HERIOT WATT UNIVERSITY

Signal Processing with Side Information

A Geometric Approach via Sparsity

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Side Information

Signal processing tasks

Denoising

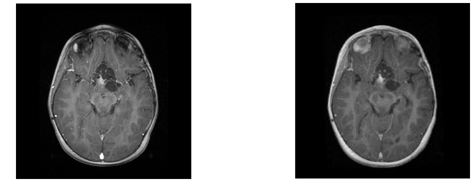
Reconstruction

Demixing (source separation)

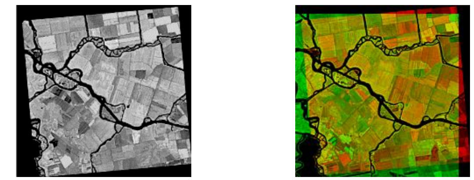
Compression

Inpainting, super-resolution, …

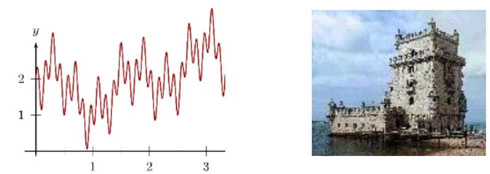
prior information



multi-modal

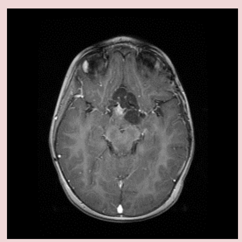


heterogeneous

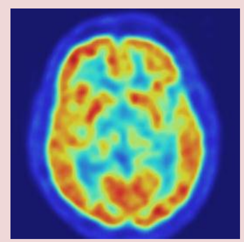


Recommender systems

Medical imaging



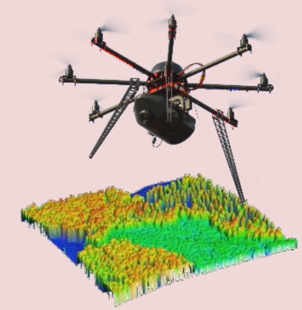
MRI



PET

Consumer electronics

Robotics

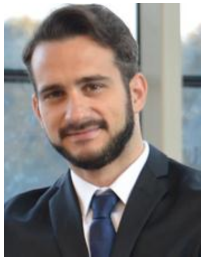


How to represent multi-modal or heterogeneous data ?

How to process it ?

Outline

- Compressed Sensing with Prior Information
- Application: Video Background Subtraction
- X-ray Image Separation
- Conclusions



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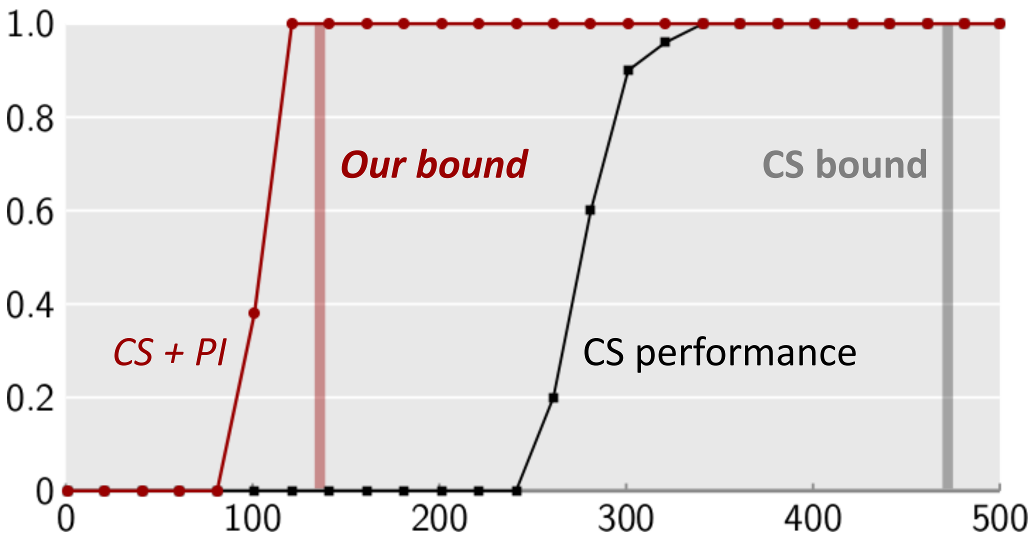
Compressed Sensing

$\|\overline{x} - x^\star\|_2 / \|x^\star\|_2 \simeq 0.45$

Sucess rate (50 trials)



number of measurements m

Compressed Sensing (CS)

70-sparse

$$b = A : m \times n \quad x^\star$$

iid Gaussian

1000

Basis pursuit

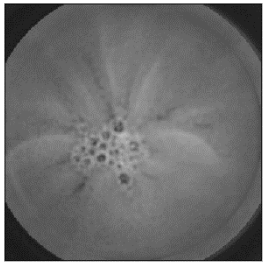
$$\widehat{x} = \underset{x}{\arg\min} \; \|x\|_1 \quad \text{s.t.} \quad b = Ax$$

What if we know $\overline{x} \sim x^\star$? ***prior information***

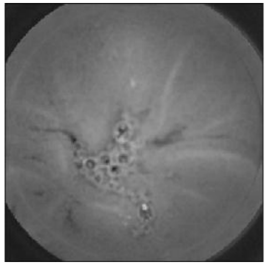
How do we integrate $\overline{x}$ in the problem?

Reconstruction guarantees?

$\overline{z} = \Psi \overline{x}$



$z^\star = \Psi x^\star$



medical images, video, …

Intuition

$Ax^\star = b$ measurements

random orientation

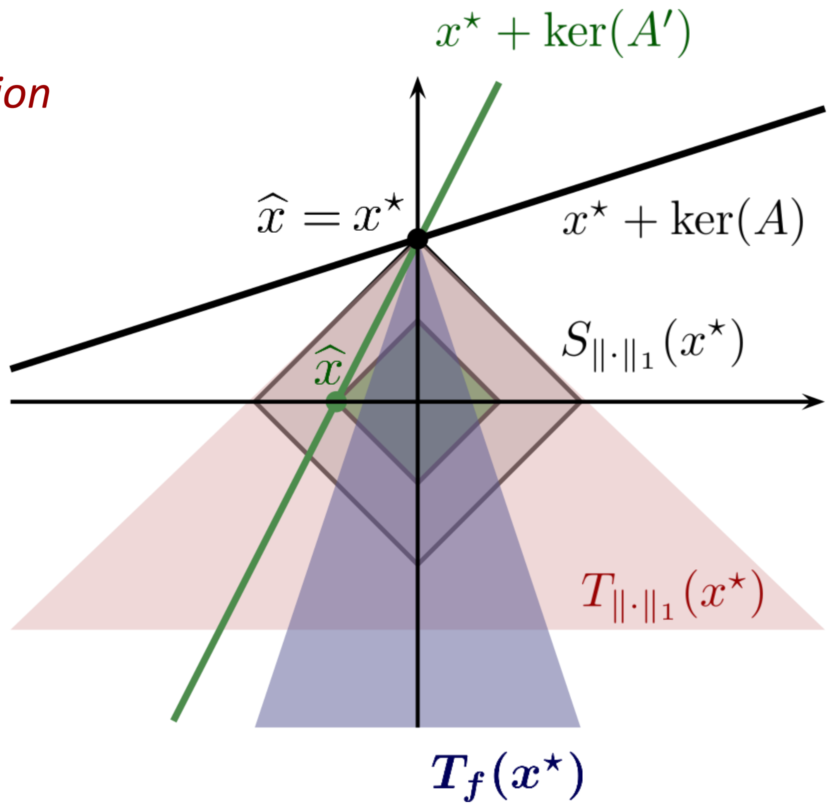
solutions of $Ax = b$: $x^\star + \ker(A)$

$$\widehat{x} = \underset{x}{\operatorname{argmin}} \ \|x\|_1 \quad \text{s.t.} \quad Ax = b$$

Tangent cone of f at $x^\star$

$$T_f(x^\star) = \operatorname{cone}\big(S_f(x^\star) - x^\star\big)$$

$\{x : f(x) \le f(x^\star)\}$



Our approach

$\overline{x}$: prior information (PI)

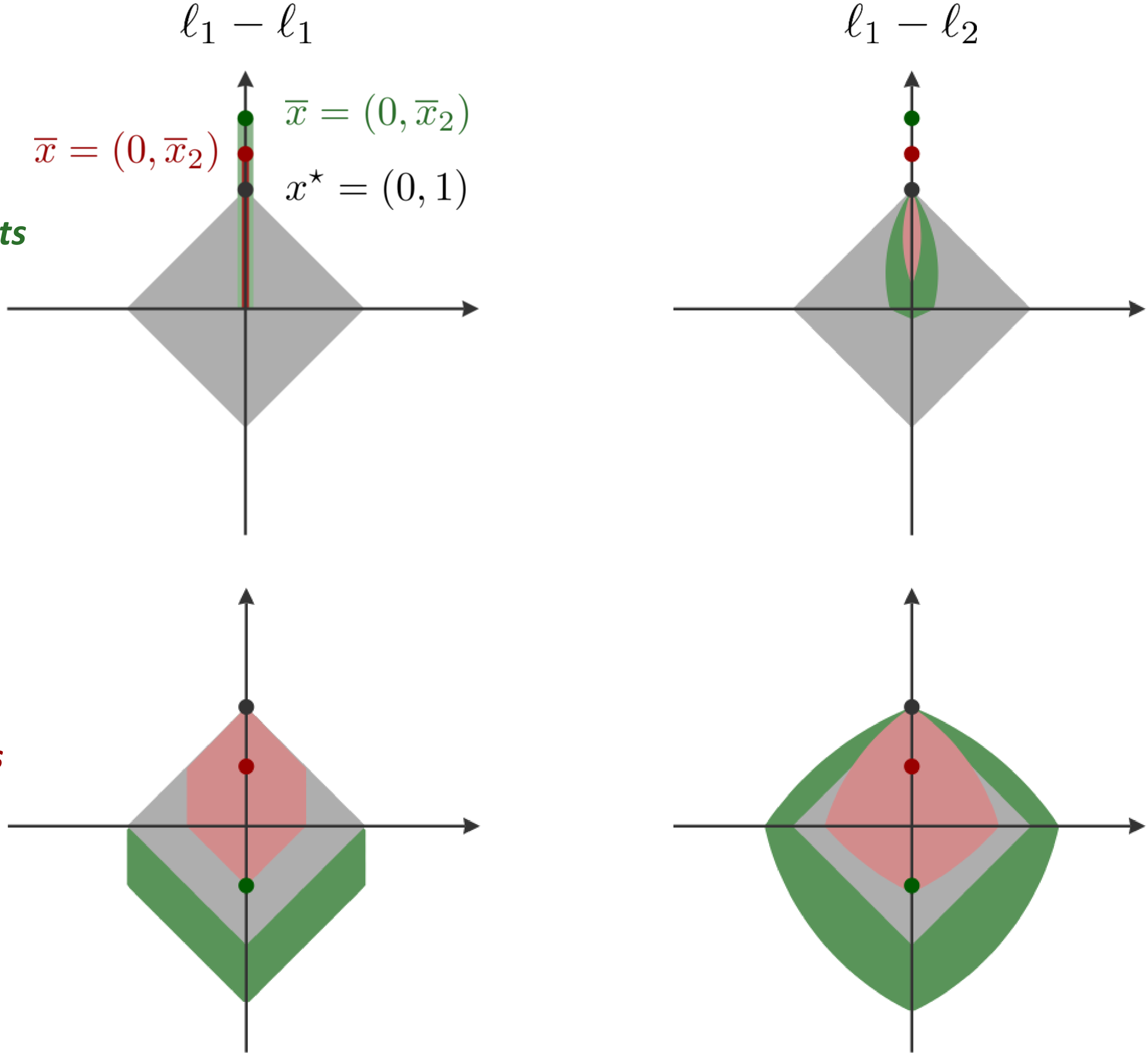
g : model for PI $\quad g(x^\star - \overline{x}) \simeq$ small

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) = \|x\|_1 + \beta\, g(x - \overline{x}) \\ \text{subject to} & Ax = b \end{array} \Rightarrow T_f(x^\star)$$

$$g_1(x - \overline{x}) = \|x - \overline{x}\|_1$$

$$g_2(x - \overline{x}) = \tfrac{1}{2}\|x - \overline{x}\|_2^2$$

$\beta = 1$



Good components

$\overline{x}_2 > x_2^\star$

Bad components

$\overline{x}_2 < x_2^\star$

L1-L1 minimization

$b = A : m \times n$ *i.i.d.* $\mathcal{N}(0, 1/m)$ $x^\star$ (s-sparse)

Theorem (BP) **[Chandrasekaran, Recht, Parrilo, Willsky, 2012]**

$$m \geq 2s \log\left(\frac{n}{s}\right) + \frac{7}{5}s + 1 \quad \Longrightarrow \quad x^\star = \underset{x}{\operatorname{argmin}} \ \|x\|_1 \ \text{ s.t. } \ b = Ax \qquad \text{w.h.p.}$$

Theorem (L1-L1 minimization) **[M, Deligiannis, Rodrigues, 2017]**

parameter-free

$\beta = 1$

$$\overline{h} := \Big|\{i \,:\, x_i^\star > 0,\, \overline{x}_i < x_i^\star\} \cup \{i \,:\, x_i^\star < 0,\, \overline{x}_i > x_i^\star\}\Big| > 0$$

$$m \geq 2\overline{h} \log\left(\frac{n}{s + \xi/2}\right) + \frac{7}{5}\left(s + \frac{\xi}{2}\right) + 1 \quad \Longrightarrow \quad x^\star = \underset{x}{\operatorname{argmin}} \ \|x\|_1 + \|x - \overline{x}\|_1 \ \text{ s.t. } \ b = Ax \qquad \text{w.h.p.}$$

$$0 \leq \overline{h} \leq s$$

$$\xi = \underbrace{\Big|\{i \,:\, \overline{x}_i \neq x_i^\star = 0\}\Big|}_{\text{support overestimation}} - \Big|\{i \,:\, \overline{x}_i = x_i^\star \neq 0\}\Big|$$

Experimental Results

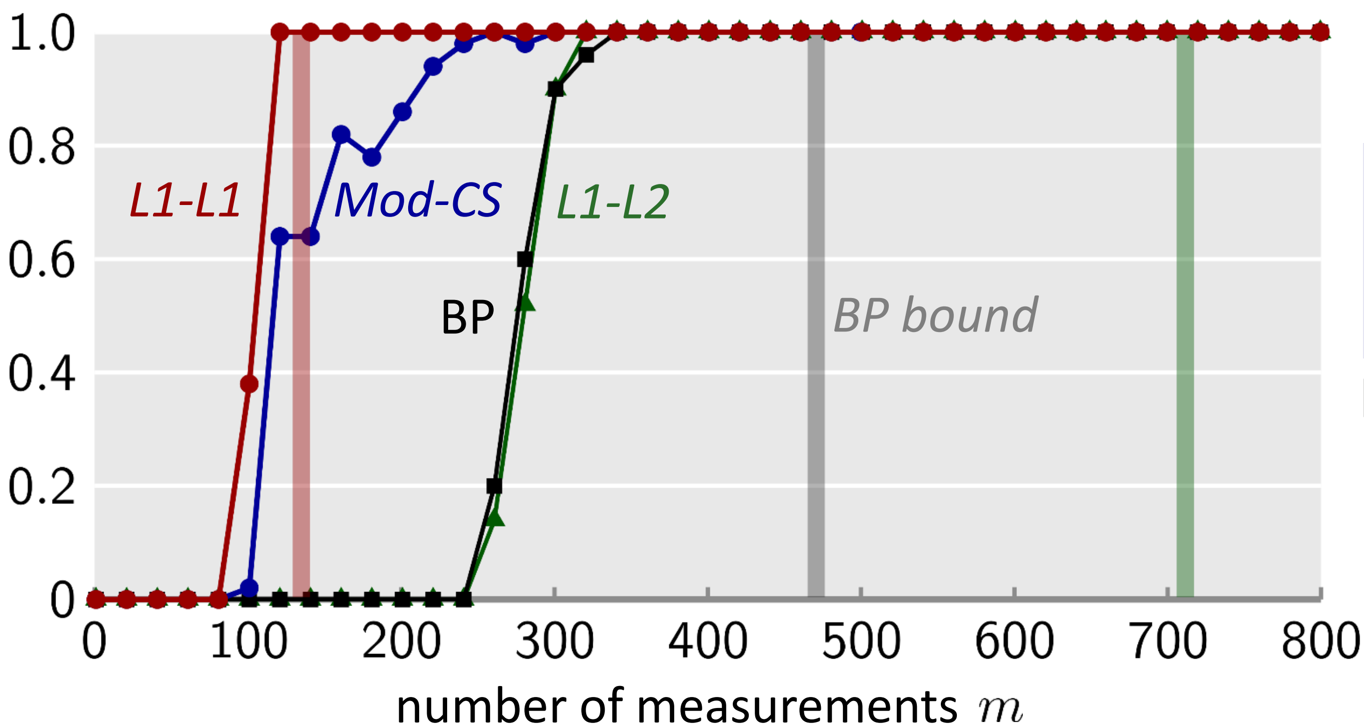
A_{ij} : Gaussian

$x^\star : \; n = 1000 \qquad s = 70 \qquad x_i^\star \sim \mathcal{N}(0,1) \qquad \|\overline{x} - x^\star\|_2 / \|x^\star\|_2 \simeq 0.45$

$\overline{x} : \; \overline{x} = x^\star + z \qquad \mathrm{card}(z) = 28 \qquad z_i \sim \mathcal{N}(0, 0.8) \qquad \overline{h} = 11 \quad \xi = -42$

$\|\hat{x} - x^\star\|_\infty / \|x^\star\|_\infty \leq 10^{-2}$

Sucess rate (50 trials)

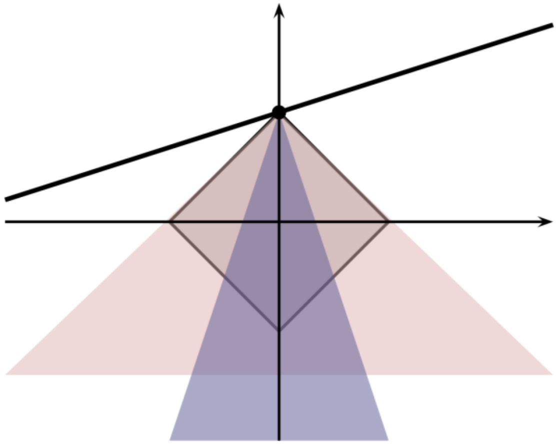


Mod-CS

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \|x_{T^c}\|_1 \\ \text{subject to} & Ax = b \end{array}$$

[Vaswani and Lu, 2010]

Summarizing



- Prior Information *can help*, but *can also hinder*
- L1-L1 works better than L1-L2 (theory and practice)
- (Computable) bounds are tight for L1-L1, but not for L1-L2
- Theory predicts optimal β ; indicates how to improve $\overline{x}$
- Limitations: Gaussian matrices; bounds depend on unknown parameters

$$m \geq 2\overline{h} \log\left(\frac{n}{s+\xi/2}\right) + \frac{7}{5}\left(s + \frac{\xi}{2}\right) + 1$$

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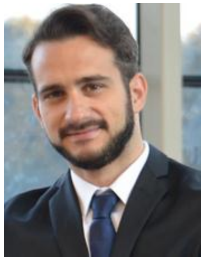

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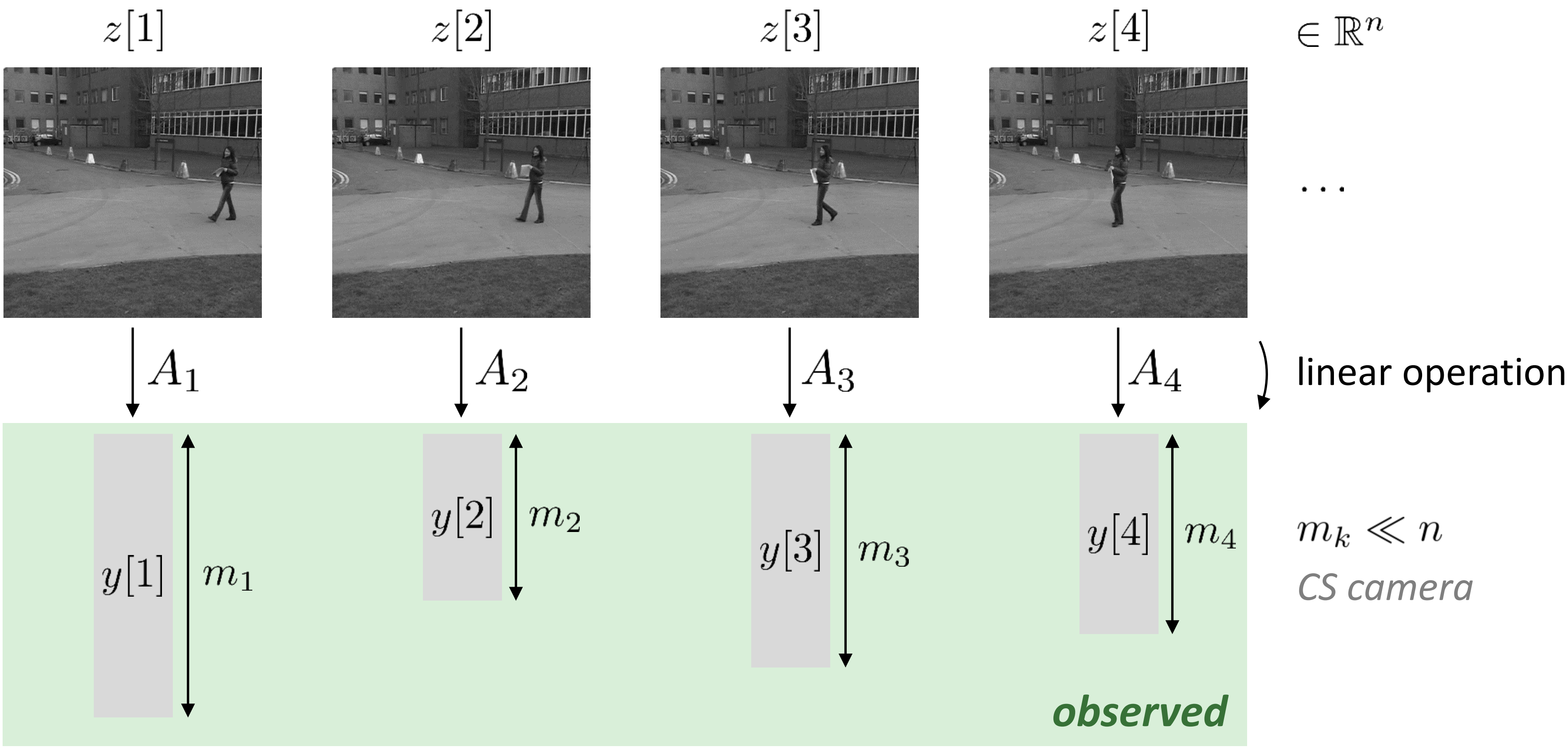
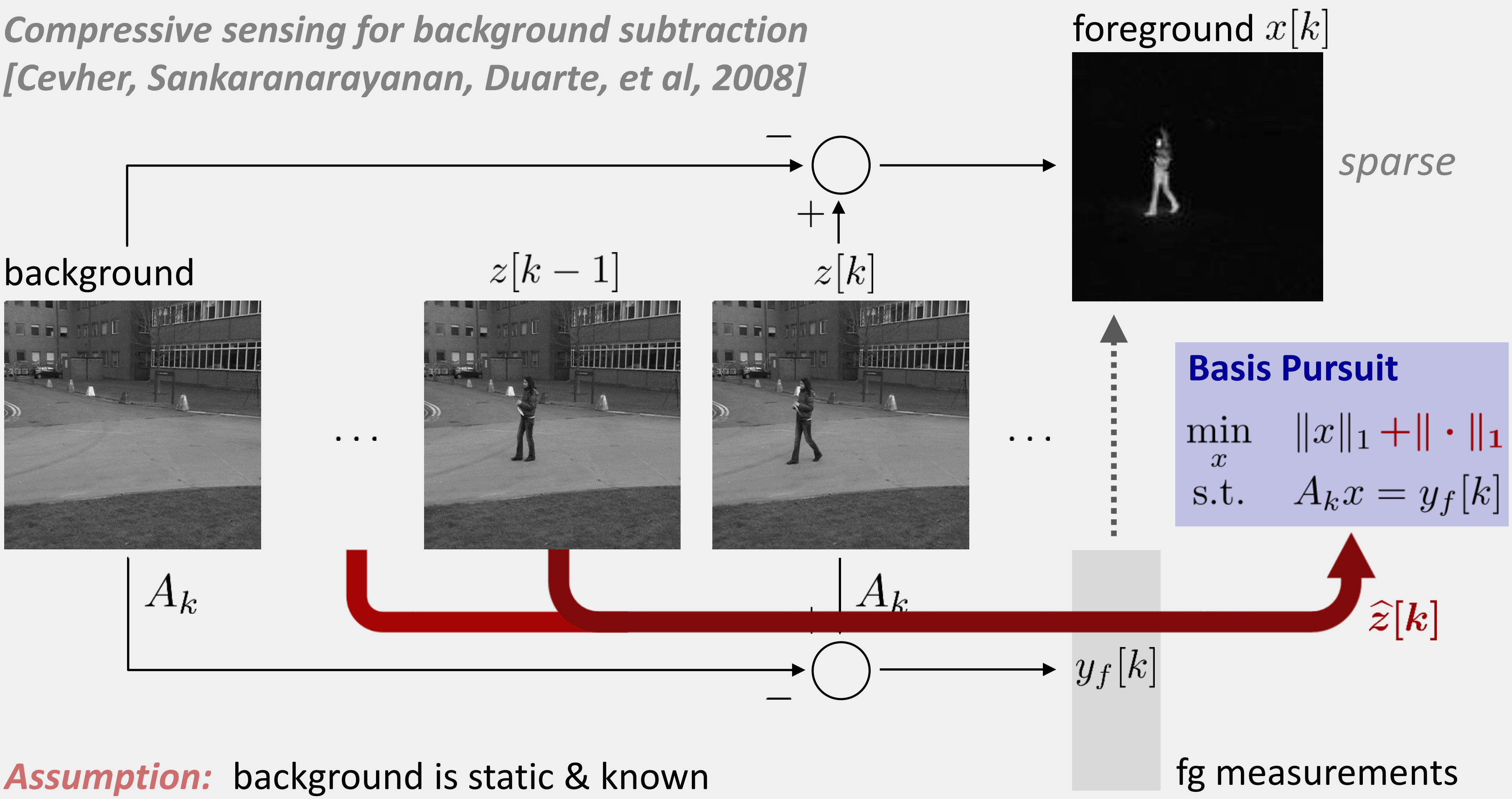
Compressive Background Subtraction



How to recover $z[k]$ *from* $y[k]$ *online ?*

How many measurements m_k *from frame* $z[k]$ *?*

Compressive sensing for background subtraction
[Cevher, Sankaranarayanan, Duarte, et al, 2008]



Assumption: background is static & known

Problems

Prior frames are ignored

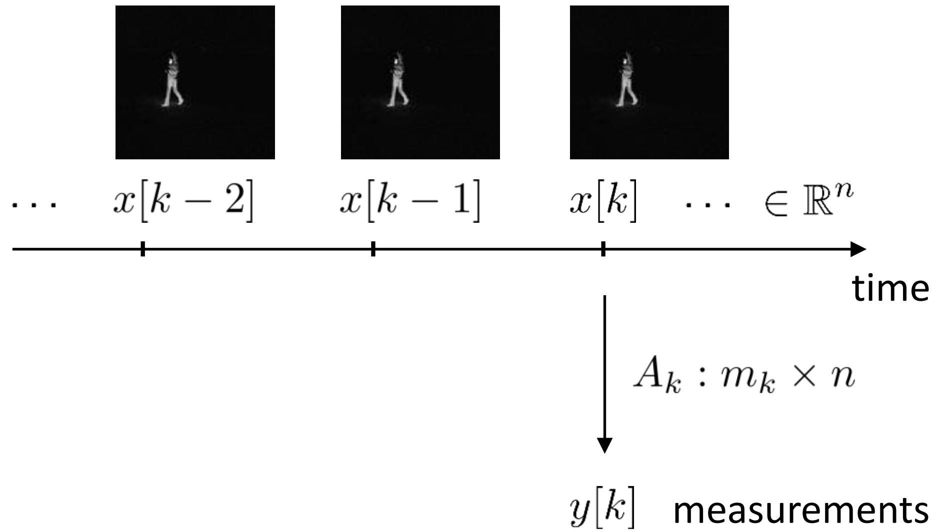
m_k : fixed; depends on foreground area

Our Approach

Estimate $z[k]$ from past frames: $\widehat{z}[k]$

Integrate $\widehat{z}[k]$ into BP via ***ℓ_1-ℓ_1 minimization***

Problem Statement



Model

sparse — arbitrary function — sparse

$$x[k] = f_k\Big(\big\{x[i]\big\}_{i=1}^{k-1}\Big) + \epsilon[k]$$

$$y[k] = A_k\, x[k]$$

Problem

Compute a minimal # of measurements m_k

Reconstruct $x[k]$ perfectly

→ *online algorithm w/ adaptive rate*

Algorithm

$$x[k] = f_k\Big(\{x[i]\}_{i=1}^{k-1}\Big) + \epsilon[k]$$

$$y[k] = A_k\, x[k]$$

Gaussian

m_k : computed at iteration $k-1$

Acquire $y[k] = A_k\, x[k]$ with $A_k : m_k \times n$

Set $w[k] = f_k\Big(\{\hat{x}[i]\}_{i=1}^{k-1}\Big)$ ***Estimate*** $x[k]$

$$\begin{aligned} \hat{x}[k] = \quad & \underset{x}{\arg\min} \quad \|x\|_1 + \big\|x - w[k]\big\|_1 \\ & \text{s.t.} \quad y[k] = A_k\, x \end{aligned}$$

L1-L1 minimization

$$\hat{\xi}_k = \big|\{i : w_i[k] \neq \hat{x}_i[k] = 0\}\big| - \big|\{i : w_i[k] = \hat{x}_i[k] \neq 0\}\big|$$

$$\hat{\overline{h}}_k = \big|\{i : \hat{x}_i[k] > 0,\ \hat{x}_i[k] > w_i[k]\} \cup \{i : \hat{x}_i[k] < 0, \hat{x}_i[k] < w_i[k]\}\big|$$

$$\hat{s}_k = \big|\{i : \hat{x}[k] \neq 0\}\big|$$

parameters of $x[k]$

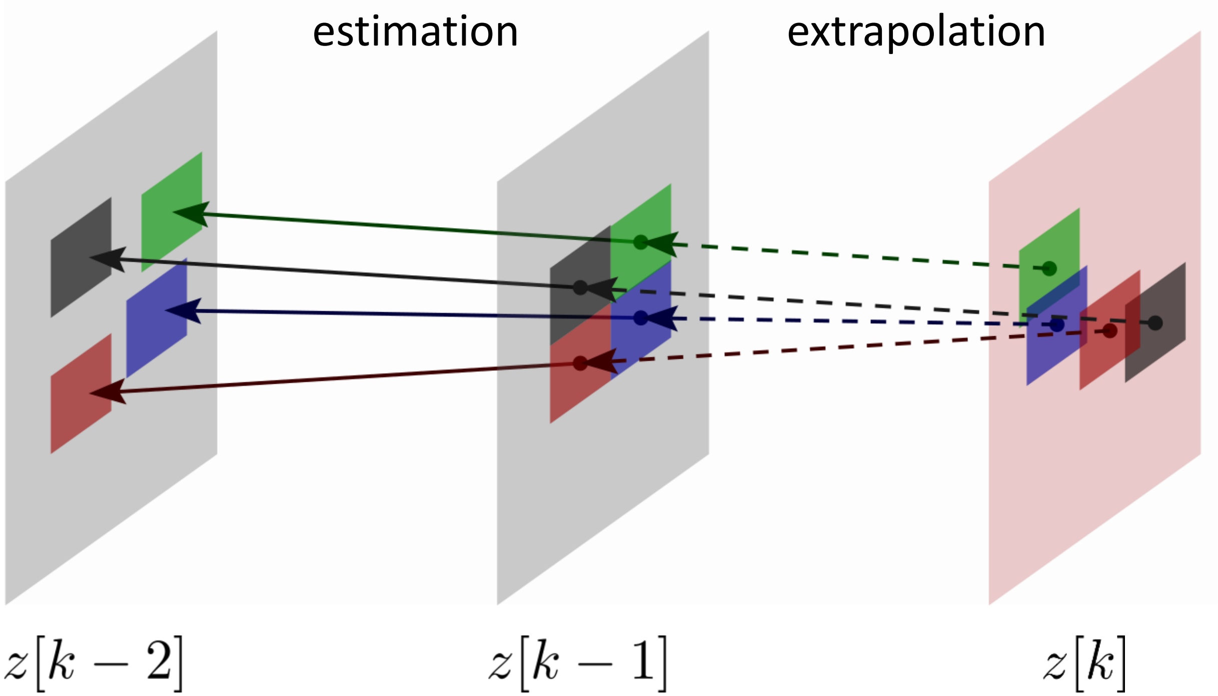
$$m_{k+1} = \left[2\hat{\overline{h}}_k \log\left(\frac{n}{\hat{s}_k + \hat{\xi}_k/2}\right) + \frac{7}{5}\left(\hat{s}_k + \frac{\hat{\xi}_k}{2}\right) + 1\right](1 + \delta_k)$$

 # measurements of $x[k+1]$

oversampling factor

$k \leftarrow k+1$ and repeat ...

Estimating a Frame



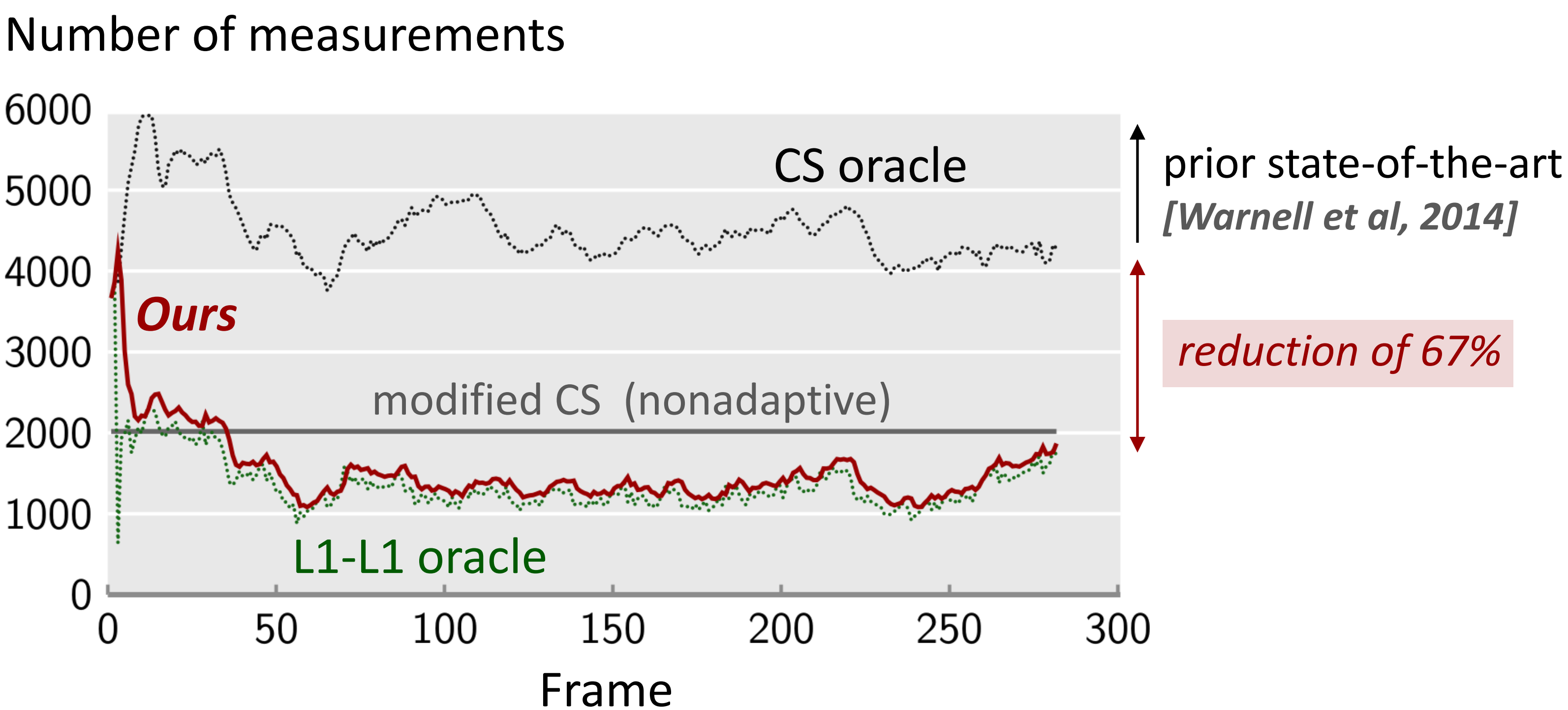
linear motion

overlap: take average; *gaps:* fill w/ average of neighbors

state-of-the-art in video coding

Experimental Results

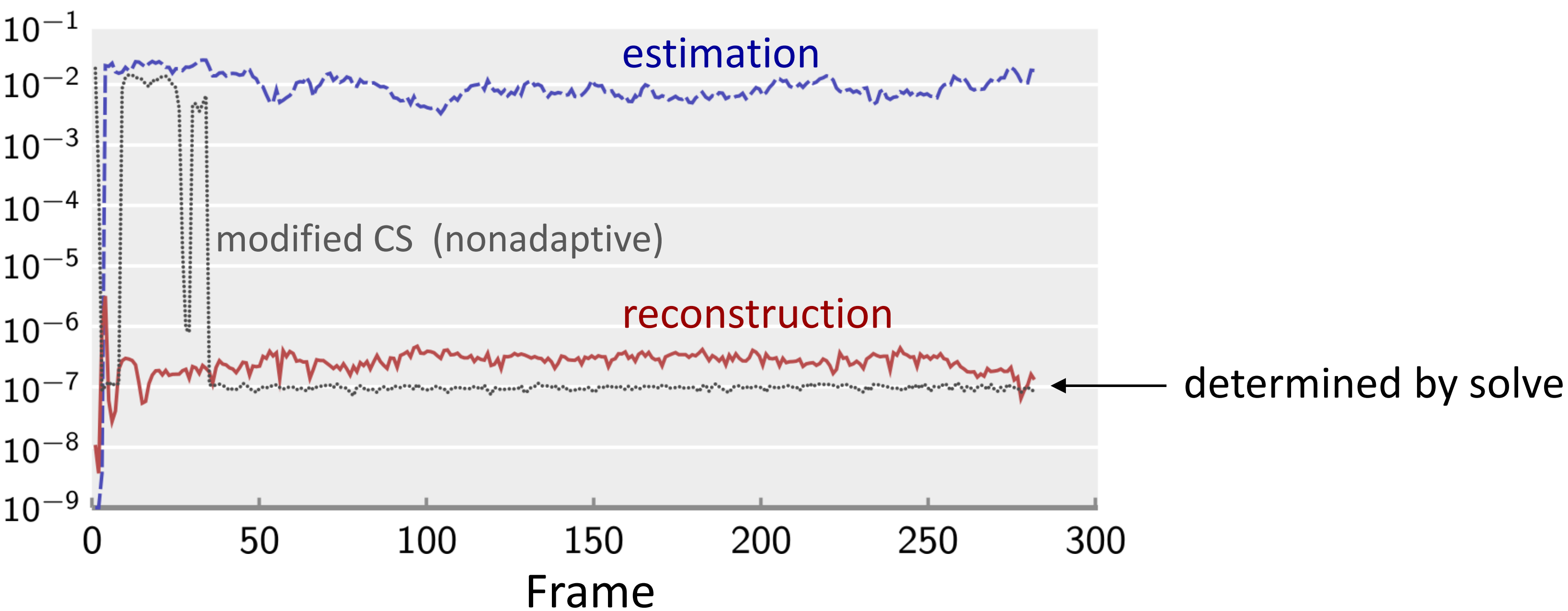
280 frames



Experimental Results

280 frames

Relative error



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- *X-ray Image Separation*
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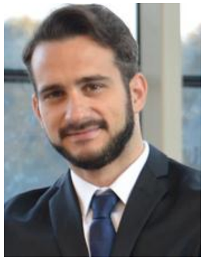

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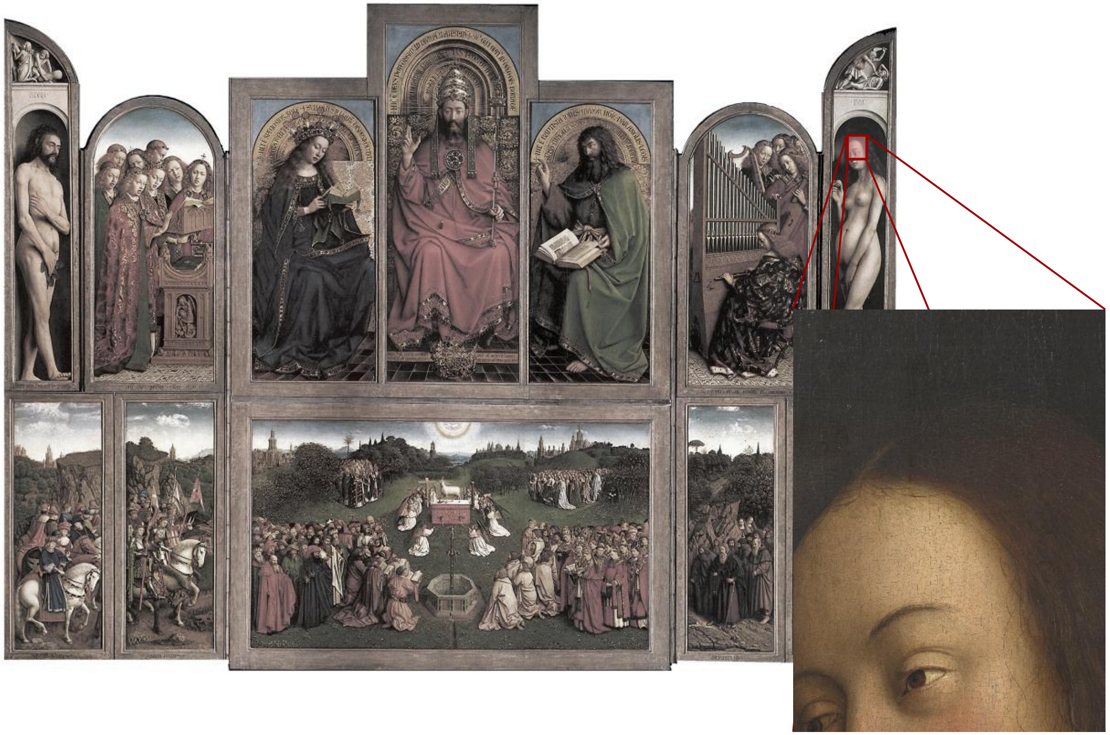
Motivation: X-Ray of Ghent Altarpiece



Mixed X-Ray

Can we use the visual images to separate the x-rays?

Approach: Coupled Dictionary Learning

Training step

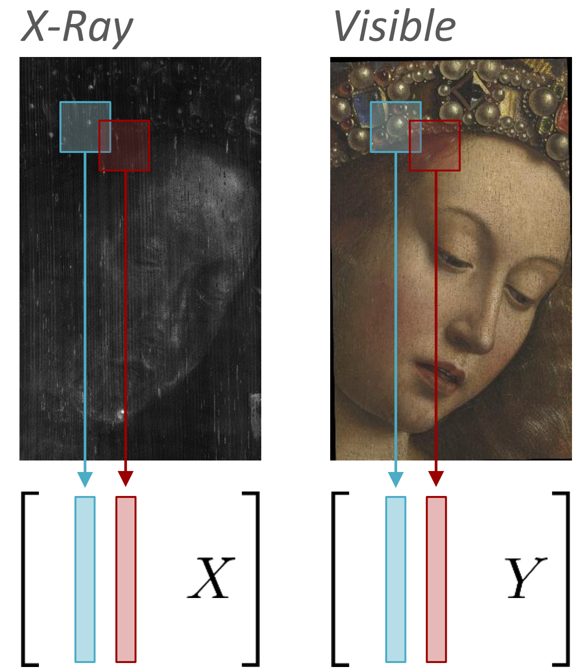
coupling

$$Y = \Psi^c Z$$
$$X = \Phi^c Z + \Phi V$$

w/ sparse columns

learn dictionaries by alternating minimization

$$\underset{\substack{\Psi^c, \Phi^c, \Phi \\ Z, V}}{\text{minimize}} \quad \|Y - \Psi^c Z\|_F^2 + \|X - \Phi^c Z - \Phi V\|_F^2$$
$$\text{subject to} \quad \text{card}(Z_i) \leq s_z\,, \quad i = 1, \ldots, T$$
$$\text{card}(V_i) \leq s_v\,, \quad i = 1, \ldots, T$$



Demixing step

mixed x-ray

$x = \Phi^c(z_1 + z_2) + 2\Phi v$

visual front

$y_1 = \Psi^c z_1$

visual back

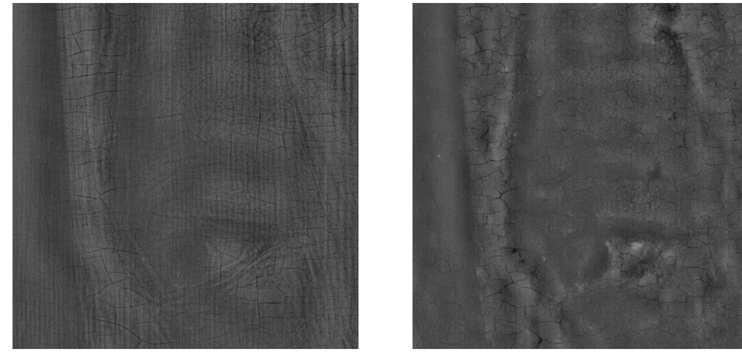
$y_2 = \Psi^c z_2$

$$\underset{z_1, z_1, v}{\text{minimize}} \quad \|z_1\|_1 + \|z_2\|_1 + \|v\|_1$$
$$\text{subject to} \quad x = \Phi^c(z_1 + z_2) + 2\Phi v$$
$$y_1 = \Psi^c z_1$$
$$y_2 = \Psi^c z_2$$

Results

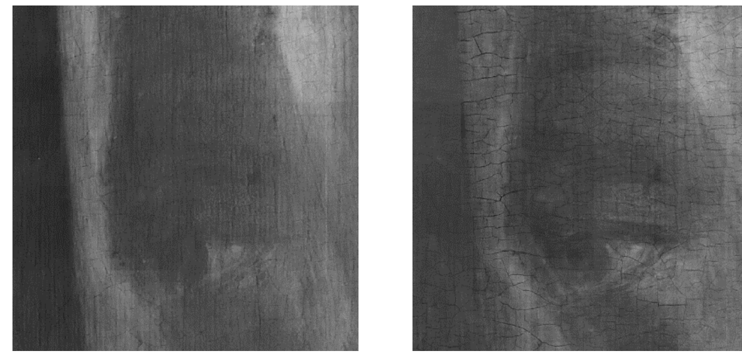
reconstructed x-rays

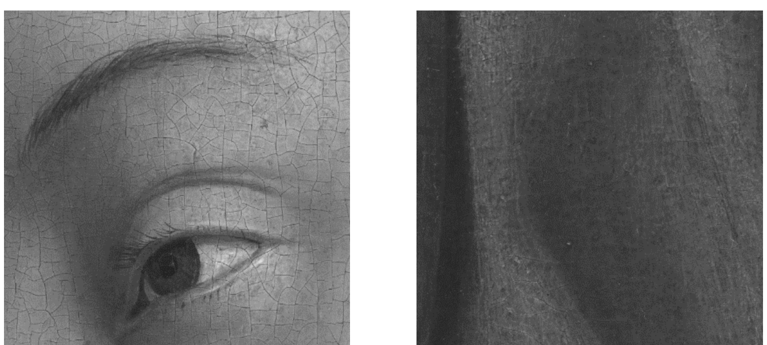
MCA ***[Bobin et al, 07']***



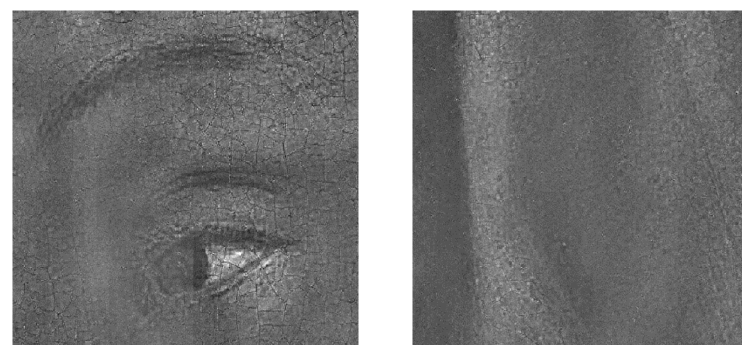
mixed x-ray

multiscale MCA w/KSVD





Ours



visuals in grayscale

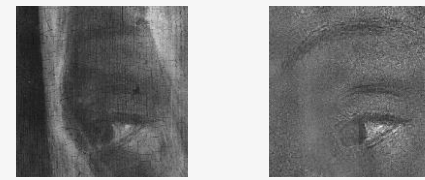
Summary / Conclusions

$$X = \Phi^c Z + \Phi V$$
$$Y = \Psi^c Z$$

data *dictionaries* *sparse columns*

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \Phi^c & \Phi \\ \Psi^c & 0 \end{bmatrix} \begin{bmatrix} Z \\ V \end{bmatrix}$$

X-ray separation



sparse

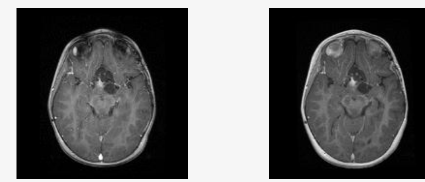
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} I & I \\ A & 0 \end{bmatrix} \begin{bmatrix} z \\ v \end{bmatrix}$$

sparse

$x = z + v$ prior information

$y = Az$ measurements

Reconstruction w/ PI



observations

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \ldots$$

multi-modal features

Low-rank model

Better models?

Guarantees?

Scalable algorithms?

Applications

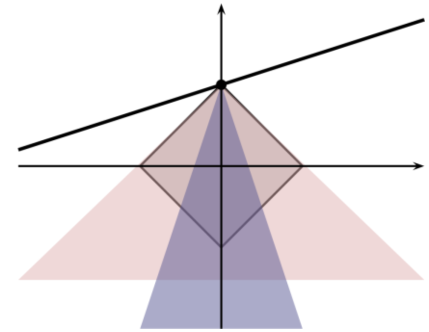
medical imaging (MRI + PET + ECG)

super-resolution (depth + visual)

SAR + microwave imaging

robotics (laser + sonar)

References



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Compressed Sensing with Prior Information: Optimal Strategies, Geometries, and Bounds
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Adaptive-Rate Reconstruction of Time-Varying Signals with Application in Compressive Foreground Extraction
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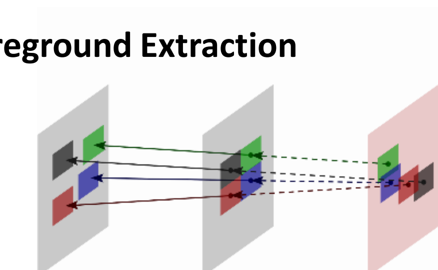

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