = v_0^\gamma(x), & \{0\}\times\mathbb{R} \end{cases}$$

where $v_0^\gamma(x) \in \mathcal{C}(\mathbb{R})$, $F(t,x) \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R})$. Then

$$v^\gamma(t,x) = \int_{\mathbb{R}} dy \rho^\gamma(t, x-y) v_0^\gamma(y) + \int_0^t ds \int_{\mathbb{R}} dy \rho^\gamma(t-s, x-y) F(s,y),$$

with $\rho^\gamma(t,x)$ the kernel defined in (16).

Proof. Let us observe that the Duhamel principle for the heat equation (i.e. the parabolic equation associated with the Laplacian) can be expressed as follows: if $u(t,x)$ is a classical solution to

$$\begin{cases} \partial_t u(t,x) = \dfrac{1}{2}\Delta_x u(t,x) + F(t,x) & (0,\infty)\times\mathbb{R},\ F \in \mathcal{C}(\mathbb{R}_+\times\mathbb{R}), \\ u(0,x) = u_0(x) & \{0\}\times\mathbb{R},\ u_0 \in \mathcal{C}(\mathbb{R}), \end{cases}$$

then its Fourier–Laplace transform satisfies

$$\tilde{u}(\mu,k) = \frac{\hat{u}(0,k) + \tilde{F}(\mu,k)}{\mu + \frac{1}{2}k^2}, \tag{60}$$

where $\left(\mu + \frac{k^2}{2}\right)^{-1}$ is the Fourier–Laplace transform of the fundamental solution of the diffusion equation, i.e. of the heat kernel.

Now let us recall that the Fourier–Laplace transform of $\rho^\gamma(t,x)$ is given by (134) (in (134) take $c_1 = 1$; see Appendix B); also, $\mu^{1-2\gamma}\,\tilde{v}^\gamma(\mu,k)$ is the Laplace transform of $D_t^\gamma(\hat{v}^\gamma(\cdot,k))$ when $0 < \gamma < \frac{1}{2}$, whereas for $\frac{1}{2} < \gamma < 1$ it is the Laplace transform of $I_t^\gamma(\hat{v}^\gamma(\cdot,k))$ (see Appendix B).

Hence

$$\begin{aligned}\mathcal{L}(\partial_t \hat{v}^\gamma(t,k)) &= \int_0^\infty dt\, e^{-\mu t}\partial_t \hat{v}(t,k)\\ &= -\hat{v}(0,k) + \mu\, \tilde{v}^\gamma(\mu,k) = -c_1 k^2 \mu^{1-2\gamma}\tilde{v}(\mu,k) + \tilde{F}(\mu,k)\\ &\Rightarrow \tilde{v}(\mu,k) = \frac{\hat{v}(0,k)+\tilde{F}(\mu,k)}{\mu + c_1 k^2\mu^{1-2\gamma}},\end{aligned}$$

which is precisely what we where looking for (see (134) and (60)). □

Lemma 3. *$\forall N \geq 1$ and $0 < \gamma < 1$, let p, q and r be real numbers greater than 1 s.t. $p^{-1} + q^{-1} = 1$, $q > \max\{N, r\}$ and $r^{-1} - q^{-1} < (2\gamma)^{-1}$. Let $v(\cdot)$ be an $\mathcal{F}_s$-adapted process in $\mathcal{C}(\mathbb{R}_+, L^r(\mathbb{R}))$ and ψ a random variable in $L^p(\mathbb{R})$. Then there exists a positive constant $C = C(q, r, \gamma)$ such that*

$$\left(E\left|\left\langle \psi, \int_{t_1}^{t_2} db(s)\rho^\gamma_{t_2-s}v(s)\right\rangle\right|^N\right)^{\frac{1}{N}} \leq C(t_2-t_1)^\nu \left(E\|\psi\|_p^\beta\right)^{\frac{1}{\beta}}\left(E\sup_{t_1\leq s\leq t_2}\|v(s)\|_r^q\right)^{\frac{1}{q}},$$

for any $t_1 \leq t_2$, where $\beta = \frac{Nq}{q-N}$ and $\nu = \frac{1}{2} - \gamma\left(\frac{1}{r} - \frac{1}{q}\right)$.

Proof (*Sketch*)**.** The proof is identical to the proof of Lemma 3.1 in [2], so we will not repeat it. The additional condition $r^{-1} - q^{-1} < (2\gamma)^{-1}$ is an integrability condition and comes from the fact that

$$\int_{t_1}^{t_2} ds\|\rho^\gamma_{t_2-s}\|^2_{r'} < \infty \Longleftrightarrow \frac{1}{r} - \frac{1}{q} < \frac{1}{2\gamma}, \tag{61}$$

where r' is such that $\frac{1}{r'} + \frac{1}{r} = 1 + \frac{1}{q}$ (see page 12 in [2]). (61) follows from the scaling property (19) in the following way:

$$\begin{aligned}\int_{t_1}^{t_2} ds\|\rho^\gamma_{t_2-s}\|^2_{r'} &= \int_{t_1}^{t_2}\frac{ds}{(t_2-s)^{2\gamma}}\left(\int_{\mathbb{R}} dy\rho_1^{r'}(y)(t_2-s)^\gamma\right)^{\frac{2}{r'}}\\ &= C\int_{t_1}^{t_2} ds(t_2-s)^{2\gamma(\frac{1}{r'}-1)} < \infty \Longleftrightarrow 2\gamma\left(\frac{1}{r'}-1\right) > -1\\ &\qquad\Longleftrightarrow \frac{1}{r} - \frac{1}{q} < \frac{1}{2\gamma}.\end{aligned}$$ □

Remark 3.1. The extra condition $\frac{1}{r} - \frac{1}{q} < \frac{1}{2\gamma}$ is automatically satisfied when $\gamma \in \left(0, \frac{1}{2}\right]$. It is non-empty only when $\gamma \in \left(\frac{1}{2}, 1\right)$.

In the remainder of this section and in the proof of Theorem 3 we will very often make use of the following simple observation (sometimes without saying it explicitly).

Note 3.1. Let (Ω, μ), (Ω', μ') be two (finite dimensional) measure spaces, $f : \Omega \times \Omega' \to \mathbb{R}$ a positive function and m a real number greater than or equal to 1. Suppose

$$F(y) := \int_\Omega d\mu(x) f(x,y) < \infty \quad \text{for a.e. } y \in \Omega \quad \text{and}$$

$$\int_{\Omega'} d\mu'(y) \left(\int_{\Omega} d\mu(x) f(x, y) \right)^m < \infty.$$

Then

$$\int_{\Omega'} d\mu'(y) \left(\int_{\Omega} d\mu(x) f(x, y) \right)^m = \int_{\Omega'} d\mu'(y) \left[F(y)^{m-1} \int_{\Omega} d\mu(x) f(x, y) \right]$$
$$= \int_{\Omega} d\mu(x) \int_{\Omega'} d\mu'(y) F(y)^{m-1} f(x, y)$$
$$\leq \left(\int_{\Omega'} d\mu'(y) F(y)^m \right)^{\frac{m-1}{m}} \int_{\Omega} d\mu(x) \left(\int_{\Omega'} d\mu'(y) f(x, y)^m \right)^{\frac{1}{m}},$$

having applied Hölder's inequality with $m/(m-1)$ and m. Looking at the first line and the last line of the above equations and dividing both sides by $\left[\int_{\Omega'} \left(\int_{\Omega} f \right)^m \right]^{\frac{m-1}{m}}$ we obtain

$$\int_{\Omega'} d\mu'(y) \left(\int_{\Omega} d\mu(x) f(x, y) \right)^m \leq \left(\int_{\Omega} d\mu(x) \left(\int_{\Omega'} d\mu'(y) f(x, y)^m \right)^{\frac{1}{m}} \right)^m. \tag{62}$$

When (Ω, μ), (Ω', μ') are just $\mathbb{R}$ equipped with the Lebesgue measure, the above inequality reads

$$\int dy \left(\int dx f(x, y) \right)^m \leq \left(\int dx \left(\int dy f(x, y)^m \right)^{\frac{1}{m}} \right)^m.$$

If instead (Ω', μ') is a probability space and (Ω, μ) is the time interval $[0, T]$ with the Lebesgue measure, inequality (62) implies that $\forall t \in [0, T]$ and $N \geq 1$, we have

$$E \sup_{t \in [0,T]} \left| \int_0^t ds f(s) \right|^N \leq E \left| \int_0^T ds |f(s)| \right|^N \leq T^N \sup_{s \in [0,T]} E |f(s)|^N. \tag{63}$$

In the remainder of this section, C is a constant that does not depend on λ or δ, although it might depend on a positive power of T. Also, in the proofs we assume for simplicity $T \geq 1$, even though all the results are true for any $T > 0$, and hence they are stated in such generality. Even if we assumed $T \geq 1$, this would not be restrictive in view of the fact that the result that we are concerned with is a long time result, more specifically $T \sim |\log \lambda|$ with $\lambda \to 0$. The case $\gamma = \frac{1}{2}$ is not explicitly considered in Lemmas 4 and 5.

Lemma 4. *$\forall N \geq 1$, $0 < \gamma < 1$ and $\zeta \in \left(0, \frac{1}{2\gamma}\right)$, there exists $C > 0$ such that*

$$\sup_{0 \leq s \leq t \leq T} \left(E \left| K_{t,s}^{(\lambda,\gamma)} \right|^N \right)^{\frac{1}{N}} \leq C T^{\zeta\gamma} \lambda^{\frac{1}{2\gamma} - 1 - \zeta}, \quad T > 0, \lambda \in (0, 1). \tag{64}$$

Also, $\forall n \geq 1$, N, γ, ζ as above,

$$\left(E \sup_{t \in [0,T]} \left| X_{(n)}^{(\lambda,\gamma)}(t) \right|^N \right)^{\frac{1}{N}} \leq C \left(1 + \lambda^{\frac{1}{2\gamma} - \zeta - 1} \right) e^{C T^{(n+1)(1-\gamma)}}, \tag{65}$$

$T > 0, \lambda \in (0, 1)$. *Moreover, for the displacement of the center we find*

$$\left(\sup_{t \in [0,T], t+\tau \leq T} E \sup_{t' \in [t, t+\tau]} \left| X_{(n)}^{(\lambda,\gamma)}(t') - X_{(n)}^{(\lambda,\gamma)}(t) \right|^N \right)^{\frac{1}{N}} \leq C \left(\tau^{\frac{1}{2}} + \tau \lambda^{\frac{1}{2\gamma} - \zeta - 1} \right) e^{CT^{(n+1)(1-\gamma)}}, \tag{66}$$

$\tau, \lambda \in (0, 1)$ *and* $T > 0$.

Proof (*Sketch*)**.** (64) follows from Lemma 3 and (48), where in Lemma 3 we have chosen $\frac{1}{r} - \frac{1}{q} = \frac{1}{2\gamma} - \zeta$, $\zeta \in \left(0, \frac{1}{2\gamma}\right)$. Having in mind Note 3.1, from (29) and (64), using (63) we have

$$\left(E \sup_{t \in [0,T]} \left| F_0^{(\lambda,\gamma)}(t) \right|^N \right)^{\frac{1}{N}} \leq C T^{1+\zeta\gamma} \lambda^{\frac{1}{2\gamma} - \zeta - 1}. \tag{67}$$

From (30), (64) and (52) we get

$$\left(\sup_{t \in [0,T]} E \left| F_n^{(\lambda,\gamma)}(t) \right|^N \right)^{\frac{1}{N}} \leq T^{\zeta\gamma} T^{n - n\gamma} \lambda^{\frac{1}{2\gamma} - \zeta - 1}, \quad n \geq 1, \tag{68}$$

so, again by (63),

$$\left(E \sup_{t \in [0,T]} \left| \int_0^t ds F_n^{(\lambda,\gamma)}(s) \right|^N \right)^{\frac{1}{N}} \leq C T^{\zeta\gamma+1} T^{n - n\gamma} \lambda^{\frac{1}{2\gamma} - \zeta - 1}. \tag{69}$$

Also, from (42) and (52),

$$\left(\sup_{t \in [0,T]} E \left| \dot{Y}_{(n)}^{(\lambda,\gamma)}(t) \right|^N \right)^{\frac{1}{N}} \leq C \left\{ \sup_{t \in [0,T]} \left(E \left| F_n^{(\lambda,\gamma)}(t) \right|^N \right)^{\frac{1}{N}} + \cdots + \sup_{t \in [0,T]} \left(E \left| F_{2n}^{(\lambda,\gamma)}(t) \right|^N \right)^{\frac{1}{N}} \right\} + C T^{n - \gamma(n+1)} \int_0^T dt \left(\sup_{s \in [0,t]} E \left| \dot{Y}_{(n)}^{(\lambda,\gamma)}(s) \right|^N \right)^{\frac{1}{N}}.$$

By the Gronwall Lemma and (68) we then obtain that $\forall n \geq 1$,

$$\left(\sup_{t \in [0,T]} E \left| \dot{Y}_{(n)}^{(\lambda,\gamma)}(t) \right|^N \right)^{\frac{1}{N}} \leq C \lambda^{\frac{1}{2\gamma} - \zeta - 1} e^{CT^{(n+1) - \gamma(n+1)}}, \tag{70}$$

and hence

$$\left(E \sup_{t \in [0,T]} \left| \int_0^t ds \dot{Y}_{(n)}^{(\lambda,\gamma)}(s) \right|^N \right)^{\frac{1}{N}} \leq C \lambda^{\frac{1}{2\gamma} - \zeta - 1} e^{CT^{(n+1) - \gamma(n+1)}}. \tag{71}$$

When $n = 1$, (65) is a straightforward consequence of (37), (67) and (71) and the fact that

$$E \sup_{t\in[0,T]} |b(t)|^N \le C T^{\frac{N}{2}}. \tag{72}$$

When $n > 1$, we first rewrite (39) as follows:

$$X_{(n)}^{(\lambda,\gamma)}(t) = b(t) + F_0^{(\lambda,\gamma)}(t) + \int_0^t ds \left[F_1^{(\lambda,\gamma)} + \cdots + F_n^{(\lambda,\gamma)}\right](s) + \int_0^t ds \dot{Y}_{(n)}^{(\lambda,\gamma)}(s), \tag{73}$$

and then (65) follows from (67), (72), (69) and (71). By acting in a similar way we find the following estimates:

$$\left(\sup_{t\in[0,T],t+\tau\le T} E \sup_{t'\in[t,t+\tau]} \left|F_0^{(\lambda,\gamma)}(t') - F_0^{(\lambda,\gamma)}(t)\right|^N\right)^{\frac{1}{N}} \le C\tau\, T^{\zeta\gamma} \lambda^{\frac{1}{2\gamma}-1-\zeta},$$

$$\left(\sup_{t\in[0,T],t+\tau\le T} E \sup_{t'\in[t,t+\tau]} \left|\int_t^{t'} ds F_n^{(\lambda,\gamma)}(s)\right|^N\right)^{\frac{1}{N}} \le C\tau\, T^{\zeta\gamma} T^{n(1-\gamma)} \lambda^{\frac{1}{2\gamma}-1-\zeta},$$

$$\left(\sup_{t\in[0,T],t+\tau\le T} E \sup_{t'\in[t,t+\tau]} \left|\int_t^{t'} ds \dot{Y}_{(n)}^{(\lambda,\gamma)}(s)\right|^N\right)^{\frac{1}{N}} \le C\tau\, T^{\zeta\gamma} T^{(2n+1)(1-\gamma)} \lambda^{\frac{1}{2\gamma}-1-\zeta}.$$

So, recalling that for the BM $b(t)$

$$E \sup_{t'\in[t,t+\tau]} |b(t') - b(t)|^N \le C\tau^{\frac{N}{2}}, \tag{74}$$

(66) follows. □

Lemma 5. $\forall N, n \ge 1$, $0 < \gamma < 1$, $\zeta \in \left(0, \frac{1}{2\gamma}\right)$, $T > 0$, $\lambda, \delta \in (0,1)$ *there exists a constant* $C > 0$ *such that*

$$\left(E \sup_{t\in[\delta,T]} \left|\int_\delta^t ds K_{s,s-\delta}^{(\lambda,\gamma)}\right|^N\right)^{\frac{1}{N}} \le C\left[\delta^{1-\gamma} + \delta^{\zeta}\lambda^{\frac{1}{2\gamma}-\zeta-1}\left(\delta^{\frac{1}{2}} + \delta\lambda^{\frac{1}{2\gamma}-\zeta-1}\right)\right] e^{CT^{(n+1)(1-\gamma)}}, \tag{75}$$

$$\left(E \sup_{t\in[\delta,T]} \left|\int_\delta^t ds \rho_{t-s}^{\gamma}(0) K_{s,s-\delta}^{(\lambda,\gamma)}\right|^N\right)^{\frac{1}{N}} \le C\left[\delta^{1-\gamma} + \delta^{\zeta}\lambda^{\frac{1}{2\gamma}-\zeta-1}\left(\delta^{\frac{1}{2}} + \delta\lambda^{\frac{1}{2\gamma}-\zeta-1}\right)\right] e^{CT^{(n+1)(1-\gamma)}}, \tag{76}$$

$$\left(E \sup_{t\in[0,T]} \left|\int_0^t ds \rho_{t-s}^{\gamma}(0) K_{s,0}^{(\lambda,\gamma)}\right|^N\right)^{\frac{1}{N}} \le C\lambda^{\frac{1}{\gamma}-2\zeta-2} e^{CT^{(n+1)(1-\gamma)}}. \tag{77}$$

Moreover, $\forall M > 0$, *we have*

$$\sup_{s\in[\delta,T]} \left(E \left| \Gamma_s^{(\lambda,\delta,\gamma)} \right|^N \right)^{\frac{1}{N}} \le C \left[\lambda^{\frac{3}{4}} + \lambda^{\frac{1}{2\gamma}-\frac{1}{4}} + \lambda^{\frac{M}{8}} \left(T^{1-2\gamma} \mathbf{1}_{\{0<\gamma<1/2\}} + \delta^{1-2\gamma} \mathbf{1}_{\{1/2<\gamma<1\}} \right) \right] e^{CT^{(n+1)(1-\gamma)}}, \tag{78}$$

where **1** *is the indicator function and*

$$\Gamma_s^{(\lambda,\delta,\gamma)} := \left\langle \varphi_s^{(\lambda)}, \int_0^{s-\delta} db(s') \rho_{s-s'}^{\gamma} \varphi_{s'}^{(\lambda)} \right\rangle - \int_0^{s-\delta} db(s') \rho_{s-s'}^{\gamma}(0). \tag{79}$$

Proof. The proof of the bounds (75)–(78) is given by following [2, pages 16–18], so it will not be very detailed. Recalling (47), we have that $\forall n \ge 1$, $\gamma \in (0,1)$ and for $0 \le s \le t$, $K_{t,s}^{(\lambda,\gamma)}$ can be expressed as

$$K_{t,s}^{(\lambda,\gamma)} = \left\langle \varphi^{(\lambda)}, \int_s^t db(s') \rho_{t-s'}^{\gamma} \varphi^{(\lambda)} \right\rangle \tag{80}$$

$$+ \left\langle \varphi^{(\lambda)}_{\lambda\left[X_{(n)}^{(\lambda,\gamma)}(t) - X_{(n)}^{(\lambda,\gamma)}(s)\right]} - \varphi^{(\lambda)}, \int_s^t db(s') \rho_{t-s'}^{\gamma} \varphi^{(\lambda)} \right\rangle \tag{81}$$

$$+ \left\langle \varphi_t^{(\lambda)}, \int_s^t db(s') \rho_{t-s'}^{\gamma} (\varphi_{s'}^{(\lambda)} - \varphi_s^{(\lambda)}) \right\rangle. \tag{82}$$

Observe also that $\forall a, b \in \mathbb{R}$ and $\forall m \ge 1$,

$$\|\varphi_b - \varphi_a\|_m \le \|\varphi'\|_m |b-a|, \quad \varphi_h := \varphi(x-h). \tag{83}$$

Let us start with proving (77). We decompose $K_{s,0}^{(\lambda,\gamma)}$ according to the prescription (80)–(82); recalling the notation (13), the term coming from (81) becomes

$$\left\langle \varphi_s^{(\lambda)} - \varphi^{(\lambda)}, \int_0^s db(s') \rho_{s-s'}^{\gamma} \varphi^{(\lambda)} \right\rangle.$$

Using Lemma 3, we have

$$\left(E \left| \left\langle \varphi_s^{(\lambda)} - \varphi^{(\lambda)}, \int_0^s db(s') \rho_{s-s'}^{\gamma} \varphi^{(\lambda)} \right\rangle \right|^N \right)^{\frac{1}{N}} \le s^{\nu} E \|\varphi_s^{(\lambda)} - \varphi^{(\lambda)}\|_p \, \|\varphi^{(\lambda)}\|_r$$

with r, p and ν to be chosen according to Lemma 3. By (83), (48) and (65), we obtain that

$$\sup_{s\in[0,t]} \left(E \left| \left\langle \varphi_s^{(\lambda)} - \varphi^{(\lambda)}, \int_0^s db(s') \rho_{s-s'}^{\gamma} \varphi^{(\lambda)} \right\rangle \right|^N \right)^{\frac{1}{N}} \le C t^{\nu} \lambda^{\frac{1}{p}-1} \lambda^{\frac{1}{r}-1} \lambda^{\frac{1}{2\gamma}-\zeta-1} e^{Ct^{(n+1)(1-\gamma)}}$$
$$\le C e^{Ct^{(n+1)(1-\gamma)}} t^{\gamma\zeta} \lambda^{\frac{1}{\gamma}-2\zeta-2},$$

having chosen $\frac{1}{r} + \frac{1}{p} - 1 = \frac{1}{2\gamma} - \zeta$. For p' and q' such that $\frac{1}{p'} + \frac{1}{q'} = 1$, we have

$$\left| \int_0^t ds \rho_{t-s}^{\gamma}(0) \left\langle \varphi_s^{(\lambda)} - \varphi^{(\lambda)}, \int_0^s db(s') \rho_{s-s'}^{\gamma} \varphi^{(\lambda)} \right\rangle \right|^N$$

$$\leq C\left(\int_0^t \frac{ds}{(t-s)^{\gamma p'}}\right)^{\frac{N}{p'}}\left(\int_0^t ds\left|\left\langle\varphi_s^{(\lambda)}-\varphi^{(\lambda)},\int_0^s db(s')\rho_{s-s'}\varphi^{(\lambda)}\right\rangle\right|^{q'}\right)^{\frac{N}{q'}},$$

and so

$$\left(E\sup_{t\in[0,T]}\left|\int_0^t ds\rho_{t-s}^{\gamma}(0)\left\langle\varphi_s^{(\lambda)}-\varphi^{(\lambda)},\int_0^s db(s')\rho_{s-s'}^{\gamma}\varphi^{(\lambda)}\right\rangle\right|^N\right)^{\frac{1}{N}}$$
$$\leq C\left|E\left(\int_0^T ds\left|\left\langle\varphi_s^{(\lambda)}-\varphi^{(\lambda)},\int_0^s db(s')\rho_{s-s'}\varphi^{(\lambda)}\right\rangle\right|^{q'}\right)^{\frac{N}{q'}}\right|^{\frac{1}{N}}$$
$$\leq C\left(E\left|\int_0^T ds\left|\left\langle\varphi_s^{(\lambda)}-\varphi^{(\lambda)},\int_0^s db(s')\rho_{s-s'}\varphi^{(\lambda)}\right\rangle\right|^{q'}\right|^N\right)^{\frac{1}{Nq'}}$$
$$\leq C\sup_{s\in[0,T]}\left(E\left|\left\langle\varphi_s^{(\lambda)}-\varphi^{(\lambda)},\int_0^s db(s')\rho_{s-s'}\varphi^{(\lambda)}\right\rangle\right|^{Nq'}\right)^{\frac{1}{Nq'}}$$
$$\leq C\lambda^{\frac{1}{\gamma}-2\zeta-2}e^{CT^{(n+1)(1-\gamma)}}.$$

The addends (80) and (82) can be examined in the same way, so we leave this to the reader. We now very briefly show how to obtain (75). We again decompose $K_{s,s-\delta}^{(\lambda,\gamma)}$ according to (80)–(82). For the term coming from (80), by exchanging the order of integration (which is now allowed) and integrating by parts, we get

$$\left(E\sup_{t\in[\delta,T]}\left|\int_\delta^t ds\left\langle\varphi^{(\lambda)},\int_{s-\delta}^s db(s')\rho_{s-s'}^{\gamma}\varphi^{(\lambda)}\right\rangle\right|^N\right)^{\frac{1}{N}}\leq C\left(\delta^{1-\gamma}+\delta^{\frac{1}{2}}\delta^{1-\gamma}\right).$$

For the term coming from (81), we have

$$\left(E\sup_{t\in[\delta,T]}\left|\int_\delta^t ds\left\langle\varphi^{(\lambda)}_{\lambda\left[X_{(n)}^{(\lambda,\gamma)}(s)-X_{(n)}^{(\lambda,\gamma)}(s-\delta)\right]}-\varphi^{(\lambda)},\int_{s-\delta}^s db(s')\rho_{s-s'}^{\gamma}\varphi^{(\lambda)}\right\rangle\right|^N\right)^{\frac{1}{N}}$$
$$\leq C\delta^{\zeta}\lambda^{\frac{1}{2\gamma}-\zeta-1}\left[\delta^{\frac{1}{2}}+\delta\lambda^{\frac{1}{2\gamma}-\zeta-1}\right]e^{CT^{(n+1)(1-\gamma)}},$$

having applied Lemma 3 with the choice $\frac{1}{r}-\frac{1}{q}=\frac{1}{2\gamma}-\zeta$, $\zeta\in\left(0,\frac{1}{2\gamma}\right)$, and (66), as well. In an analogous way, for the term coming from (82) we obtain

$$\left(E\sup_{t\in[\delta,T]}\left|\int_\delta^t ds\left\langle\varphi_s^{(\lambda)},\int_{s-\delta}^s db(s')\rho_{s-s'}^{\gamma}\left(\varphi_{s'}^{(\lambda)}-\varphi_{s-\delta}^{(\lambda)}\right)\right\rangle\right|^N\right)^{\frac{1}{N}}$$
$$\leq C\delta^{\zeta}\lambda^{\frac{1}{2\gamma}-\zeta-1}\left[\delta^{\frac{1}{2}}+\delta\lambda^{\frac{1}{2\gamma}-\zeta-1}\right]e^{CT^{(n+1)(1-\gamma)}}.$$

(76) results from applying the same technique again so we will not present the proof.

In order to prove (78), let us express $\Gamma_s^{(\lambda,\delta,\gamma)}$ as

$$\Gamma_s^{(\lambda,\delta,\gamma)}=\int dx\varphi\left(x-X_{(n)}^{(\lambda,\gamma)}(t)\right)I_t^{(\lambda,\delta,\gamma)}(x),$$

where

$$I_t^{(\lambda,\delta,\gamma)}(x) := \int_0^{t-\delta} db(s) \int_{\mathbb{R}} dy\varphi(y) \left\{ \rho_{t-s}^{\gamma} \left[\lambda(x - y - X_{(n)}^{(\lambda,\gamma)}(s)) \right] - \rho_{t-s}^{\gamma}(0) \right\}.$$

By a change of variables and using the scaling property (19), we can rewrite as follows:

$$I_t^{(\lambda,\delta,\gamma)}(x) = \int_0^{t-\delta} db(s)\rho_{t-s}^{\gamma}(0) \int_R dy\varphi(y) \left\{ \frac{1}{c(\gamma)} \rho_1^{\gamma} \left[\frac{\lambda\left(x - y - X_{(n)}^{(\lambda,\gamma)}(s)\right)}{(t-s)^{\gamma}} \right] - 1 \right\},$$

where $c(\gamma)$ is defined in (20). We now use the bounds (22) and (23). More precisely, setting $z = \lambda\left(x - y - X_{(n)}^{(\lambda,\gamma)}(s)\right)/(t-s)^{\gamma}$, we estimate the integrand above in the following way:

$$\begin{cases} \left| \dfrac{\rho_1^{\gamma}(z)}{c(\gamma)} - 1 \right| \le C & \text{when } |x| > \lambda^{-1/8} \\ \left| \dfrac{\rho_1^{\gamma}(z)}{c(\gamma)} - 1 \right| \le C|z| & \text{when } |x| \le \lambda^{-1/8}. \end{cases}$$

So, following [2, pages 15–16], we apply the Burkholder inequality [16] and we get

$$\begin{aligned} E\,|I_t(x)|^N &\le C\,\mathbf{1}_{\{|x|>\lambda^{-1/8}\}} \left| \int_0^{t-\delta} \frac{ds}{(t-s)^{2\gamma}} \right|^{\frac{N}{2}} \\ &\quad + C\,\mathbf{1}_{\{|x|\le\lambda^{-1/8}\}} E \left| \int_0^{t-\delta} \frac{ds}{(t-s)^{2\gamma}} \left(\int_{\mathbb{R}} dy\varphi(y) \frac{\lambda\left|x - y - X_{(n)}^{(\lambda,\gamma)}(s)\right|}{(t-s)^{\gamma}} \right)^2 \right|^{\frac{N}{2}} \\ &\le C\mathbf{1}_{\{|x|>\lambda^{-1/8}\}} \left| \mathbf{1}_{\{0<\gamma<1/2\}} t^{1-2\gamma} + \mathbf{1}_{\{1/2<\gamma<1\}} \delta^{1-2\gamma} \right|^{\frac{N}{2}} \\ &\quad + C\mathbf{1}_{\{|x|\le\lambda^{-1/8}\}} \lambda^N \left(\lambda^{-N/8} + 1 + E \sup_{s\in[0,T]} \left| X_{(n)}^{(\lambda,\gamma)} \right|^N \right) \left| \int_0^{t-\delta} \frac{ds}{(t-s)^{4\gamma}} \right|^{\frac{N}{2}} \\ &\le C\mathbf{1}_{\{|x|>\lambda^{-1/8}\}} \left| \mathbf{1}_{\{0<\gamma<1/2\}} t^{1-2\gamma} + \mathbf{1}_{\{1/2<\gamma<1\}} \delta^{1-2\gamma} \right|^{\frac{N}{2}} \\ &\quad + C\mathbf{1}_{\{|x|\le\lambda^{-1/8}\}} \lambda^N \left(\lambda^{-N/8} + 1 + \lambda^{\frac{1}{2\gamma}-\zeta-1} \right) e^{CT^{(n+1)(1-\gamma)}}, \end{aligned}$$

where in the last inequality we have used (65). If we choose $\zeta = 1/8$ in the above, we obtain

$$\begin{aligned} &\left(E \left| \int_{\{|x|\le\lambda^{-1/8}\}} \varphi\left(x - X_{(n)}^{(\lambda,\gamma)}(t)\right) I_t^{(\lambda,\delta,\gamma)} \right|^N \right)^{\frac{1}{N}} \\ &\le C\,\varphi_{\frac{N}{N-1}} \left(E \int_{\{|x|\le\lambda^{-1/8}\}} dx \left| I_t^{(\lambda,\delta,\gamma)} \right|^N \right)^{\frac{1}{N}} \\ &\le C\left(\lambda^{3/4} + \lambda^{\frac{1}{2\gamma}-\frac{1}{4}} \right) e^{CT^{(n+1)(1-\gamma)}}. \end{aligned}$$

Moreover, for any $M > 0$ we have

$$\left(E\left|\int_{\{|x|>\lambda^{-1/8}\}} \varphi\left(x - X_{(n)}^{(\lambda,\gamma)}(t)\right) I_t^{(\lambda,\delta,\gamma)}\right|^N\right)^{\frac{1}{N}}$$

$$\leq \lambda^{\frac{M}{8}} \left[E\left(\int_{\{|x|>\lambda^{-1/8}\}} dx\varphi\left(x - X_{(n)}^{(\lambda,\gamma)}(t)\right)^{\frac{2N}{2N-1}} (1+x^2)^{\frac{1}{2N-1}} |x|^{\frac{2NM}{2N-1}}\right)^{2N-1}\right]^{\frac{1}{2N}}$$

$$\cdot \left(E\int_{\{|x|>\lambda^{-1/8}\}} dx \frac{\left|I_t^{(\lambda,\delta,\gamma)}(x)\right|^{2N}}{1+x^2}\right)^{\frac{1}{2N}}$$

$$\leq \lambda^{M/8}\left(\mathbf{1}_{\{0<\gamma<1/2\}} t^{1-2\gamma} + \mathbf{1}_{\{1/2<\gamma<1\}} \delta^{1-2\gamma}\right).$$

This concludes the proof of (78). □

Lemma 6. *$\forall 0 \leq s \leq t$, $\lambda \in (0,1)$, $\beta \in (0,1]$, $n, N \geq 1$ and $\gamma \in (0,1)$, we have*

$$\left(E\left(\sup_{0\leq s\leq t\leq T} (t-s)^{(1+\beta)\gamma} |P_{t,s}^{(\lambda,\gamma)} - \rho_{t-s}^{\gamma}(0)|\right)^N\right)^{\frac{1}{N}} \leq C\lambda^{\beta}\lambda^{\frac{1}{2\gamma}-\zeta-1} e^{CT^{(n+1)(1-\gamma)}}. \tag{84}$$

Also, $\forall \delta \in (0,1)$ and for any $Q > 0$, we have

$$\left(E\sup_{t\in[\delta,T]}\left(\int_0^{t-\delta} ds |P_{t,s}^{(\lambda,\gamma)} - \rho_{t-s}^{\gamma}(0)|\right)^N\right)^{\frac{1}{N}} \leq C\left(\lambda^{\frac{1}{2\gamma}-\zeta} + \lambda^Q\right) e^{CT^{(n+1)(1-\gamma)}}. \tag{85}$$

Sketch of Proof. Using the definition of $P_{t,s}^{(\lambda,\gamma)}$ (27), by a change of variables and the scaling property (19), we have

$$|P_{t,s}^{(\lambda,\gamma)} - \rho_{t-s}^{\gamma}(0)| \leq \rho_{t-s}^{\gamma}(0) \iint dxdy\, \varphi(x)\varphi(y)$$
$$\times \left|\frac{1}{c(\gamma)}\rho_1^{\gamma}\left[\frac{\lambda(x-y+X_{(n)}^{(\lambda,\gamma)}(t) - X_{(n)}^{(\lambda,\gamma)}(s))}{(t-s)^{\gamma}}\right] - 1\right|. \tag{86}$$

From (22), then

$$|P_{t,s}^{(\lambda,\gamma)} - \rho_{t-s}^{\gamma}(0)|$$
$$\leq C\rho_{t-s}^{\gamma}(0) \iint dxdy\varphi(x)\varphi(y) \frac{\left|x-y+X_{(n)}^{(\lambda,\gamma)}(t) - X_{(n)}^{(\lambda,\gamma)}(s)\right|^{\beta}\lambda^{\beta}}{(t-s)^{\gamma\beta}}.$$

We now want to use (66) in order to conclude; however, (66) holds only for $N \geq 1$ whereas β is in the range $\beta \in (0,1]$. We do not want to choose $\beta = 1$ (see (109) and comments after it);

hence we first apply the Young inequality with $p = \frac{1}{\beta}$ and get

$$|P_{t,s}^{(\lambda,\gamma)} - \rho_{t-s}^{\gamma}(0)| \le C\rho_{t-s}^{\gamma}(0) \times \iint dxdy\varphi(x)\varphi(y)\frac{\left(|x|+|y|+\left|X_{(n)}^{(\lambda,\gamma)}(t)-X_{(n)}^{(\lambda,\gamma)}(s)\right|+1\right)\lambda^{\beta}}{(t-s)^{\gamma\beta}},$$

and now (84) is a straightforward consequence of (66). To get (85), we use again the bounds (22) and (23), this time in the following way: setting $z = \lambda\left(x - y + X_{(n)}^{(\lambda,\gamma)}(t) - X_{(n)}^{(\lambda,\gamma)}(s)\right)/(t-s)^{\gamma}$, we estimate

$$\begin{cases} \left|\dfrac{\rho_1^{\gamma}(z)}{c(\gamma)} - 1\right| \le C & \text{when } |x| > \lambda^{-1} \\ \left|\dfrac{\rho_1^{\gamma}(z)}{c(\gamma)} - 1\right| \le C|z| & \text{when } |x| \le \lambda^{-1}. \end{cases}$$

So, from (86) we have

$$\begin{aligned} &\int_0^{t-\delta} ds|P_{t,s}^{(\lambda,\gamma)} - \rho_{t-s}^{\gamma}(0)| \\ &\le C\int_0^{t-\delta}\frac{ds}{(t-s)^{2\gamma}}\iint \varphi(x)\varphi(y)\mathbf{1}_{\{|x|\le\lambda^{-1}\}}\left[\lambda\left(x+y+X_{(n)}^{(\lambda,\gamma)}(t)-X_{(n)}^{(\lambda,\gamma)}(s)\right)\right] \\ &\quad + C\int_0^{t-\delta}\frac{ds}{(t-s)^{\gamma}}\iint \varphi(x)\varphi(y)\mathbf{1}_{\{|x|>\lambda^{-1}\}}C \\ &\le C\mathbf{1}_{\{|x|\le\lambda^{-1}\}}\lambda\int_0^{t-\delta}\frac{ds}{(t-s)^{2\gamma}}\left(C+\left|X_{(n)}^{(\lambda,\gamma)}(t)-X_{(n)}^{(\lambda,\gamma)}(s)\right|\right) \\ &\quad + C\mathbf{1}_{\{|x|>\lambda^{-1}\}}\int_0^{t-\delta}\frac{ds}{(t-s)^{\gamma}}\left(\int\varphi(x)|x|^{2Q}\right)^{\frac{1}{2}}\left(\int_{\{|x|>\lambda^{-1}\}}\frac{\varphi(x)}{|x|^{2Q}}\right)^{\frac{1}{2}}. \end{aligned}$$

(85) now follows from (65). □

4. Proof of Theorem 3

We recall that C is a positive constant that does not depend on λ and δ, though it might depend on a positive power of T. Also, for simplicity, all the proofs are presented for $T \ge 1$, even though the statements are clearly still valid for any $T > 0$. Since it has already been treated in [2], the case $\gamma = 1/2$ is not explicitly considered.

The intuitive idea that motivates the structure of the proof is based on the observation that, "morally", things go as if $P_{t,s}^{(\lambda,\gamma)}$ were converging to $\rho_{t-s}^{\gamma}(0)$ as $\lambda \to 0$ (see Lemma 6); formally, this can be obtained by thinking that, as $\lambda \to 0$, $\varphi_t^{(\lambda)} \to \delta_0$. While such an idea is not hard to turn into a rigorous argument, one of the main technical difficulties is encountered when trying to do the same thing to get some intuition as regards what $K_{s,0}^{(\lambda,\gamma)}$ ought to converge to. If in the definition of $K_{s,0}^{(\lambda,\gamma)}$ we replace $\varphi_t^{(\lambda)}$ with δ_0 and exchange the order of integration, we find that $K_{s,0}^{(\lambda,\gamma)}$ should converge to $\int_0^t db(s)\rho_{t-s}^{\gamma}(0)$. The problem is that we are not allowed to exchange the order of integration (see the comment after (3.5) in [2]) and that $\int_0^t db(s)\rho^{\gamma}_{t-s}(0)$ is not well defined as a process in $\mathcal{C}(\mathbb{R}_+)$ when $\gamma \geq \frac{1}{2}$. In the same way, $\forall n \geq 1$, $F_n^{(\lambda,\gamma)}$ is well defined for any $\gamma \in (0,1)$, whereas the object that it converges to is not (see (116) and (45)).

The proof goes as follows. $\forall n \geq 1$ we introduce the process $\eta^{\gamma}_{(n)}(t)$, a solution to the equation

$$\eta^{\gamma}_{(n)}(t) = G^{\gamma}_{(n)}(t) + (-1)^{n+1}\int_0^t ds\eta^{\gamma}_{(n)}(s)\mathbb{K}^{*(n+1)}(t-s), \quad 0<\gamma<\frac{n}{n+1} \tag{87}$$

where

$$G^{\gamma}_{(n)}(t) := \sum_{\nu=n}^{2n}(-1)^{\nu+1}\int_0^t db(u)\mathbb{K}^{*(\nu+1)}_{\gamma}(t-u), \quad n\geq 1, 0<\gamma<\frac{n}{n+1}. \tag{88}$$

We now observe that Lemma 1 can be applied to $\xi^{\gamma}_{(n)}$, defined in (44), and $\eta^{\gamma}_{(n)}$. In this case the forcing terms are $A^{\gamma}_{(n)}$ and $G^{\gamma}_{(n)}$, respectively, and we can easily prove that they are related through (57). We can in fact show that the ith addend of $A^{\gamma}_{(n)}$ is related to the ith addend of $G^{\gamma}_{(n)}$ through (57); all we need to show is that $\forall \nu \in 0,\ldots,n$,

$$(-1)^{\nu}\left(\mathbb{K}^{*(\nu)}_{\gamma} * b * \mathbb{K}^{*(n+1)}_{\gamma}\right)(t) = (-1)^{n+1}\int_0^t ds(-1)^{\nu+n+1}\int_0^s db(u)\mathbb{K}^{*(\nu+1)}_{\gamma}(s-u),$$

which is a straightforward consequence of the definition of $\mathbb{K}^{*(m)}_{\gamma}$ given in (25), together with the following equality:

$$\left(\mathbb{K}^{*(n+1)}_{\gamma} * b\right)(t) = \int_0^t ds\int_0^s db(u)\mathbb{K}^{*(n)}_{\gamma}(s-u), \quad n\geq 1. \tag{89}$$

Hence, Lemma 1 gives

$$(-1)^{n+1}\left(\xi^{\gamma}_{(n)} * \mathbb{K}^{*(n+1)}_{\gamma}\right)(t) = \int_0^t ds\eta^{\gamma}_{(n)}(s). \tag{90}$$

Recall that the definition of $X^{(\lambda,\gamma)}_{(n)}$ is given by (39) for $n\geq 2$ and by (37) when $n=1$. Using (90), we look at the difference between $X^{(\lambda,\gamma)}_{(n)}$ and $\xi^{\gamma}_{(n)}$:

$$X^{(\lambda,\gamma)}_{(n)}(t) - \xi^{\gamma}_{(n)}(t) = F_0^{(\lambda,\gamma)} + \int_0^t ds\rho^{\gamma}_{t-s}(0)b(s) \tag{91a}$$

$$+\sum_{j=1}^{n-1}\left[\int_0^t dsF_j^{(\lambda,\gamma)}(s) - (-1)^{j+1}\left(\mathbb{K}^{*(j+1)} * b\right)(t)\right] \tag{91b}$$

$$+\left[\int_0^t ds\dot{Y}^{(\lambda,\gamma)}_{(n)}(s) - (-1)^{n+1}\int_0^t ds\xi^{\gamma}_{(n)}(s)\mathbb{K}^{*(n+1)}(t-s)\right] \tag{91c}$$

$$= F_0^{(\lambda,\gamma)} + \int_0^t ds\rho^{\gamma}_{t-s}(0)b(s) \tag{91d}$$

$$+\sum_{j=1}^{n-1}\left[\int_0^t dsF_j^{(\lambda,\gamma)}(s) - (-1)^{j+1}\left(\mathbb{K}^{*(j+1)} * b\right)(t)\right] \tag{91e}$$

$$+\int_0^t ds\left(\dot{Y}^{(\lambda,\gamma)}_{(n)}(s)-\eta^{\gamma}_{(n)}(s)\right), \tag{91f}$$

where for $n = 1$ the sum in (91b) (and in (91e)) is understood to be equal to zero. As we have already said, we want to prove that $\forall n \geq 1$, $X^{(\lambda,\gamma)}_{(n)}$ converges to $\xi^{\gamma}_{(n)}$ for $\gamma \in \left(0, \frac{n}{n+1}\right)$. To this end, let us further expand the integrand in (91f), using the fact that $\dot{Y}^{(\lambda,\gamma)}_{(n)}$ solves Eq. (42):

$$\left(\dot{Y}^{(\lambda,\gamma)}_{(n)}-\eta^{\gamma}_{(n)}\right)(t)=R^{(\lambda,\gamma)}_{(n)}(t)+(-1)^{(n+1)}\int_0^t ds\left(\dot{Y}^{\gamma}_{\lambda}-\eta^{\gamma}\right)(s)\mathbb{K}^{*(n+1)}(t-s) \tag{92}$$

where

$$\begin{aligned}R^{(\lambda,\gamma)}_{(n)}(t)&:=\sum_{j=n}^{2n}F^{(\lambda,\gamma)}_{j}(t)-G^{\gamma}_{(n)}(t)\\&\quad+(-1)^{(n+1)}\int_0^t ds\,\dot{Y}^{(\lambda,\gamma)}_{(n)}(s)\left[P^{*(n+1)}_{t,s}-\mathbb{K}^{*(n+1)}(t-s)\right],\end{aligned} \tag{93}$$

and $G^{\gamma}_{(n)}(t)$ is defined in (88).
Let $\delta \in (0, 1)$. From now on we assume that $t \geq \delta$.

Remark 4.1. We omit to study the case $t < \delta$ because it can be treated in the same way as it is dealt with in [2], where it is presented explicitly; see in particular (3.23), (3.44) and (3.45) in [2]. In other words, what we actually show is that the estimates in (106), (113)–(117) and (121) are valid when the supremum is taken over the interval $[\delta, T]$ (more precisely, in the case of (113)–(117) and (121) the supremum should be over $[\lambda^a, T]$, because at that point δ will have been chosen to be equal to λ^a; see the lines before (117)). Though, by acting as in [2], we can show that the same estimate holds true when the supremum is taken over the whole interval $[0, T]$. Hence from now on we will assume that $t \geq \delta$ in order to streamline the notation and the presentation of the proof.

Using the definition of $K^{(\lambda,\gamma)}_{s,0}$, we obtain the following decomposition:

$$\begin{aligned}&\int_0^t ds\rho^{\gamma}_{t-s}(0)K^{(\lambda,\gamma)}_{s,0}\\&=\int_0^{\delta} ds\rho^{\gamma}_{t-s}(0)K^{(\lambda,\gamma)}_{s,0}+\int_{\delta}^t ds\rho^{\gamma}_{t-s}(0)\left\langle\varphi^{(\lambda)}_s,\int_0^{s-\delta}db(s')\rho^{\gamma}_{s-s'}\varphi^{(\lambda)}_{s'}\right\rangle\\&\quad+\int_{\delta}^t ds\rho^{\gamma}_{t-s}(0)\left\langle\varphi^{(\lambda)}_s,\int_{s-\delta}^{s}db(s')\rho^{\gamma}_{s-s'}\varphi^{(\lambda)}_{s'}\right\rangle.\end{aligned}$$

We now use the above decomposition to rewrite the difference between $F^{(\lambda,\gamma)}_1$ and $\int_0^t db(s)\mathbb{K}^{*(2)}_{\gamma}(t-s)$. For $\gamma \in (0, 1/2)$,

$$\begin{aligned}&\left|F^{(\lambda,\gamma)}_1(t)-\int_0^t db(s)\mathbb{K}^{*(2)}_{\gamma}(t-s)\right|^N\\&=\left|\int_0^t ds\,P^{(\lambda,\gamma)}_{t,s}K^{(\lambda,\gamma)}_{s,0}-\int_0^t db(s)\mathbb{K}^{*(2)}_{\gamma}(t-s)\right|^N\end{aligned} \tag{94}$$

$$\le C \left| \int_0^t ds \left(P_{t,s}^{(\lambda,\gamma)} - \rho_{t-s}^{\gamma}(0) \right) K_{s,0}^{(\lambda,\gamma)} \right|^N \tag{95}$$

$$+ C \left| \int_0^\delta ds \rho_{t-s}^{\gamma}(0)\, K_{s,0}^{(\lambda,\gamma)} \right|^N$$

$$+ C \left| \int_\delta^t ds \rho_{t-s}^{\gamma}(0) \left\langle \varphi_s^{(\lambda)}, \int_0^{s-\delta} db(s') \rho_{s-s'}^{\gamma} \varphi_{s'}^{(\lambda)} \right\rangle \right.$$

$$\left. - \int_\delta^t ds \rho_{t-s}^{\gamma}(0) \int_0^{s-\delta} db(s') \rho_{s-s'}^{\gamma}(0) \right|^N$$

$$+ C \left| \int_\delta^t ds \rho_{t-s}^{\gamma}(0) \int_0^{s-\delta} db(s') \rho_{s-s'}^{\gamma}(0) - \int_0^t db(s) \mathbb{K}^{*(2)}(t-s) \right|^N \tag{96}$$

$$+ C \left| \int_\delta^t ds \rho_{t-s}^{\gamma}(0) \left\langle \varphi_s^{(\lambda)}, \int_{s-\delta}^{s} db(s') \rho_{s-s'}^{\gamma} \varphi_{s'}^{(\lambda)} \right\rangle \right|^N$$

$$\le C \left| \int_0^t ds \left(P_{t,s}^{(\lambda,\gamma)} - \rho_{t-s}^{\gamma}(0) \right) K_{s,0}^{(\lambda,\gamma)} \right|^N \tag{97}$$

$$+ C \left| \int_0^\delta ds \rho_{t-s}^{\gamma}(0)\, K_{s,0}^{(\lambda,\gamma)} \right|^N + C \left| \int_\delta^t ds \rho_{t-s}^{\gamma}(0) \Gamma_s^{(\lambda,\delta,\gamma)} \right|^N \tag{98}$$

$$+ C \left| \int_\delta^t ds \rho_{t-s}^{\gamma}(0) K_{s,s-\delta}^{(\lambda,\gamma)} \right|^N + C \left| \Psi_{(1)}^{(\delta,\gamma)}(t) \right|^N, \tag{99}$$

where in the last inequality we used the definition of $\Gamma_s^{(\lambda,\delta,\gamma)}$ given in (79) and we set $\Psi_{(1)}^{(\delta,\gamma)}(t)$ to be the difference in (96), namely

$$\Psi_{(1)}^{(\delta,\gamma)}(t) := \int_\delta^t ds \rho_{t-s}^{\gamma}(0) \int_0^{s-\delta} db(s') \rho_{s-s'}^{\gamma}(0) - \int_0^t db(s) \mathbb{K}^{*(2)}(t-s),$$
$$\gamma \in (0, 1/2).$$

For $n \ge 1$, we define

$$\Psi_{(n+1)}^{(\delta,\gamma)}(t) := \int_0^t ds \rho_{t-s}^{\gamma}(0)\, \Psi_{(n)}^{(\delta,\gamma)}(s), \quad \gamma \in \left(0, \frac{n}{n+1}\right). \tag{100}$$

In the same way, by using (31), (50) and (100), we have

$$\left| F_2^{(\lambda,\gamma)}(t) + \int_0^t db(s) \mathbb{K}^{*(3)}(t-s) \right|^N \tag{101}$$

$$\le C \left| \int_0^t ds \int_0^s ds' \rho_{s-s'}^{\gamma}(0) \left(P_{t,s}^{(\lambda,\gamma)} - \rho_{t-s}^{\gamma}(0) \right) K_{s',0}^{(\lambda,\gamma)} \right|^N \tag{102}$$

$$+ C \left| \int_0^t ds \rho_{t-s}^{\gamma}(0) \int_0^\delta ds' \rho_{s-s'}^{\gamma}(0) K_{s',0}^{(\lambda,\gamma)} \right|^N$$

$$+ C \left| \int_0^t ds \rho_{t-s}^{\gamma}(0) \int_\delta^s ds' \rho_{s-s'}^{\gamma}(0) \Gamma_{s'}^{(\lambda,\delta,\gamma)} \right|^N \tag{103}$$

$$+ C\left|\int_0^t ds\rho^{\gamma}_{t-s}(0)\int_{\delta}^{s} ds'\rho^{\gamma}_{s-s'}(0)K^{(\lambda,\gamma)}_{s',s'-\delta}\right|^N + C\left|\Psi^{(\delta,\gamma)}_{(2)}(t)\right|^N. \tag{104}$$

Remark 4.2. We will show that the terms in (98) and the first addend in (99) (and hence also the addends in (103) and the first addend in (104)) are small for $\gamma \in (0, 1)$ (see (106), (107) and (76)). The reason why we need to iterate the equation for $X^{(\lambda,\gamma)}$ and ξ^{γ} an infinite number of times comes from $\Psi^{(\delta,\gamma)}_{(n)}$ (see (45) and (105)). We will in fact prove that

$$\left(E\sup_{t\in[\delta,T]}\left|\Psi^{(\delta,\gamma)}_{(n)}(t)\right|^N\right)^{\frac{1}{N}} \le C\delta^{n-(n+1)\gamma}. \tag{105}$$

Also, we will show that (97) is small when $\gamma \in \left(0, \frac{1}{2}\right)$ and (102) is small for $\gamma \in \left(\frac{1}{2}, 1\right)$; see (110) and (113).

Let us now address the points mentioned in Remark 4.2, in the same order in which we listed them.

For $p, q > 1$ s.t. $p^{-1} + q^{-1} = 1$ and $p\gamma < 1$, we have

$$\left|\int_0^{\delta} ds\rho^{\gamma}_{t-s}(0)\, K^{(\lambda,\gamma)}_{s,0}\right|^N \le C\left|\int_0^{\delta}\frac{ds}{(t-s)^{p\gamma}}\right|^{\frac{N}{p}}\left|\int_0^{\delta} ds|K^{(\lambda,\gamma)}_{s,0}|^q\right|^{\frac{N}{q}}.$$

Since $t \ge \delta$,

$$\int_0^{\delta}\frac{ds}{(t-s)^{p\gamma}} \le \int_0^{\delta}\frac{ds}{(\delta-s)^{p\gamma}} = C\delta^{1-p\gamma},$$

and hence

$$\begin{aligned} E\sup_{t\in[\delta,T]}\left|\int_0^{\delta} ds\rho^{\gamma}_{t-s}(0)\, K^{(\lambda,\gamma)}_{s,0}\right|^N &\le C\delta^{\frac{1-p\gamma}{p}N} E\left|\int_0^{\delta} ds|K^{(\lambda,\gamma)}_{s,0}|^q\right|^{\frac{N}{q}} \\ &\le C\delta^{\frac{1-p\gamma}{p}N}\left(E\left|\int_0^{\delta} ds|K^{(\lambda,\gamma)}_{s,0}|^q\right|^N\right)^{\frac{1}{q}} \le C\delta^{\frac{1-p\gamma}{p}N}\delta^{\frac{N}{q}}\sup_{s\in[0,T]}\left(E|K^{(\lambda,\gamma)}_{s,0}|^{Nq}\right)^{\frac{1}{q}}, \end{aligned}$$

where in the last inequality we used Note 3.1. If we choose $p = \frac{\gamma+1}{2\gamma}$ and $q = \frac{\gamma+1}{1-\gamma}$, by using (64) we get

$$E\sup_{t\in[0,T]}\left|\int_0^{\delta} ds\rho^{\gamma}_{t-s}(0)\, K^{(\lambda,\gamma)}_{s,0}\right|^N \le C\delta^{1-\gamma}\lambda^{\frac{1}{2\gamma}-\zeta-1}, \quad \gamma \in (0, 1). \tag{106}$$

By the same sort of trick as was used to get (106), we also get

$$\left|\int_{\delta}^{s} ds'\rho^{\gamma}_{s-s'}(0)\Gamma^{(\lambda,\delta,\gamma)}_{s'}\right|^N \le C\sup_{s'\in[\delta,s]}\left(E\left|\Gamma^{(\lambda,\delta,\gamma)}_{s'}\right|^{Nq}\right)^{\frac{1}{q}}.$$

Therefore, using (78), we have

$$E\sup_{s\in[\delta,T]}\left|\int_{\delta}^{s} ds'\rho^{\gamma}_{s-s'}(0)\Gamma^{(\lambda,\delta,\gamma)}_{s'}\right|^N \le C\left[\lambda^{\frac{3}{4}} + \lambda^{\frac{1}{2\gamma}-\frac{1}{4}}\right]e^{CT^{(n+1)(1-\gamma)}}$$

$$+\left[\lambda^{\frac{M}{8}}(T^{1-2\gamma}\mathbf{1}_{\{0<\gamma<1/2\}}+\delta^{1-2\gamma}\mathbf{1}_{\{1/2<\gamma<1\}})\right]e^{CT^{(n+1)(1-\gamma)}}, \quad \gamma\in(0,1). \tag{107}$$

Notice that on the right hand side of the above equation, n appears because $X_{(n)}^{(\lambda,\gamma)}$ is contained in the definition of $\Gamma_s^{(\lambda,\delta,\gamma)}$; see (78), (47) and the comment after it.

As for the first term in (99) (or the first term in (104)), we just use (76) in Lemma 5. In order to prove (105), we show in some detail how the estimate for $\Psi_{(1)}^{(\lambda,\gamma)}$ is obtained; the way that one gets (105) for $n \geq 1$ should then be obvious from the definition (100) and using (25). Recalling that we are assuming that $t \geq \delta$, using (24) and exchanging the order of integration in the definition of $\Psi_{(1)}^{(\lambda,\gamma)}$ we have

$$\begin{aligned}\Psi_{(1)}(t)^{(\lambda,\gamma)} &= \int_0^{t-\delta} db(s)\int_{s+\delta}^{t} ds'\rho_{t-s'}^{\gamma}(0)\rho_{s'-s}^{\gamma}(0)-\int_0^{t} db(s)\int_{s}^{t} ds'\rho_{t-s'}^{\gamma}(0)\rho_{s'-s}^{\gamma}(0)\\ &= -\int_0^{t-\delta} db(s)\int_{s}^{s+\delta} ds'\rho_{t-s'}^{\gamma}(0)\rho_{s'-s}^{\gamma}(0)\\ &\quad -\int_{t-\delta}^{t} db(s)\int_{s}^{t} ds'\rho_{t-s'}^{\gamma}(0)\rho_{s'-s}^{\gamma}(0).\end{aligned} \tag{108}$$

Now we can estimate the two terms in (108) separately. In both cases we first make a further change of variables and then integrate the stochastic integral by parts. We show how to handle the first; for the second the procedure is the same:

$$\begin{aligned}&\left|\int_0^{t-\delta} db(s)\int_{s}^{s+\delta} ds'\rho_{t-s'}^{\gamma}(0)\rho_{s'-s}^{\gamma}(0)\right| = \left|\int_0^{t-\delta} db(s)\int_0^{\delta} du\rho_{t-s-u}^{\gamma}(0)\rho_u^{\gamma}(0)\right|\\ &\leq \left|b(t-\delta)\int_0^{\delta} du\rho_{\delta-u}^{\gamma}(0)\rho_u^{\gamma}(0)\right|\\ &\quad+\sup_{s\in[0,t-\delta]}|b(s)|\left|\int_0^{t-\delta} ds\frac{\partial}{\partial s}\int_0^{\delta} du\rho_{t-s-u}^{\gamma}(0)\rho_u^{\gamma}(0)\right|\\ &\leq \left|b(t-\delta)\int_0^{\delta} du\rho_{\delta-u}^{\gamma}(0)\rho_u^{\gamma}(0)\right|\\ &\quad+\sup_{s\in[0,t-\delta]}|b(s)|\left|\int_0^{\delta} du\rho_{\delta-u}^{\gamma}(0)\rho_u^{\gamma}(0)-\int_0^{\delta} du\rho_{t-u}^{\gamma}(0)\rho_u^{\gamma}(0)\right|.\end{aligned}$$

Notice now that from (24),

$$\int_0^{\delta} du\rho_{\delta-u}^{\gamma}(0)\rho_u^{\gamma}(0) = C\delta^{1-2\gamma}$$

and, since $t \geq \delta$,

$$\int_0^{\delta}\rho_{t-u}^{\gamma}(0)\rho_u^{\gamma}(0) = C\int_0^{\delta}\frac{du}{(t-u)^{\gamma}u^{\gamma}} \leq C\int_0^{\delta}\frac{du}{(\delta-u)^{\gamma}u^{\gamma}} = \int_0^{\delta} du\rho_{\delta-u}^{\gamma}(0)\rho_u^{\gamma}(0).$$

So, after dealing with the second term in (108) in an analogous way, (105) follows on using (72).

Let us now turn to (97) and (102). Let $\beta > 0$; then for (97), applying the Hölder inequality, we have

$$\left|\int_0^t ds \left(P_{t,s}^{(\lambda,\gamma)} - \rho_{t-s}^{\gamma}(0)\right) K_{s,0}^{(\lambda,\gamma)}\right|^N$$
$$\leq \sup_{0\leq s\leq t}\left\{\left|P_{t,s}^{(\lambda,\gamma)} - \rho_{t-s}^{\gamma}(0)\right|^N (t-s)^{\gamma(1+\beta)N}\right\}$$
$$\times \left|\int_0^t ds \left|K_{s,0}^{(\lambda,\gamma)}\right|^p\right|^{\frac{N}{p}} \left|\int_0^t \frac{ds}{(t-s)^{\gamma q(1+\beta)}}\right|^{\frac{N}{q}}. \tag{109}$$

Looking at the last integral in (109), we need to impose the integrability condition $\beta < -1+1/\gamma$. Taking the supremum for $t \in [0, T]$, and the expectation of both sides, using (64) and (84), we then obtain that for $\gamma \in \left(0, \frac{1}{2}\right)$ and for any $N \geq 1$,

$$E \sup_{t\in[0,T]}\left|\int_0^t ds \left(P_{t,s}^{(\lambda,\gamma)} - \rho_{t-s}^{\gamma}(0)\right) K_{s,0}^{(\lambda,\gamma)}\right|^N \leq C\lambda^{\frac{1}{\gamma}-2\zeta-\frac{3}{2}} e^{CT^{(n+1)(1-\gamma)}}, \tag{110}$$

where we have chosen $\beta = 1/2$ in (84). We can make such a choice for β because when we study the difference in (94), and hence (97), we take $\gamma \in \left(0, \frac{1}{2}\right)$; see Remark 4.2. When we consider (102), we cannot mimic what we have done for (97); in fact from (109) we get that the left hand side of (110) is bounded by $\lambda^{\beta+\frac{1}{\gamma}-\zeta-2} \exp(CT^{(n+1)(1-\gamma)})$. When we impose the integrability condition $\beta < -1 + 1/\gamma$ and $\beta + \frac{1}{\gamma} - \zeta - 2 > 0$, $\beta \in (0, 1]$, we find that these two conditions together cannot be satisfied for all $\gamma \in (0, 1)$ (actually they hold at most for $\gamma \in (0, 2/3)$). So, when $\gamma \in \left(\frac{1}{2}, 1\right)$ we need to do something else.

$$\left|\int_0^t ds \int_0^s ds' \rho_{s-s'}^{\gamma}(0) \left(P_{t,s}^{(\lambda,\gamma)} - \rho_{t-s}^{\gamma}(0)\right) K_{s',0}^{(\lambda,\gamma)}\right|^N \tag{111}$$
$$\leq C\left|\int_0^{t-\delta} ds \int_0^s ds' \rho_{s-s'}^{\gamma}(0) \left|P_{t,s}^{(\lambda,\gamma)} - \rho_{t-s}^{\gamma}(0)\right| K_{s',0}^{(\lambda,\gamma)}\right|^N$$
$$+ C\left|\int_{t-\delta}^{t} ds \int_0^s ds' \rho_{s-s'}^{\gamma}(0) \left|P_{t,s}^{(\lambda,\gamma)} - \rho_{t-s}^{\gamma}(0)\right| K_{s',0}^{(\lambda,\gamma)}\right|^N$$
$$\leq C \sup_{s\in[0,T]}\left|\int_0^s ds' \rho_{s-s'}^{\gamma}(0) K_{s',0}^{(\lambda,\gamma)}\right|^N$$
$$\cdot \left(\sup_{t\in[\delta,T]}\left|\int_0^{t-\delta} ds \left|P_{t,s}^{(\lambda,\gamma)} - \rho_{t-s}^{\gamma}(0)\right| + \int_{t-\delta}^t \rho_{t-s}^{\gamma}(0)\right|^N\right), \tag{112}$$

where in the last inequality we have used (86) and then (23). By (77) and (85), we then have

$$\left(E \sup_{t\in[0,T]}\left|\int_0^t ds \int_0^s ds' \rho_{s-s'}^{\gamma}(0) \left|P_{t,s}^{(\lambda,\gamma)} - \rho_{t-s}^{\gamma}(0)\right| K_{s',0}^{(\lambda,\gamma)}\right|^N\right)^{\frac{1}{N}}$$
$$\leq C\lambda^{\frac{1}{2\gamma}-\zeta-1}\left(\lambda^{\frac{1}{2\gamma}-\zeta} + \delta^{1-\gamma}\right) e^{CT^{(n+1)(1-\gamma)}}. \tag{113}$$

If in (105), (106), (107) and (76) we choose $\delta = \lambda$ and $M > 0$, recalling (110) we have that for $\gamma \in \left(0, \frac{1}{2}\right)$ and $\forall N \geq 1$, $\exists b(\gamma) > 0$ s.t.

$$\left(E \sup_{t\in[0,T]} \left| F_1^{(\lambda,\gamma)}(t) - \int_0^t db(s)\mathbb{K}_\gamma^{*(2)}(t-s)\right|^N\right)^{\frac{1}{N}} \leq C\lambda^{b(\gamma)} e^{CT^{2(1-\gamma)}}. \tag{114}$$

Via (31) and (50), this implies that for $n \geq 1$, $\gamma \in \left(0, \frac{1}{2}\right)$ and $\forall N \geq 1$, $\exists\, b(\gamma) > 0$ s.t.

$$\left(E \sup_{t\in[0,T]} \left| F_n^{(\lambda,\gamma)}(t) - (-1)^{(n+1)} \int_0^t db(s)\mathbb{K}_\gamma^{*(n+1)}(t-s)\right|^N\right)^{\frac{1}{N}} \leq C\lambda^{b(\gamma)} e^{CT^{2(1-\gamma)}}. \tag{115}$$

On the other hand, if in (105)–(107) and (76) we choose $\delta = \lambda^a$, with $a = \frac{2\gamma-1}{2\gamma(1-\gamma)}$, and $M > \frac{4(2\gamma-1)^2}{\gamma(1-\gamma)}$, recalling (113), we find that $\forall n \geq 2$, $\frac{1}{2} < \gamma < \frac{n}{n+1}$ and $N \geq 1$, $\exists\, l(\gamma) > 0$ s.t.

$$\left(E \sup_{t\in[0,T]} \left| F_n^{(\lambda,\gamma)}(t) - (-1)^{(n+1)} \int_0^t db(s)\mathbb{K}^{*(n+1)}(t-s)\right|^N\right)^{\frac{1}{N}} \leq C\lambda^{l(\gamma)} e^{CT^{(n+1)(1-\gamma)}}. \tag{116}$$

Note 4.1. We want to stress that the above estimate (116) is needed only for $n \geq 2$ and $\frac{1}{2} < \gamma < \frac{n}{n+1}$, whereas (115) is valid for any $n \geq 1$ and $\gamma \in \left(0, \frac{1}{2}\right)$. In other words we will not need an estimate on $\left|F_1^{(\lambda,\gamma)}(t) - \int_0^t db(s)\mathbb{K}_\gamma^{*(2)}(t-s)\right|$ for $\gamma > \frac{1}{2}$.

Set now

$$\Psi_{(0)}(t)^{(\delta,\gamma)} := \int_0^t ds b(s)\rho_{t-s}^\gamma(0) - \int_\delta^t ds \int_0^{s-\delta} db(s')\rho_{s-s'}^\gamma(0);$$

then

$$\left|F_0^{(\lambda,\gamma)} + \int_0^t ds b(s)\rho_{t-s}^\gamma(0)\right|^N \leq C\left|\int_0^\delta ds K_{s,0}^{(\lambda)}\right|^N + C\left|\int_\delta^t ds \Gamma_s^{(\lambda,\delta,\gamma)}\right|^N + C\left|\int_\delta^t ds K_{s,s-\delta}^{(\lambda,\gamma)}\right|^N + C\left|\Psi_{(0)}(t)^{(\delta,\gamma)}\right|^N.$$

It is easy to prove that

$$\left(E \sup_{t\in[\delta,T]} \left|\Psi_{(0)}(t)^{(\delta,\gamma)}\right|^N\right)^{\frac{1}{N}} \leq C\delta^{1/2}.$$

So by (64), (78) and (75), on choosing again $\delta = \lambda^a, a = \mathbf{1}_{\{0<\gamma<1/2\}} + \frac{2\gamma-1}{2\gamma(1-\gamma)}\mathbf{1}_{\{1/2<\gamma<1\}}$ and $M > 0 \cdot \mathbf{1}_{\{0<\gamma<1/2\}} + \frac{4(2\gamma-1)^2}{\gamma(1-\gamma)}\mathbf{1}_{\{1/2<\gamma<1\}}$, we get that $\forall n \geq 1$, $0 < \gamma < \frac{n}{n+1}$ and $\forall N \geq 1$,

$\exists\, m(\gamma) > 0$ s.t.

$$\left(E \sup_{t\in[0,T]} \left| F_0^{(\lambda,\gamma)}(t) - \int_0^t dsb(s)\rho_{t-s}^{\gamma}(0) \right|^N \right)^{\frac{1}{N}} \le C\lambda^{m(\gamma)} e^{CT^{(n+1)(1-\gamma)}}. \tag{117}$$

We will also need the following estimate:

$$\left(E \sup_{t\in[\delta,T]} \left| \int_0^t ds\, \Psi_{(n)}^{(\delta,\gamma)}(s) \right|^N \right)^{\frac{1}{N}} \le C\delta^{(n+1)(1-\gamma)}. \tag{118}$$

This inequality can be worked out with calculations analogous to those needed to obtain (105), and hence we omit them; roughly speaking, looking at (105), (118) is correct thanks to the further integration. Also, it is what one would expect in view of the fact that $\int_0^t dsb(s)\mathbb{K}^{*(n+1)}(t-s)$ is defined for any $\gamma \in (0,1)$, as opposed to $\int_0^t db(s)\mathbb{K}^{*(n+1)}(t-s)$. With this remark in mind, it is easily seen that, with the same steps as led to an estimate on

$$\left| F_n^{(\lambda,\gamma)}(t) - (-1)^{(n+1)} \int_0^t db(s)\mathbb{K}^{*(n+1)}(t-s) \right|,$$

using this time (113) and (118), we have that $\forall n \ge 1$, $\gamma \in \left(0, \frac{n}{n+1}\right)$ and $\forall N \ge 1$, $\exists \tau = \tau(\gamma, N) > 0$ s.t.

$$\lim_{\lambda\to 0} E \sup_{t \le \tau |\log\lambda|^{\frac{1}{(n+1)(1-\gamma)}}} \left| \int_0^t F_n^{(\lambda,\gamma)}(s) - (-1)^{(n+1)} \int_0^t ds\, b(s)\mathbb{K}^{*(n+1)}(t-s) \right|^N = 0. \tag{119}$$

The last ingredient that we will need in order to conclude is the following estimate: $\forall n \ge 1$, $\gamma \in (0, \frac{n}{n+1})$ and $\forall N \ge 1$, $\exists\, d(\gamma) > 0$ s.t.

$$\left(E \sup_{t\in[0,T]} \left| \int_0^t ds\, \dot{Y}_{(n)}^{(\lambda,\gamma)}(s) \left[P_{t,s}^{*(n+1)} - \mathbb{K}^{*(n+1)}(t-s) \right] \right|^N \right)^{\frac{1}{N}} \le C\lambda^{d(\gamma)} e^{CT^{(n+1)(1-\gamma)}}, \tag{120}$$

which is obtained by combining (84) and (71) when $n = 1$; when $n \ge 2$, we act like in (111)–(112) and then use (85) and (71).

From the definition of $R_{(n)}^{(\lambda,\gamma)}$ given in (93), using (115), (116) and (120), it is straightforward to see that $\exists\, \tilde{d}(\gamma) > 0$ s.t.

$$\left(E \sup_{t\in[0,T]} \left| R_{(n)}^{(\lambda,\gamma)}(t) \right|^N \right)^{\frac{1}{N}} \le C\lambda^{\tilde{d}(\gamma)} e^{CT^{(n+1)(1-\gamma)}}, \tag{121}$$

for any $n \geq 1$, $\gamma \in \left(0, \frac{n}{n+1}\right)$ and $N \geq 1$. Hence, the Gronwall Lemma applied to (92) gives that $\forall n \geq 1$, $\gamma \in \left(0, \frac{n}{n+1}\right)$ and $N \geq 1$, $\exists \tau = \tau(\gamma, N) > 0$ s.t.

$$\lim_{\lambda \to 0} E \sup_{t \leq \tau |\ln \lambda|^{\frac{1}{(n+1)(1-\gamma)}}} \left| \left(\dot{Y}_\lambda^\gamma - \eta^\gamma \right)(t) \right|^N = 0. \tag{122}$$

Finally, looking at (91d)–(91f), thanks to (117), (119) and (122), Theorem 3 is proven.

5. Proof of Theorem 2

In the diffusive case, the integral equation (2) is explicitly solvable. To our knowledge, (14) cannot be solved for $\gamma \neq \frac{1}{2}$. However, considering the associated Green function, that is, the solution of

$$F^\gamma(t) = 1 - \int_0^t ds \rho_{t-s}^\gamma(0) F^\gamma(s), \quad 0 < \gamma < 1, \tag{123}$$

one gets

$$\xi^\gamma(t) = \int_0^t db(s) F^\gamma(t-s), \quad 0 < \gamma < 1. \tag{124}$$

Notice that the theory of Volterra integral equations for kernels with bounded iterates implies that the solution to (123) is unique, as commented at the beginning of Section 3, after the statement of Lemma 1.

Lemma 7. *For any* $0 < \gamma < 1$, *the following holds:*

$$\lim_{t \to \infty} t^{1-\gamma} F^\gamma(t) = \frac{\sin(\pi\gamma)}{\pi c(\gamma)}, \tag{125}$$

where $c(\gamma)$ *is defined in* (20).

Remark 5.1. Since $c(1/2) = (2\pi)^{-1/2}$, Lemma 7 is an extension of Theorem 2.2 in [2]. When $\gamma = 1/2$, it provides an alternative proof of such a theorem.

Proof of Lemma 7. By taking the Laplace transform of (123) we obtain that the Green function F^γ has the Laplace transform

$$(F^\gamma)^\#(\mu) = \frac{\mu^{-\gamma}}{\mu^{1-\gamma} + c(\gamma)\Gamma(1-\gamma)}. \tag{126}$$

Provided that $F^\gamma(t)$ is monotone decreasing, the Tauberian theorem for densities (see e.g. [7]) gives

$$\lim_{t \to \infty} t^{1-\gamma} F^\gamma(t) = \frac{1}{\Gamma(\gamma)} \lim_{\mu \to 0} \mu^\gamma (F^\gamma)^\#(\mu).$$

Therefore the only thing that we need to show is that $F^\gamma(t)$ is monotone decreasing. We recall that a function is completely monotone if and only if its even derivatives are positive and the odd ones are negative. Furthermore, a function is the Laplace transform of a positive measure if and only if it is completely monotone (see again [7]). We think of $dF^\gamma(t)$ as an (a priori signed) measure on $\mathbb{R}_+$ and introduce

$$\Phi^\#(\mu) := -\int_0^\infty e^{-\mu t} dF^\gamma(t) = 1 - \mu (F^\gamma)^\#(\mu).$$

By (126) we have

$$\Phi^\#(\mu) = \frac{c(\gamma)\Gamma(1-\gamma)}{\mu^{1-\gamma} + c(\gamma)\Gamma(1-\gamma)}.$$

The function $(0,\infty) \ni \mu \longrightarrow \mu^{1-\gamma}$ is positive and has completely monotone derivatives. For $A > 0$ the function $(0,\infty) \ni x \longrightarrow A(A+x)^{-1}$ is completely monotone. Hence (see [7]), the function $\Phi^\#(\mu)$ is completely monotone and we are done. □

Proof of Theorem 2. By (124) we get

$$E\left[\xi^\gamma(t)\right]^2 = \int_0^t (F^\gamma(s))^2 ds,$$

so (15) is straightforward. In order to prove the invariance principle in Theorem 1, we first need to prove tightness of the process $\xi^\gamma_\epsilon(t)$. From (124) to (125) a few computations show that for each $\gamma \in \left(\frac{1}{2}, 1\right)$ there exists a constant $C = C(\gamma)$ such that

$$\lim_{\epsilon \to 0} E(\xi^\gamma_\epsilon(t) - \xi^\gamma_\epsilon(s))^2 \le C(t-s)^{2\gamma-1}.$$

Since ξ^γ_ϵ is a Gaussian process, we can first obtain a bound on the higher moments, thus getting tightness from Kolmogorov's criterion. Finally, the convergence of the finite dimensional distributions follows from the convergence of the covariance, deduced from (124) to (125). □

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Appendix A. Existence and uniqueness

In this section we sketch the proof of the existence, uniqueness and continuity of the solution of the system (32).

Theorem 4. *Let B be the Banach space of vectors $(X,h) \in \mathbb{R} \times L^2(\mathbb{R})$ with the norm*

$$\|(X,h)\|_B := \sqrt{|X|^2 + \|h\|_2^2}.$$

Let us consider the following Cauchy problem with initial datum $(X_0,h_0) \in B$:

$$\begin{cases} X(t) = X_0 + b(t) + \int_0^t ds\ \Upsilon(X(s),h(s)) \\ h(t) = \rho^\gamma_t h_0 - \int_0^t db(s) \rho^\gamma_{t-s} \varphi_{X(s)} - \int_0^t ds\ \Upsilon(X(s),h(s)) \rho^\gamma_{t-s} \varphi_{X(s)}, \end{cases} \tag{127}$$

where $\Upsilon : B \to \mathbb{R}$ is bounded and globally Lipschitz; recall that φ is a probability density in the Schwartz class of test functions and $\varphi_X = \varphi(x - X)$.

Then for any $(X_0, h_0) \in B$ there exists a unique solution to (127); such a solution, $(X(t), h(t))$, belongs to $\mathcal{C}(\mathbb{R}_+; B)$ and is such that

$$\sup_{t\in[0,T]} E\|(X(t), h(t))\|_B^2 < \infty \quad \forall T > 0. \tag{128}$$

Uniqueness holds in the following sense: if $(\bar{X}(t), \bar{h}(t))$ is another continuous solution satisfying (128), then

$$P\left(\sup_{t\in[0,T]} \|(X(t), h(t)) - (\bar{X}(t), \bar{h}(t))\|_B^2 = 0\right) = 1 \quad \forall T > 0.$$

Proof. We prove existence by Picard iterations, uniqueness by using the Gronwall Lemma and continuity by using Kolmogorov's criterion. For the time being, ρ_t^γ is either (16) or (18), so $\gamma \in (0, 1)$.

Existence: construct the sequence $\{(X^{(n)}(t), h^{(n)}(t))\}$ such that $(X_t^{(0)}, h_t^{(0)}) = (X_0, \rho_t^\gamma h_0)$ and, for $n \geq 1$,

$$\begin{cases} X^{(n)}(t) = X_0 + b(t) + \int_0^t ds\ \Upsilon(X^{(n-1)}(s), h^{(n-1)}(s)) \\ h^{(n)}(t) = \rho_t h_0 - \int_0^t db(s)\rho_{t-s}^\gamma \varphi_{X^{(n-1)}(s)} - \int_0^t ds\rho_{t-s}^\gamma \beta(X^{(n-1)}(s), h^{(n-1)}(s)), \end{cases}$$

where we set $\beta(X, h) := \Upsilon(X, h)\varphi_X$; notice that for a suitable constant $K > 1$ we have

$$|\Upsilon(X, h)|^2 + \|\beta(X, h)\|_2^2 + \|\varphi_X\|_2^2 \leq K$$
$$|\Upsilon(X, h) - \Upsilon(Y, g)| + \|\beta(X, h) - \beta(Y, g)\|_2 + \|\varphi_X - \varphi_Y\|_2 \leq K\|(X, h) - (Y, g)\|_B,$$

for any (X, h) and (Y, g) in B. Hence

$$E\|(X^{(1)}(t), h^{(1)}(t)) - (X^{(0)}(t), h^{(0)}(t))\|_B^2 \leq 2K^2\left(t + t^2\right);$$

moreover, by the Cauchy–Schwarz inequality,

$$E|X^{(n+1)}(t) - X^{(n)}(t)|^2 \leq t \int_0^t dsE|\Upsilon(X^{(n)}(s), h^{(n)}(s)) - \Upsilon(X^{(n-1)}(s), h^{(n-1)}(s))|^2,$$

for $n \geq 1$. Similarly,

$$E\|h^{(n+1)}(t) - h^{(n)}(t)\|_2^2 \leq 2E\int_0^t ds\ \left\|\rho_{t-s}^\gamma[\varphi_{X^{(n)}(s)} - \varphi_{X^{(n-1)}(s)}]\right\|_2^2$$
$$+ 2tE\int_0^t ds\|\rho_{t-s}^\gamma[\Upsilon(X^{(n)}(s), h^{(n)}(s)) - \Upsilon(X^{(n-1)}(s), h^{(n-1)}(s))]\|_2^2.$$

As ρ_t^γ is a probability density, and because $\|\rho_t^\gamma\varphi\|_2 \leq \|\rho_t^\gamma\|_1\|\varphi\|_2$, ρ_t^γ is contractive on $L^2(\mathbb{R})$; therefore

$$E\|(X^{(n+1)}(t), h^{(n+1)}(t)) - (X^{(n)}(t), h^{(n)}(t))\|_B^2$$
$$\leq 2K^2(1 + t)\int_0^t dsE\|(X^{(n)}(s), h^{(n)}(s)) - (X^{(n-1)}(s), h^{(n-1)}(s))\|_B^2.$$

Iterating, we end up with

$$E\|(X^{(n+1)}(t), h^{(n+1)}(t)) - (X^{(n)}(t), h^{(n)}(t))\|_B^2 \leq \frac{[2K^2(t+t^2)]^{n+1}}{n!},$$

which gives uniform convergence on compacts $[0, T]$ of the sequence $\left(X^{(n)}(t), h^{(n)}(t)\right)$ to a limiting process, $(X(t), h(t))$. Such a process is therefore an $\mathcal{F}_t$-adapted solution to (127).

Uniqueness: by what we have done so far, it is clear that one can find a suitable $c(t)$ uniformly bounded on compacts such that if $\left(\bar{X}(t), \bar{h}(t)\right)$ is another solution, then

$$E\|(X(t), h(t)) - (\bar{X}(t), \bar{h}(t))\|_B^2 \leq c(t)\int_0^t ds E\|(X(t), h(t)) - (\bar{X}(t), \bar{h}(t))\|_B^2,$$

and hence uniqueness follows by the Gronwall Lemma; (128) is then a consequence of continuity, which we are going to prove.

Continuity: as $b(t)$ is a.s. continuous and $\beta(X, h)$ bounded, $X(t)$ is a.s. continuous. In order to prove continuity for $h(t)$ we first need to prove that for any $g \in L^2(\mathbb{R})$,

$$\lim_{t\to 0}\|\rho_t^\gamma g - g\|_2 = 0.$$

In fact, using the scaling property of the kernel and the Jensen inequality (weighted version), we get

$$\begin{aligned}\|\rho_t^\gamma g - g\|_2^2 &= \int_{\mathbb{R}} dx \left[\int_{\mathbb{R}} dw \rho_1^\gamma(w)\left(g(x - wt^\gamma) - g(x)\right)\right]^2 \\ &\leq \int_{\mathbb{R}} dx \int_{\mathbb{R}} dw \rho_1^\gamma(w)\left(g(x - wt^\gamma) - g(x)\right)^2 \\ &= \int_{\mathbb{R}} dw \rho_1^\gamma(w)\, \|T_{wt^\gamma} g - g\|_2^2\end{aligned}$$

where T_τ, $\tau \in \mathbb{R}$, is the translation $(T_\tau g)(x) = g(x - \tau)$. Let us study the integrand:

$$\|T_\tau g - g\|_2^2 = C\|\widehat{T_\tau g} - \hat{g}\|_2^2 = \int_{\mathbb{R}} d\xi |e^{-i\xi\tau}\,\hat{g}(\xi) - \hat{g}(\xi)|^2$$

$\Rightarrow \lim_{t\to 0}\|T_{wt^\gamma} g - g\|_2^2 = 0$ for a.e. w and

$$\rho_1^\gamma(w)\,\|T_{wt^\gamma} g - g\|_2^2 \leq C\rho_1^\gamma(w)\|g\|_2^2 \in L^1(\mathbb{R}),$$

so we can apply the dominated convergence theorem and conclude.

We are left with the continuity of $k(t) := h(t) - \rho_t^\gamma h_0$.

$$\begin{aligned}-k(t+\delta) + k(t) &= \int_0^t db(s)\,(\rho_{t+\delta-s}^\gamma - \rho_{t-s}^\gamma)\varphi_{X(s)} + \int_t^{t+\delta} db(s)\,\rho_{t+\delta-s}^\gamma\varphi_{X(s)} \\ &\quad + \int_0^t ds\ \Upsilon(X(s), h(s))\,(\rho_{t+\delta-s}^\gamma - \rho_{t-s}^\gamma)\varphi_{X(s)} \\ &\quad + \int_t^{t+\delta} ds\ \Upsilon(X(s), h(s))\,\rho_{t+\delta-s}^\gamma\varphi_{X(s)}.\end{aligned}$$

From now on we treat the cases $0 < \gamma < \frac{1}{2}$ and $\frac{1}{2} < \gamma < 1$ separately.

Let us start with the superdiffusion:

$$E\|k(t+\delta)-k(t)\|_2^4 \le C(A_1+A_2+A_3+A_4),$$

where

$$A_1 := E\left\|\int_0^t ds\,(\rho^\gamma_{t+\delta-s}-\rho^\gamma_{t-s})\varphi_{X(s)}\right\|_2^4,$$
$$A_2 := E\left\|\int_t^{t+\delta} ds\,\rho^\gamma_{t+\delta-s}\varphi_{X(s)}\right\|_2^4,$$
$$A_3 := E\left\|\int_0^t db(s)\,(\rho^\gamma_{t+\delta-s}-\rho^\gamma_{t-s})\varphi_{X(s)}\right\|_2^4,$$
$$A_4 := E\left\|\int_t^{t+\delta} db(s)\,\rho^\gamma_{t+\delta-s}\varphi_{X(s)}\right\|_2^4.$$

We need to estimate all the above terms:

$$\begin{aligned}
A_1 &\le CE\left[\int_0^t ds\|(\rho^\gamma_{t+\delta-s}-\rho^\gamma_{t-s})\varphi_{X(s)}\|_2\right]^4 \\
&= CE\left[\int_0^t ds\|(\rho^\gamma_{s+\delta}-\rho^\gamma_{s})\varphi\|_2\right]^4 \\
&= CE\left[\int_0^t ds\left(\int_{\mathbb{R}} dx\left(\int_{\mathbb{R}} dz\rho^\gamma_1(z)[\varphi(x-z(s+\delta)^\gamma)-\varphi(x-zs^\gamma)]\right)^2\right)^{\frac{1}{2}}\right]^4 \\
&\le CE\left[\int_0^t ds\int_{\mathbb{R}} dz\rho^\gamma_1(z)\|\varphi_{z(s+\delta)^\gamma}-\varphi_{zs^\gamma}\|_2\right]^4 \\
&\le CE\left[\int_0^t ds\int_{\mathbb{R}} dz\rho^\gamma_1(z)|z|\delta^\gamma\right]^4 \le Ct^4\delta^{4\gamma},
\end{aligned}$$

having used the scaling property (19) and (83).

$$\begin{aligned}
A_2 &\le E\left[\int_{\mathbb{R}} dx\delta\int_t^{t+\delta}(\rho^\gamma_{t+\delta-s}\varphi_{X(s)})^2 ds\right]^2 \\
&= \delta^2 E\left(\int_0^\delta ds\|\rho^\gamma_s\varphi_{X(t+\delta-s)}\|_2^2\right)^2 \le C\delta^4,
\end{aligned}$$

having used the Cauchy–Schwartz inequality and the contractivity.

In order to find estimates for the last two terms, let us choose $\psi(x)=\sqrt{1+|x|}$ so that $\forall f\in L^2(\mathbb{R})$, $\|f\|_2^4 \le \|\psi^{-2}\|_2^2\|f\psi\|_4^4$. Hence, via the Burkholder inequality and again Cauchy–Schwartz, we get

$$\begin{aligned}
A_3 &\le \|\psi^{-2}\|_2^2\,E\left\|\int_0^t db(s)\psi(\rho^\gamma_{t+\delta-s}-\rho^\gamma_{t-s})\varphi_{X(s)}\right\|_4^4 \\
&\le CE\left\|\int_0^t ds\left[\psi(\rho^\gamma_{t+\delta-s}-\rho^\gamma_{t-s})\varphi_{X(s)}\right]^2\right\|_2^2
\end{aligned}$$

$$\le Ct \int_0^t ds E \int_{\mathbb{R}} dx\, \psi(x+X(s))^4 \left[(\rho^{\gamma}_{t+\delta-s} - \rho^{\gamma}_{t-s})\varphi\right]^4 (x)$$

$$\le Ct \left(1 + E \sup_{u\in[0,t]} |X(u)|^2\right) \int_0^t ds \|\psi(\rho^{\gamma}_{s+\delta} - \rho^{\gamma}_{s})\varphi\|_4^4,$$

having used $\psi(x+X)^4 \le (1+|X|^2)\psi^4(x)$. Let us look at the integrand: since $\psi(x) \le \psi(y) + \sqrt{|x-y|}$, we have

$$\|\psi(\rho^{\gamma}_{s+\delta} - \rho^{\gamma}_{s})\varphi\|_4^4 \le C\|(\rho^{\gamma}_{s+\delta} - \rho^{\gamma}_{s})(\psi\varphi)\|_4^4 \tag{129}$$

$$+ C\int_{\mathbb{R}} dx \left[\int_{\mathbb{R}} dy(\rho^{\gamma}_{s+\delta}(x-y) - \rho^{\gamma}_{s}(x-y))\sqrt{|x-y|}\varphi(y)\right]^4. \tag{130}$$

The first addend can be estimated similarly to what we have done for A_1, so we get

$$\|(\rho^{\gamma}_{s+\delta} - \rho^{\gamma}_{s})(\psi\varphi)\|_4^4 \le C\delta^{4\gamma};$$

for the second, after applying Cauchy–Schwartz on the integrand, we find

$$(130) \le C\int_{\mathbb{R}} dx \left\{\left(\int_{\mathbb{R}} dy(\rho^{\gamma}_{s+\delta} - \rho^{\gamma}_{s})(x-y)\,|x-y|\right)^2 \times \left(\int_{\mathbb{R}} dy(\rho^{\gamma}_{s+\delta} - \rho^{\gamma}_{s})(x-y)\varphi^2(y)\right)^2\right\}$$

$$\le C\left(\int_{\mathbb{R}} dz \rho^{\gamma}_{1}(z)|z|((s+\delta)^{\gamma} - s^{\gamma})\right)^2 \|(\rho^{\gamma}_{s+\delta} - \rho^{\gamma}_{s})\varphi^2\|_2^2 \le C\delta^{4\gamma},$$

and we end up with

$$A_3 \le Ct^2\left(1 + E\sup_{u\in[0,T]}|X(u)|^2\right)\delta^{4\gamma}.$$

For A_4, analogously,

$$A_4 \le C\delta\left(1 + E\sup_{u\in[0,T]}|X(u)|^2\right)\int_0^{\delta} ds\|\psi\rho^{\gamma}_{s}\varphi\|_4^4$$

$$\le C\delta\left(1 + E\sup_{u\in[0,T]}|X(u)|^2\right) \times \int_0^{\delta} ds\left\{\|\psi\varphi\|_4^4 + \int_{\mathbb{R}} dx\left(\int_{\mathbb{R}} dy\rho^{\gamma}_{s}(x-y)\sqrt{|x-y|}\varphi(y)\right)^4\right\}.$$

Now the integral on the second line is estimated from above by

$$\int_0^{\delta} ds\left\{\|\psi\varphi\|_4^4 + \left(\int_{\mathbb{R}} dz\rho^{\gamma}_{s}(z)|z|\right)^2 \|\rho^{\gamma}_{s}\varphi^2\|_2^2\right\},$$

and so

$$A_4 \le C\delta^2\left(1 + E\sup_{u\in[0,T]}|X(u)|^2\right).$$

Proving continuity in the subdiffusive case is slightly more delicate; let us write

$$E\|k(t+\delta)-k(t)\|_2^{2N} \le C(\mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3+\mathcal{A}_4),$$

where $N=N(\gamma)$ is to be specified in the following and

$$\mathcal{A}_1 := E\left\|\int_0^t ds\,(\rho^{\gamma}_{t+\delta-s}-\rho^{\gamma}_{t-s})\varphi_{X(s)}\right\|_2^{2N},$$

$$\mathcal{A}_2 := E\left\|\int_t^{t+\delta} ds\,\rho^{\gamma}_{t+\delta-s}\varphi_{X(s)}\right\|_2^{2N},$$

$$\mathcal{A}_3 := E\left\|\int_0^t db(s)\,(\rho^{\gamma}_{t+\delta-s}-\rho_{t-s})\varphi_{X(s)}\right\|_2^{2N},$$

$$\mathcal{A}_4 := E\left\|\int_t^{t+\delta} db(s)\,\rho^{\gamma}_{t+\delta-s}\varphi_{X(s)}\right\|_2^{2N}.$$

For $\mathcal{A}_2$,

$$\mathcal{A}_2 \le C\delta^{2N}\left|\int_0^{\delta}\|\rho_s^{\gamma}\varphi\|_2^2\right|^{2N} \le C\,\delta^{4N},$$

so we need $N>\frac{1}{4}$.

For $\mathcal{A}_3$, let us choose again $\psi(x)=\sqrt{1+|x|}$ as an auxiliary function; then $\forall N>0$, $\|\psi^{-2}\|^N_{\frac{N}{N-1}}<\infty$ and $\forall f\in L^2(\mathbb{R})$, $\|f\|_2^{2N}\le\|\psi^{-2}\|^N_{\frac{N}{N-1}}\|f\psi\|^{2N}_{2N}$. Via the Burkholder inequality, using $\psi^{2N}(x+X)\le C\,(1+|X(u)|^N)\psi^{2N}(x)$ and working as we did for A_3, we get

$$\begin{aligned}
\mathcal{A}_3 &\le CE\left\|\int_0^t ds\,\psi^2\,[(\rho^{\gamma}_{t+\delta-s}-\rho^{\gamma}_{t-s})\varphi_{X(s)}]^2\right\|_N^N \qquad (131)\\
&\le C\,t^{N-1}\left(1+E\sup_{u\in[0,t]}|X(u)|^N\right)\int_0^t ds\int_{\mathbb{R}} dx\,\psi^{2N}(x)\left|(\rho^{\gamma}_{s+\delta}-\rho^{\gamma}_s)\varphi\right|^{2N}(x)\\
&\le C\,t^{N-1}\left(1+E\sup_{u\in[0,t]}|X(u)|^N\right)\int_0^t ds\int_{\mathbb{R}} dx\left|\psi(x)\int_s^{s+\delta}d\tau\rho'^{\gamma}_{\tau}\,\varphi\right|^{2N}\\
&\le C\,t^{N-1}\left(1+E\sup_{u\in[0,t]}|X(u)|^N\right)\\
&\quad\times\int_0^t ds\int_{\mathbb{R}} dx\left|\psi(x)\int_s^{s+\delta}d\tau\,\frac{d}{d\tau}\int_0^{\tau}du\frac{\rho_u^{\gamma}\varphi''}{(\tau-u)^{1-2\gamma}}\right|^{2N}\\
&= C\,t^{N-1}\left(1+E\sup_{u\in[0,t]}|X(u)|^N\right)\\
&\quad\times\int_0^t ds\int_{\mathbb{R}} dx\left|\psi(x)\left[\int_0^{s+\delta}\frac{du\,\rho_u^{\gamma}\varphi''}{(s+\delta-u)^{1-2\gamma}}-\int_0^s\frac{du\,\rho_u^{\gamma}\varphi''}{(s-u)^{1-2\gamma}}\right]\right|^{2N}\\
&\le C\,t^{N-1}\left(1+E\sup_{u\in[0,t]}|X(u)|^N\right)
\end{aligned}$$

$$\times \left\{ \int_0^t ds \int_{\mathbb{R}} dx \left| \psi(x) \int_0^s du \rho_u^\gamma \varphi'' \left(\frac{1}{(s+\delta-u)^{1-2\gamma}} - \frac{1}{(s-u)^{1-2\gamma}} \right) \right|^{2N} + \int_0^t ds \int_{\mathbb{R}} dx \left| \psi(x) \int_s^{s+\delta} du \frac{\rho_u^\gamma \varphi''}{(s+\delta-u)^{1-2\gamma}} \right|^{2N} \right\}$$

$$\leq C\, t^{N-1} \left(1 + E \sup_{u\in[0,t]} |X(u)|^N \right) [\mathcal{A}_{3a} + \mathcal{A}_{3b}], \tag{132}$$

where

$$\begin{aligned}
\mathcal{A}_{3a} = & \int_0^t ds \int_{\mathbb{R}} dx \left| \int_0^s du \left(\frac{1}{(s+\delta-u)^{1-2\gamma}} - \frac{1}{(s-u)^{1-2\gamma}} \right) \rho_u^\gamma \varphi'' \psi \right|^{2N} \\
& + \int_0^t ds \int_{\mathbb{R}} dx \left| \int_0^s du \left(\frac{1}{(s+\delta-u)^{1-2\gamma}} - \frac{1}{(s-u)^{1-2\gamma}} \right) \right. \\
& \left. \times \int_{\mathbb{R}} dy \rho_u^\gamma (x-y) \varphi''(y) \sqrt{|x-y|} \right|^{2N} \\
\mathcal{A}_{3b} = & \int_0^t ds \int_{\mathbb{R}} dx \left| \int_s^{s+\delta} du \frac{1}{(s+\delta-u)^{1-2\gamma}} \rho_u^\gamma \varphi'' \psi \right|^{2N} \\
& + \int_0^t ds \int_{\mathbb{R}} dx \left| \int_s^{s+\delta} du \frac{1}{(s+\delta-u)^{1-2\gamma}} \int_{\mathbb{R}} dy \rho_u^\gamma (x-y) \varphi''(y) \sqrt{|x-y|} \right|^{2N}.
\end{aligned}$$

We claim that

$$\begin{aligned}
& \int_{\mathbb{R}} dx \left| \max_{0\leq u\leq s+\delta} (\rho_u^\gamma \varphi'' \psi)(x) \right|^{2N} < \infty, \\
& \int_{\mathbb{R}} dx \left| \max_{0\leq u\leq s+\delta} (\rho_u^\gamma (\cdot) \sqrt{\cdot} * \varphi'')(x) \right|^{2N} < \infty.
\end{aligned}$$

Indeed, $\rho_u^\gamma \varphi'' \psi$ is continuous in u, so the maximum in $(132)_1$ is attained at, say, $\tilde{u}$ and $\|\rho_{\tilde{u}}^\gamma \varphi'' \psi\|_{2N}^{2N} \leq C$. The maximum in $(132)_2$ is reached at the second extremum $(s+\delta)$; in fact

$$\int_{\mathbb{R}} dx \left(\int_{\mathbb{R}} dy \rho_u^\gamma (x-y) \sqrt{|x-y|}\, \varphi''(y) \right)^{2N} \leq C \int_{\mathbb{R}} dx \left(\int_{\mathbb{R}} dz \rho_1^\gamma (z) |z|^N u^\gamma \right)^{2N}.$$

Therefore,

$$\mathcal{A}_{3a} \leq C \int_0^t ds \left| \int_0^s du \left(\frac{1}{(s+\delta-u)^{1-2\gamma}} - \frac{1}{(s-u)^{1-2\gamma}} \right) \right|^{2N} = C(t) \delta^{(1-2\gamma)2N},$$

with $C(t)$ bounded on compacts and

$$\mathcal{A}_{3b} \leq C \int_0^t ds \left| \int_s^{s+\delta} du \frac{1}{(s+\delta-u)^{1-2\gamma}} \right|^{2N} = C t \delta^{4N\gamma}.$$

In order to apply the Kolmogorov criterion we need $4N\gamma > 1$ and $(1-2\gamma)2N > 1$. For $\mathcal{A}_1$ and $\mathcal{A}_4$,

$$\mathcal{A}_1 \leq C \left\| \int_0^t ds \psi(x)(\rho^\gamma_{t+\delta-s} - \rho^\gamma_{t-s})\varphi_{X(s)} \right\|_{2N}^{2N}$$
$$\leq C t^{2N-1} E \int_{\mathbb{R}} dx \int_0^t ds |\psi(x)(\rho^\gamma_{t+\delta-s} - \rho^\gamma_{t-s})\varphi_{X(s)}|^{2N},$$

which is exactly (131).

$$\mathcal{A}_4 \leq C \left(1 + E \sup_{u\in[0,t]} |X(u)|^{2N} \right) \delta^{2N-1} \int_0^\delta \|\psi\rho^\gamma_s \varphi\|^{2N}_{2N};$$

with analogous calculations, the integrand on the right hand side is bounded, and hence

$$\mathcal{A}_4 \leq C \left(1 + E \sup_{u\in[0,t]} |X(u)|^{2N} \right) \delta^{2N}.$$

To conclude, requiring

$$\begin{cases} N \geq \dfrac{1}{2(1-2\gamma)} & \text{if } \gamma \geq \dfrac{1}{4} \\ N \geq \dfrac{1}{4\gamma} & \text{if } \gamma \leq \dfrac{1}{4}, \end{cases}$$

continuity follows. □

Appendix B. Motivation

In the introduction we have briefly discussed the choice of the operators of fractional differentiation and of the fractional Laplacian. In this appendix, we want to show how the operators D_t^γ and I_t^γ naturally arise in the context of anomalous diffusion and explain in some more detail the link with CTRWs.

We want to determine an operator A s.t.

$$\begin{cases} \partial_t \rho_t^\gamma(x) = A\, \rho_t^\gamma(x) \\ \rho_t^\gamma(0) = \delta_0, \end{cases}$$

with $\rho^\gamma(t,x)$ enjoying the following three properties:

$$\int_{\mathbb{R}} dx \rho_t^\gamma(x) = 1, \qquad \int_{\mathbb{R}} dx \rho_t^\gamma(x)\, x = 0 \quad \text{and} \quad \int_{\mathbb{R}} dx \rho_t^\gamma(x)\, x^2 \sim t^{2\gamma} \tag{133}$$

(notice that for $\gamma = \frac{1}{2}$ we recover the diffusion equation with $A = \Delta$). We recall that $\hat{f}$, $f^\#$ and $\tilde{f}$ denote the Fourier, the Laplace and the Fourier–Laplace transforms of the function f, respectively.

By (133), the following must hold:

$$\hat{\rho}_t^\gamma(k) = 1 - \frac{1}{2} c t^{2\gamma} k^2 + o(k^2) \quad \text{and}$$
$$\tilde{\rho}^\gamma(\mu,k) = \frac{1}{\mu} - \frac{ck^2}{2\mu^{2\gamma+1}} \Gamma(2\gamma+1) = \frac{1}{\mu}(1 - c_1 \mu^{-2\gamma} k^2),$$

where $c_1 = \frac{1}{2}c\Gamma(2\gamma+1)$. In definitions (7) and (8) the constant c_1 should appear; we just set it equal to 1 both for simplicity and because we are not interested, in this context, in estimating the "anomalous diffusion" constant.

We can assume that the expression for $\tilde{\rho}^\gamma(\mu,k)$ is valid in the regime $\mu^{-2\gamma}k^2 \ll 1$. Actually, condition $(133)_3$ is meant for an infinitely wide system and for long times. In other words, if Λ is the region where the particle moves, we claim that

$$\lim_{t\to\infty}\lim_{\Lambda\to\mathbb{R}} \frac{\int_\Lambda dx \rho_t^\gamma(x)\, x^2}{t^{2\gamma}} = \text{const.}$$

This means that we are interested in the case $k \ll \mu$. Of course one can in principle find an infinite number of functions s.t. $\tilde{\rho}^\gamma(\mu,k) = \frac{1}{\mu}(1-c_1\epsilon)$ for $\epsilon = \mu^{-2\gamma}k^2$. One possible choice is

$$\tilde{\rho}^\gamma(\mu,k) = \frac{1}{\mu(1+c_1\epsilon)} = \mu^{\gamma-1}\frac{\mu^\gamma}{\mu^{2\gamma}+(c_1k)^2} = \frac{1}{\mu+c_1k^2\mu^{1-2\gamma}}, \tag{134}$$

which leads to an integro-differential equation and, when $\gamma = \frac{1}{2}$, it coincides with the Fourier–Laplace transform of a Gaussian density.

We now find the operator whose fundamental solution is $\tilde{\rho}^\gamma(\mu,k)$. We have

$$\mathcal{L}(\partial_t\hat{\rho}^\gamma(\cdot,k))(\mu) = -1+\mu\tilde{\rho}^\gamma(\mu,k) = -c_1k^2\mu^{1-2\gamma}\tilde{\rho}^\gamma(\mu,k).$$

Let $p = 2\gamma-1$ and $\phi_p(t) = \frac{t^{p-1}}{\Gamma(p)}$; then we need to distinguish two cases in order to study the right hand side of the above equation:

when $0<\gamma<\frac{1}{2}$ one can easily check that

$$\mathcal{L}(\phi_p * \hat{\rho}^\gamma(k,\cdot)) = \tilde{\rho}^\gamma(\mu,k)\mu^{-p}$$

which implies that

$$\tilde{\rho}^\gamma(\mu,k)\mu^{1-2\gamma} \quad \text{is the Laplace transform of } \frac{1}{\Gamma(2\gamma-1)}\int_0^t ds\frac{\hat{\rho}^\gamma(s,k)}{(t-s)^{2-2\gamma}};$$

when $\frac{1}{2}<\gamma<1$, instead, a straightforward calculation shows that

$$\mathcal{L}[\partial_t(\phi_{p+1} * \hat{\rho}^\gamma(k,\cdot))] = \tilde{\rho}^\gamma(\mu,k)\mu^{-p}$$

and so

$$\tilde{\rho}^\gamma(\mu,k)\mu^{1-2\gamma} \quad \text{is the Laplace transform of } \frac{1}{\Gamma(2\gamma)}\frac{d}{dt}\int_0^t ds\frac{\hat{\rho}^\gamma(s,k)}{(t-s)^{1-2\gamma}}.$$

Finally, taking the inverse Fourier transform, we get that $\rho^\gamma(t,x)$ satisfies (7) when $0<\gamma<\frac{1}{2}$ and (8) when $\frac{1}{2}<\gamma<1$. Moreover, the explicit expression for $\rho_t^\gamma(x)$ holds true: by (134) we get that

$$\tilde{\rho}^\gamma(\mu,k) = \int_R dx\, e^{ikx}\frac{\mu^{\gamma-1}}{2\sqrt{c_1}}e^{-\frac{\mu^\gamma}{\sqrt{c_1}}|x|}$$

and hence

$$\rho^\#(x,\mu) = \frac{\mu^{\gamma-1}}{2\sqrt{c_1}}e^{-\frac{\mu^\gamma}{\sqrt{c_1}}|x|}$$

and now, by the inverse Laplace formula, we obtain (16). Obviously, the expression (16) has been deduced after having chosen (134) among all possible candidates for $\tilde{\rho}^{\gamma}$ and this choice can now be justified in view of the link with CTRWs.

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