Civil Engineering 2 Mathematics Autumn 2011

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Systems of ODEs

Notation: $t \in \mathbb{R}$ is the independent variable and $y(t)$ is a vector valued function, i.e. $y(t) \in \mathbb{R}^n$. $\frac{dy}{dt}, \dot{y}, y'$ are equivalent notatations and denote the derivative of y w.r.t. t. A is a $n \times n$ matrix with constant coefficients.

• Power of a matrix A

A **diagonal.** You can easily check that if $A = \mathrm{diag}\{\lambda_1, \dots, \lambda_n\}$, i.e.

$$A = \begin{vmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \dots & 0 & \lambda_n \end{vmatrix} \Rightarrow A^k = \begin{vmatrix} \lambda_1^k & 0 & 0 & \dots & 0 \\ 0 & \lambda_2^k & & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \dots & 0 & \lambda_n^k \end{vmatrix}, \quad \forall k \geq 0.$$

A **diagonalizable.** In this case there exists a matrix C s.t. $C^{-1}AC = \Delta$ with Δ a diagonal matrix having the eigenvalues of A on the main diagonal. Hence

$$A = C\Delta C^{-1}.$$

This expression for A is particularly useful to our purposes, indeed

$$A^2 = C\Delta C^{-1}\, C\Delta C^{-1} = C\Delta^2 C^{-1},$$

and we know how to calculate Δ^2 because Δ is diagonal. Can you guess what happens for higher powers?

$$A^k = C\Delta^k C^{-1}, \quad \forall k \geq 0.$$

Some special matrices, J**,** N_2 **and** N_3**.**

$$J = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \quad \text{check that} \quad J^{2k} = (-1)^k I, \quad J^{2k+1} = (-1)^k J. \qquad (1)$$

$$N_2 = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \quad \text{check that} \quad N_2^2 = 0. \qquad (2)$$

$$N_3 = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix} \quad \text{check that} \quad N_3^2 = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \quad \text{and} \quad N_3^3 = 0.$$

• Matrix exponential and Systems of ODEs

Theorem 0.1. *Let A be an $n \times n$ matrix with constant coefficients. Then*

$$\begin{cases} y' = Ay \\ y(t_0) = y_0, \quad y_0 \in \mathbb{R}^n \end{cases} \quad \Rightarrow \quad y(t) = e^{A(t-t_0)} y_0, \ \ \forall t \in \mathbb{R}.$$

We are very familiar with the exponential of a real number but...how to calculate the exponential of a matrix? You will remember that, if x is a real number (a scalar) then

$$e^x = \sum_{k=0}^{+\infty} \frac{x^k}{k!}.$$

Does this suggest anything?

Definition 0.1. *If A is a $n \times n$ matrix with constant coefficients then*

$$e^A = \sum_{k=0}^{+\infty} \frac{A^k}{k!}.$$

Properties of e^A.

i) $e^0 = I$

ii) If A and B commute then $\quad e^{A+B} = e^A e^B$

iii) $\left(e^A\right)^{-1} = e^{-A}$

iv) $e^{CBC^{-1}} = Ce^B C^{-1}$

Explicit calculation of e^A.

A **diagonal.**

$$A = \left|\begin{array}{ccccc} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ & & & & \\ 0 & & \dots & 0 & \lambda_n \end{array}\right| \quad \Rightarrow \quad e^A = \left|\begin{array}{ccccc} e^{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & e^{\lambda_2} & & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ & & & & \\ 0 & & \dots & 0 & e^{\lambda_n} \end{array}\right|.$$

A **diagonalizable.** As before $A = C\Delta C^{-1}$. Hence

$$e^A = Ce^\Delta C^{-1}$$

and we know how to calculate e^Δ because Δ is diagonal.

Example of lucky case 1

$$A = \begin{vmatrix} a & b \\ -b & a \end{vmatrix}, \qquad a, b \in \mathbb{R}.$$

A can be written as $A = aI + bJ$ and I and J commute, so $e^A = e^{aI} e^{bJ}$. e^{aI} is easy, so let's look at e^{bJ}. Using (1):

$$\begin{aligned} e^{bJ} &= \sum_{k=0}^{+\infty} \frac{1}{(2k)!} J^{2k} b^{2k} + \sum_{k=0}^{+\infty} \frac{1}{(2k+1)!} J^{2k+1} b^{2k+1} \\ &= I \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k)!} b^{2k} + J \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!} b^{2k+1} \\ &= \cos bI + \sin bJ. \end{aligned}$$

Putting everything together

$$e^A = e^a \begin{vmatrix} \cos b & \sin b \\ -\sin b & \cos b \end{vmatrix}.$$

Example of lucky case 2 Let N be the matrix defined in (2). Then

$$e^N = I + N.$$