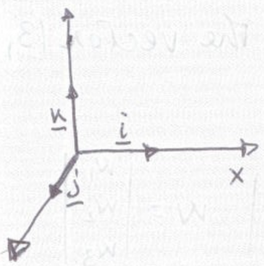
VECTOR CALCULUS

- Let us begin from the beginning:
- The function $f(x) = x^2$ is a scalar function; it is called this way because to every value of the scalar variable x it associates the value x^2, which is still a scalar. In symbols:

$$f: \text{scalar} \longrightarrow \text{scalar} \qquad \underline{\text{scalar function}}$$

(this notation means: f takes a scalar into a scalar or, equivalently, to every scalar it associates another scalar)

- If $\underline{i}, \underline{j}, \underline{k}$ are the unit vectors
$\underline{i} = (1,0,0)$ $\quad \underline{j} = (0,1,0)$ $\quad \underline{k} = (0,0,1)$, for every



vector $v = (v_1, v_2, v_3) = v_1 \underline{i} + v_2 \underline{j} + v_3 \underline{k}$

for example $v = (1, \pi, -3) = \underline{i} + \pi \underline{j} - 3\underline{k}$

- When talking about PDEs we have met scalar functions of two variables, e.g. $u(x,y) = x+y$.

One can also have scalar functions of 3 variables

$$h\underbrace{(x,y,z)}_{} = x^2 + y - z$$

If we think of (x,y,z) as a vector, then the function h associates to every vector (x,y,z) a scalar number x^2+y-z

$$h: \text{vector} \longrightarrow \text{scalar} \qquad \underline{\text{scalar field}}$$

h is also said to be a scalar field. So for example the solution of a PDE is a scalar field.

But one can also think of more exotic objects. Consider for example the function

$$\underline{A}(x,y,z) = (x+y, z, -3x) = (x+y)\underline{i} + z\underline{j} - 3x\underline{k}$$

To every vector (x,y,z) it associates another vector (Think for example of the velocity of a 3D fluid)

$$\underline{A} : \text{vector} \longrightarrow \text{vector} \qquad \underline{\text{vector field}}$$

$\underline{A}$ is called a vector field. In the example above

$$\underline{A}(1,2,0) = (1+2)\underline{i} + 0\underline{j} - 3\underline{k} = 3\underline{i} - 3\underline{k} = \begin{vmatrix} 3 \\ 0 \\ 3 \end{vmatrix}$$

So to the vector $(1,2,0)$, $\underline{A}$ associates the vector $(3,0,3)$.

<u>Operations between vectors</u>

if v and w are two vectors, say $v = \begin{vmatrix} v_1 \\ v_2 \\ v_3 \end{vmatrix}$ $w = \begin{vmatrix} w_1 \\ w_2 \\ w_3 \end{vmatrix}$

then the <u>scalar product</u> between v and w is denoted by $v \cdot w$ and it is given by

$$v \cdot w = \begin{vmatrix} v_1 \\ v_2 \\ v_3 \end{vmatrix} \cdot \begin{vmatrix} w_1 \\ w_2 \\ w_3 \end{vmatrix} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

The <u>vector product</u> $v \times w$ is given by

$$v \times w = \det \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \underline{i}(v_2 w_3 - v_3 w_2) - \underline{j}(v_1 w_3 - v_3 w_1) + \underline{k}(v_1 w_2 - v_2 w_1)$$

REM The result of the scalar product is a scalar,
The result of the vector product is a vector.

Also $v \cdot w = w \cdot v$

but $v \times w \neq w \times v$, in particular

$v \times w = - w \times v$. (check it)

EX $(3x \underline{i} - y \underline{j} + \underline{k}) \times (z \underline{i} - \underline{j} + y \underline{k}) =$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3x & -y & 1 \\ z & -1 & y \end{vmatrix} = \underline{i}(-y^2+1) - \underline{j}(3xy - z) + \underline{k}(-3x + yz)$$

Reminder: the length of a vector $v = (x, y, z)$ is

$$|v| = \sqrt{x^2 + y^2 + z^2} = (v \cdot v)^{1/2}.$$

- If we have a function of one variable, $f(x)$, we can consider $\frac{df}{dx}(x)$, its derivative.

When we have a scalar function of many variables, that is, a scalar field $\phi(x, y, z)$, we define

$$\text{grad}\, \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right).$$

In other words grad ϕ, the gradient of ϕ, is the vector containing all the partial derivatives of ϕ.

~~grad ϕ is denoted by ∇ nabla~~

~~∇ : scalar field $\to$ vector field~~

<u>EX</u>: $\phi(x,y,z) = 3x + \frac{y^2}{2} - zx$

$\text{grad}\,\phi = (3, 1, -x)$

- Denote $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. Then grad ϕ can also be denoted $\text{grad}\,\phi = \nabla\phi$. In conclusion

∇ : scalar field $\longrightarrow$ vector field

- Some more definitions:
- If ϕ is a scalar field then

$$\nabla \cdot (\nabla\phi) = \begin{vmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial\phi}{\partial x} \\ \frac{\partial\phi}{\partial y} \\ \frac{\partial\phi}{\partial z} \end{vmatrix} = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

Usually we denote $\nabla^2\phi = \nabla \cdot (\nabla\phi)$
or, briefly, $\nabla^2 = \nabla \cdot \nabla$

∇^2 is called <u>Laplacian</u>. So the Laplace equation

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ can be rewritten as $\nabla^2 u = 0$.

- If $\underline{A}(x,y,z)$ is a vector field then the <u>divergence of $\underline{A}$</u> is given by

$$\text{div}\,\underline{A} = \nabla \cdot \underline{A} = \begin{vmatrix} \partial_x \\ \partial_y \\ \partial_z \end{vmatrix} \cdot \begin{vmatrix} A_1 \\ A_2 \\ A_3 \end{vmatrix} = \partial_x A_1 + \partial_y A_2 + \partial_z A_3$$

where A_1, A_2, A_3 are the compoments of $\underline{A}$.

• If $\underline{A}$ is a vector field then

$$\text{curl } \underline{A} = \nabla \times \underline{A} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ A_1 & A_2 & A_3 \end{vmatrix} =$$

$$= \underline{i}\left(\partial_y A_3 - \partial_z A_2\right) - \underline{j}\left(\partial_x A_3 - \partial_z A_1\right) + \underline{k}\left(\partial_x A_2 - \partial_y A_1\right)$$

<u>REMARK</u> ; $\nabla = \begin{vmatrix} \partial x \\ \partial y \\ \partial z \end{vmatrix}$, ϕ scalar field, $\underline{A}$ vect. field

∇ : scalar field $\longrightarrow$ vector field

$\nabla \cdot \underline{A} = \text{div } \underline{A}$: vector field $\longrightarrow$ scalar field

$\nabla^2 \phi = \nabla \cdot (\nabla \phi) = \text{div}(\text{grad } \phi)$: scalar field $\longrightarrow$ scalar field

$\nabla \times \underline{A} = \text{curl } \underline{A}$: vector field $\longrightarrow$ vector field

<u>EX 1</u> : given the scalar field $\phi(x,y,z) = z^3 x + e^y$

$$\nabla \phi = (z^3, e^y, 3z^2 x) = z^3 \underline{i} + e^y \underline{j} + 3z^2 x \underline{k}$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 + e^y + 6zx$$

Notice that $\nabla \phi$ is a vector field, so we can calculate

$(\nabla \phi) \times \underline{A}$ where $\underline{A} = 3\underline{i} - 2x\underline{j} + \underline{k}$

$$(\nabla \phi) \times \underline{A} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ z^3 & e^y & 3z^2 x \\ 3 & -2x & 1 \end{vmatrix} = \underline{i}\left(e^y + 6x^2 z^2\right) - \underline{j}\left(z^3 - 9z^2 x\right) + \underline{k}\left(-2x z^3 - 3e^y\right)$$

<u>REM</u> : Roughly speaking: $v = (1,2,9)$ has constant components and it's called vector. $\underline{B} = (3x, 2y, x+y-z)$ has variable components and it's called vector field.

EX: prove that for any vector field $\underline{B} = \begin{vmatrix} B_1 \\ B_2 \\ B_3 \end{vmatrix}$

one has $\nabla \cdot (\nabla \times \underline{B}) = 0$.

$$\nabla \times \underline{B} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ B_1 & B_2 & B_3 \end{vmatrix} = \underline{i}\left(\partial_y B_3 - \partial_z B_2\right) - \underline{j}\left(\partial_x B_3 - \partial_z B_1\right) + \underline{k}\left(\partial_x B_2 - \partial_y B_1\right)$$

$$\nabla \cdot (\nabla \times \underline{B}) = \begin{vmatrix} \partial_x \\ \partial_y \\ \partial_z \end{vmatrix} \cdot \begin{vmatrix} \partial_y B_3 - \partial_z B_2 \\ \partial_z B_1 - \partial_x B_3 \\ \partial_x B_2 - \partial_y B_1 \end{vmatrix} =$$

$$= \partial_{xy} B_3 - \partial_{zx} B_2 + \partial_{zy} B_1 - \partial_{xy} B_3 + \partial_{zx} B_2 - \partial_{zy} B_1 = 0$$

• Def: a function (or better, a scalar field) such that $\nabla^2 u = 0$ is said to be HARMONIC.