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PSEUDOPARABOLIC VARIATIONAL INEQUALITIES WITHOUT INITIAL CONDITIONS

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We consider a pseudoparabolic variational inequality in a cylindrical domain semibounded in a variable t. Under certain conditions imposed on the coefficients of the inequality, we prove theorems on the unique existence of a solution for a class of functions with exponential growth as $t \to \infty$.

1. Numerous practical problems (filtration of liquid in media with double porosity, liquid transport in soils, diffusion in a cracked medium with absorption or partial saturation, etc.) lead to the investigation of boundary-value problems for pseudoparabolic equations. The general theory of equations of this type is presented in [1–3]. The Cauchy problem and mixed problems for pseudoparabolic equations and systems of equations were studied in [4–8]; problems without initial conditions were investigated in [9, 10] and other works. Parabolic variational inequalities without initial conditions were studied in [12, 13].

In the present paper, we investigate conditions for the unique existence of a solution of a pseudoparabolic inequality without initial conditions.

We use the following notation: $\Omega \subset \mathbb{R}^n$ is a bounded domain, $Q_T = \Omega \times (-\infty; T]$, $T < \infty$, $Q_{t_1, t_2} = \Omega \times (t_1; t_2)$, $-\infty < t_1 < t_2 \leq T$, V is a closed subspace that is compactly and continuously imbedded in $L^2(\Omega)$, $\overset{\circ}{H}{}^1(\Omega) \subset V \subset H^1(\Omega)$, V^* is the space dual to V, K is a convex closed subset in V that contains the zero element,

$$L^r_{\text{loc}}((-\infty; T], B) = \{ u(x,t) \colon u(x,t) \in L^r((t_1, T], B) \}, \qquad 1 \leq r \leq \infty,$$

for all $t_1 \in (-\infty; T]$, B is a Banach space, and

$$W = \{ w(x,t) \colon w(x,t) \in L^2_{\text{loc}}((-\infty; T], V), w_t \in L^2_{\text{loc}}((-\infty; T], V^*) \}.$$

In the domain Q_T, we consider the problem of finding a solution of the following pseudoparabolic variational inequality:

$$\int\limits_{Q_{t_1, t_2}} \Bigg(v_t (v - u) + \sum_{i,j=1}^{n} b_{ij}(x,t) v_{x_i, t} (v_{x_j} - u_{x_j})$$

$$+ \sum_{i,j=1}^{n} a_{ij}(x,t) u_{x_i} (v_{x_j} - u_{x_j}) + \frac{1}{2} \sum_{i,j=1}^{n} b_{ij,t}(x,t) (v_{x_i} - u_{x_i})(v_{x_j} - u_{x_j})$$

$$+ \lambda \sum_{i,j=1}^{n} b_{ij}(x,t) (v_{x_i} - u_{x_i})(v_{x_j} - u_{x_j}) + \sum_{i=1}^{n} c_i(x,t) u_{x_i} (v - u)$$

$$+ \lambda (v-u)^2 + c_0(x,t) u (v-u) - f(x,t)(v-u) \Bigg) e^{2\lambda t} \, dx \, dt$$

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$$\geq \frac{1}{2}\int_{\Omega}\left(\sum_{i,j=1}^{n} b_{ij}(x,t_2)(v_{x_i}(x,t_2)-u_{x_i}(x,t_2))\,(v_{x_j}(x,t_2)-u_{x_j}(x,t_2))\right.$$

$$\left.+\,(v(x,t_2)-u(x,t_2))^2\right)e^{2\lambda t_2}\,dx$$

$$-\frac{1}{2}\int_{\Omega}\left(\sum_{i,j=1}^{n} b_{ij}(x,t_1)(v_{x_i}(x,t_1)-u_{x_i}(x,t_1))\,(v_{x_j}(x,t_1)-u_{x_j}(x,t_1))\right.$$

$$\left.+\,(v(x,t_1)-u(x,t_1))^2\right)e^{2\lambda t_1}\,dx, \qquad \lambda\in\mathbb{R}^1. \tag{1}$$

Definition 1. *A solution of inequality (1) is understood as a function $u(x,t)$ with the following properties:*

(i) $u\in L^{\infty}_{\mathrm{loc}}((-\infty;T],V)$, u, $u_x\in W$;

(ii) $u\in K$ *for almost all* $t\in(-\infty;T]$;

(iii) $u(x,t)$ *satisfies inequality (1) for almost all* $t_1, t_2\in(-\infty;T]$ *and for an arbitrary function* $v(x,t)$ *such that* v, $v_x\in W$ *and* $v\in K$ *for almost all* $t\in(-\infty;T]$.

2. First, we consider the problem of uniqueness of a solution of inequality (1). Denote by γ_1 the following number:

$$\gamma_1 = \sup_{Q_T}\sum_{i=1}^{n} c_i^2(x,t).$$

We assume that the coefficients of inequality (1) satisfy the conditions

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq a_0\sum_{i=1}^{n}\xi_i^2, \qquad a_0>0, \tag{2}$$

$$b_0\sum_{i=1}^{n}\xi_i^2 \leq \sum_{i,j=1}^{n} b_{ij}(x,t)\xi_i\xi_j \leq b^0\sum_{i=1}^{n}\xi_i^2, \qquad b_0>0, \tag{3}$$

$$c_0(x,t) \geq \gamma_0 > 0 \tag{4}$$

for almost all $(x,t)\in Q_T$ and all $\xi\in\mathbb{R}^N$.

Theorem 1. *Suppose that the coefficients of inequality (1) satisfy conditions (2)–(4),* $a_{ij}, b_{ij}, b_{ijt}, c_i\in L^2(Q_T)$, *and* $2(2a_0-b^1)\gamma_0>\gamma_1$. *Then inequality (1) cannot have more than one solution that satisfies the condition*

$$\lim_{t \to -\infty} \int_{\Omega} \left(u^2(x,t) + \sum_{i=1}^{n} u_{x_i}^2(x,t) \right) e^{\beta t} dx = 0,$$

where

$$\beta = \frac{1}{2b^0} \left(a_0 - \frac{b^1}{2} + \gamma_0 - \sqrt{\left(a_0 - \frac{b^1}{2} - \gamma_0 \right)^2 + \gamma_1} \right).$$

Proof. Assume that there exist two solutions u_1 and u_2 of inequality (1). Since $u_k, u_{kx_j} \in W$, $k = 1, 2$, $j = 1, \ldots, n$, by virtue of Theorem 1.17 in [2] we have $u_k, u_{kx_j t} \in C((-\infty; T]; L^2(\Omega))$, and the following integrals are meaningful:

$$\int_{Q_{t_1,t_2}} u_k u_{kt} dx dt, \qquad \int_{Q_{t_1,t_2}} \sum_{i,j=1}^{n} u_{kx_i t} u_{kx_j t} dx dt, \quad i = 1, 2.$$

Consider the operators A and B defined for arbitrary functions $w_1(x,t), w_2(x,t) \in W$ and almost all $t \in (-\infty; T]$ by the following equalities:

$$\langle Aw_1, w_2 \rangle(t) = \int_{\Omega} \left(\sum_{i,j=1}^{n} a_{ij} w_{1x_i} w_{2x_j} + \sum_{i=1}^{n} c_i w_{1x_i} w_2 + c_0 w_1 w_2 \right) dx,$$

$$\langle Bw_1, w_2 \rangle(t) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} b_{ijt} w_{1x_i} w_{2x_j} dx.$$

Let us show that $A - B$ is a monotone operator. By virtue of the conditions of the theorem and the elementary estimate

$$ab \le \frac{a^2\delta}{2} + \frac{b^2}{2\delta}, \qquad \delta > 0,$$

we obtain the following inequality:

$$\langle (A-B)(w_1 - w_2), w_1 - w_2 \rangle(t) \ge \int_{\Omega} \left(\left(a_0 - \frac{b^1}{2} - \frac{\gamma_1 \delta_0}{2} \right) \sum_{i=1}^{n} (w_{1,x_i} - w_{2,x_i})^2 \left(\gamma_0 - \frac{1}{2\delta_0} \right) (w_1 - w_2)^2 \right) dx$$

$$\ge \beta \int_{\Omega} \left(\sum_{i,j=1}^{n} b_{ij}(x,t)(w_{1,x_i} - w_{2,x_j})(w_{1,x_i} - w_{2,x_j}) + (w_1 - w_2)^2 \right) dx, \qquad (5)$$

where $w_1(x,t)$ and $w_2(x,t)$ are arbitrary functions from W.

Estimate (5) yields

$$\langle (A-B)(w_1 - w_2), w_1 - w_2 \rangle(t) \ge 0$$

for arbitrary $w_1, w_2 \in W$, i.e., the operator $A - B$ is monotone.

Consider functions u^i for which

$$u^i,\, u_x^i \in L^2_{\mathrm{loc}}((-\infty;T];V)\cap C\big((-\infty;T];L^2(\Omega)\big),\quad i=1,2,$$

and the following inequality holds:

$$\int\limits_{Q_{t_1,t_2}} \left((v_t - f)(v-u) + \sum_{i,j=1}^{n} b_{ij} v_{x_i t}(v_{x_j} - u_{x_j}) \right) e^{2\lambda t}\, dx\, dt$$

$$\geq -\lambda \int\limits_{Q_{t_1,t_2}} \left((v-u)^2 + \sum_{i,j=1}^{n} b_{ij}(v_{x_i} - u_{x_i})(v_{x_j} - u_{x_j}) \right) e^{2\lambda t}$$

$$-\frac{1}{2} \int\limits_{Q_{t_1,t_2}} \sum_{i,j=1}^{n} b_{ijt}(v_{x_i} - u_{x_i})(v_{x_j} - u_{x_j}) e^{2\lambda t}\, dx\, dt$$

$$+\frac{1}{2}\int\limits_{\Omega}\left(\sum_{i,j=1}^{n} b_{ij}(v_{x_i}(x,t_2) - u_{x_i}(x,t_2))\,(v_{x_j}(x,t_2) - u_{x_j}(x,t_2))\right.$$

$$\left. + (v(x,t_2) - u(x,t_2))^2 \right) e^{2\lambda t_2}\, dx$$

$$-\frac{1}{2}\int\limits_{\Omega}\left(\sum_{i,j=1}^{n} b_{ij}(v_{x_i}(x,t_1) - u_{x_i}(x,t_1))\,(v_{x_j}(x,t_1) - u_{x_j}(x,t_1))\right.$$

$$\left. + (v(x,t_1) - u(x,t_1))^2 \right) e^{2\lambda t_1}\, dx.$$

It is easy to show that, for $f = f_1$ and $f = f_2$, respectively, such functions satisfy the following estimate:

$$\int\limits_{Q_{t_1,t_2}} (f_1 - f_2)(u^1 - u^2) e^{2\lambda t}\, dx\, dt$$

$$\geq -\lambda \int\limits_{Q_{t_1,t_2}} \left(\sum_{i,j=1}^{n} b_{ij}\big(u^1_{x_i} - u^2_{x_i}\big)\big(u^1_{x_j} - u^2_{x_j}\big) + (u^1 - u^2)^2 \right) e^{2\lambda t}\, dx\, dt$$

$$+\frac{1}{2}\int\limits_{\Omega}\left(\sum_{i,j=1}^{n} b_{ij}\big(u^1_{x_i}(x,t_2) - u^2_{x_i}(x,t_2)\big)\big(u^1_{x_j}(x,t_2) - u^2_{x_j}(x,t_2)\big)\right.$$

$$\left. + \big(u^1(x,t_2) - u^2(x,t_2)\big)^2 \right) e^{2\lambda t_2}\, dx - \frac{1}{2}\int\limits_{\Omega}\left(\sum_{i,j=1}^{n} b_{ij}\big(u^1_{x_i}(x,t_1)\right.$$

$$- u^2_{x_i}(x,t_1)\Big)\Big(u^1_{x_j}(x,t_1)-u^2_{x_j}(x,t_1)\Big)+\Big(u^1(x,t_1)-u^2(x,t_1)\Big)^2\Big)e^{2\lambda t_1}\,dx$$

$$-\frac{1}{2}\int\limits_{Q_{t_1,t_2}}\left(\sum_{i,j=1}^{n} b_{ijt}\Big(u^1_{x_i}-u^2_{x_i}\Big)\Big(u^1_{x_j}-u^2_{x_j}\right)e^{2\lambda t}\,dx\,dt. \tag{6}$$

Relation (6) yields

$$\frac{1}{2}\int\limits_{t_1}^{t_2}\frac{d}{dt}\left(\int\limits_{\Omega}\left[\big(u^1(x,t)-u^2(x,t)\big)^2+\sum_{i,j=1}^{n} b_{ij}\Big(u^1_{x_i}-u^2_{x_i}\Big)\Big(u^1_{x_j}-u^2_{x_j}\Big)\right]e^{2\lambda t}\,dx\right)dt$$

$$\leq \lambda\int\limits_{Q_{t_1,t_2}}\left[\sum_{i,j=1}^{n} b_{ijt}\Big(u^1_{x_i}-u^2_{x_i}\Big)\Big(u^1_{x_j}-u^2_{x_j}\Big)+(u^1-u^2))^2\right]e^{2\lambda t}\,dx\,dt$$

$$+\int\limits_{Q_{t_1,t_2}}(f_1-f_2)(u^1-u^2)e^{2\lambda t}\,dx\,dt$$

$$+\frac{1}{2}\int\limits_{Q_{t_1,t_2}}\sum_{i,j=1}^{n} b_{ijt}\Big(u^1_{x_i}-u^2_{x_i}\Big)\Big(u^1_{x_j}-u^2_{x_j}\Big)e^{2\lambda t}\,dx\,dt. \tag{7}$$

By setting $f_1=f-Au_1$ and $f_2=f-Au_2$ in (7), we get

$$\frac{1}{2}\int\limits_{t_1}^{t_2}\frac{d}{dt}\left(\int\limits_{\Omega}\left(\sum_{i,j=1}^{n} b_{ij}\Big(u_{1x_i}-u_{2x_i}\Big)\Big(u_{1x_j}-u_{2x_j}\Big)+(u_1-u_2)^2\right)e^{2\lambda t}\,dx\right)dt$$

$$\leq \lambda\int\limits_{Q_{t_1,t_2}}\left(\sum_{i,j=1}^{n} b_{ij}\Big(u_{1x_i}-u_{2x_i}\Big)\Big(u_{1x_j}-u_{2x_j}\Big)+(u_1-u_2)^2\right)e^{2\lambda t}\,dx\,dt$$

$$+\int\limits_{t_1}^{t_2}\langle A(u_2-u_1),u_2-u_1\rangle e^{2\lambda t}\,dx$$

$$+\frac{1}{2}\int\limits_{Q_{t_1,t_2}}\sum_{i,j=1}^{n} b_{ij,t}\Big(u_{1x_i}-u_{2x_i}\Big)\Big(u_{1x_j}-u_{2x_j}\Big)e^{2\lambda t}\,dx\,dt.$$

Carrying out differentiation with respect to t on the left-hand side of the last inequality, we obtain the estimate

$$\frac{1}{2}\int\limits_{t_1}^{t_2}e^{2\lambda t}\frac{d}{dt}\left(\int\limits_{\Omega}\left(\sum_{i,j=1}^{n} b_{ij}(x,t)\Big(u_{1x_i}-u_{2x_i}\Big)\Big(u_{1x_j}-u_{2x_j}\Big)+(u_1-u_2)^2\right)dx\right)dt$$

$$+\int\limits_{t_1}^{t_2}\langle (A-B)(u_2-u_1),u_2-u_1\rangle e^{2\lambda t}\,dt \;\leq\; 0. \tag{8}$$

We denote

$$y(t) = \int_{\Omega}\left(\sum_{i,j=1}^{n} b_{ij}(x,t)\left(u_{1x_i}-u_{2x_i}\right)\left(u_{1x_j}-u_{2x_j}\right)+(u_1-u_2)^2\right)dx$$

and use estimate (5). Then relation (8) yields

$$\int_{t_1}^{t_2}\left[y'(t)+\beta y(t)\right]e^{2\lambda t}\,dt \le 0 \tag{9}$$

for all $t_1, t_2 \in (-\infty; T]$, $t_1 < t_2$. It follows from (9) that, for almost all $t \in (-\infty; T]$, we have $y'(t)+\beta y(t) \le 0$.

Multiplying the last inequality by $e^{\beta t}$ and integrating from t_1 to t_2, we get

$$\int_{t_1}^{t_2}\frac{d}{dt}\left(y(t)e^{\beta t}\right)dt \le 0,$$

which yields

$$y(t_2)e^{\beta t_2} \le y(t_1)e^{\beta t_1}. \tag{10}$$

Passing to the limit as $t_1 \to \infty$ in (10) and using the condition of the theorem, we obtain $y(t_2)e^{\beta t_2} \le 0$ for all $t_2 \in (-\infty; T]$. Estimate (10) implies that

$$\int_{\Omega}\left(\sum_{i,j=1}^{n} b_{ij}(x,t_2)\left(u_{1x_i}-u_{2x_i}\right)\left(u_{1x_j}-u_{2x_j}\right)+(u_1-u_2)^2\right)dx = 0$$

for all $t_2 \in (-\infty; T]$, i.e., $u_1(x,t) = u_2(x,t)$ almost everywhere in Q_T. The theorem is proved.

3. Let us establish conditions for the existence of a solution of inequality (1). We set

$$\alpha_1 = \frac{\gamma_1}{2\left(\alpha_0+\sqrt{\alpha_0^2+\gamma_1 b^0}\right)},$$

where $\alpha_0 = a_0 - \gamma_0 b^0 - \frac{b^1}{2}$ for $\gamma_1 > 0$, and $\alpha_1 = 0$ for $\gamma_1 = 0$.

Theorem 2. *Suppose that the coefficients of inequalities (1) satisfy conditions (2)–(4),* $a_{ij}, b_{ij}, b_{ijt}, c_i \in L^2(Q_T)$, *and the functions* $t \to a_{ij}(x,t)$, $t \to b_{ijt}(x,t)$, $t \to c_i(x,t)$, $i,j = 1,\ldots,n$, $t \to c_0(x,t)$, *and* $t \to f(x,t)$ *are continuous on* $(-\infty; T]$ *for almost all* $x \in \Omega$. *Also assume that there exists a number* $\lambda, \lambda < \gamma_0 - \alpha_1$ *such that* $f(x,t)e^{\lambda t} \in L^2(Q_T)$. *Then there exists a solution* $u(x,t)$ *of inequality (1) such that*

$$\lim_{t\to-\infty}\int_{\Omega}\left[\sum_{i=1}^{n} u_{x_i}^2(x,t)+u^2(x,t)\right]e^{2\lambda t}dx = 0. \tag{11}$$

Proof. Consider the following auxiliary problem in the domain $Q_{t_0, T}$:

$$u_t + \sum_{i,j=1}^{n} \left(b_{ij}(x,t)u_{x_i,t}\right)_{x_j} + A(t)u + \frac{1}{\varepsilon}\mathcal{B}(u) = f_{t_0}(x,t), \tag{12}$$

$$u(x, t_0) = 0, \tag{13}$$

where $\varepsilon > 0$, $t_0 \in (-\infty; T]$, $\mathcal{B}(u) = J(u - P_K(u))$, J is the operator of duality between the spaces V and V^*, P_K is the operator of projection of V onto K, and

$$f_{t_0} = \begin{cases} f(x,t), & (x,t) \in Q_{t_0, T}, \\ 0, & (x,t) \in Q_{t_0}. \end{cases}$$

It is known [13] that the operator $\mathcal{B}$ is monotone, bounded, and continuous in the Lipschitz sense. By virtue of the conditions of the theorem, the reasoning presented above, and the results of [10], there exists a solution $u(x, t)$ of problem (12), (13) such that

$$u \in L^2((t_0, T), V), \quad u_t \in L^2\left((t_0, T), V^*\right), \quad u_{x_i} \in L^2((t_0, T), V),$$

$$u_{x_i, t} \in L^2\left((t_0, T), V^*\right), \quad i = 1, \ldots, n.$$

Consider a sequence of functions $\{u^{k,\varepsilon}(x,t)\}$ that are solutions of problem (12), (13) for $t_0 = T - k$, $k = 1, 2, \ldots$, and extend every function $u^{k,\varepsilon}(x,t)$ by zero to the domain Q_{T-k}. Then the following equality holds:

$$\int_{Q_\tau} \Bigg(u_t^{k,\varepsilon} u^{k,\varepsilon} + \sum_{i,j=1}^{n} b_{ij}(x,t) u_{t,x_i}^{k,\varepsilon} u_{x_j}^{k,\varepsilon} + \sum_{i,j=1}^{n} a_{ij}(x,t) u_{x_i}^{k,\varepsilon} u_{x_j}^{k,\varepsilon}$$

$$+ \sum_{i=1}^{n} c_i(x,t) u_{x_i}^{k,\varepsilon} u^{k,\varepsilon} + c_0(x,t)(u^{k,\varepsilon})^2 + \frac{1}{\varepsilon}\mathcal{B}(u^{k,\varepsilon}) u^{k,\varepsilon}$$

$$- f_{T-k}(x,t) u^{k,\varepsilon} \Bigg) e^{2\lambda t}\, dx\, dt = 0, \quad k \in \mathbb{N}, \quad \tau \le T. \tag{14}$$

This yields the estimate

$$\frac{1}{2}\int_{\Omega} \left[b_0 \sum_{i=1}^{n} \left(u_{x_i}^{k,\varepsilon}\right)^2 + \left(u^{k,\varepsilon}\right)^2 \right] e^{2\lambda t}\, dx + \int_{Q_\tau} \Bigg(\left(a_0 - \lambda b^0 - \frac{b^1}{2} - \frac{\gamma_1 \delta_0}{2} \right) \sum_{i=1}^{n} \left(u_{x_i}^{k,\varepsilon}\right)^2$$

$$+ \left(\gamma_0 - \lambda - \frac{1}{2\delta_0} - \frac{\delta_1}{2} \right) \left(u^{k,\varepsilon}\right)^2 + \frac{1}{\varepsilon}\mathcal{B}(u^{k,\varepsilon}) u^{k,\varepsilon} \Bigg) e^{2\lambda t}\, dx\, dt$$

$$\le \frac{1}{2\delta_1} \int_{Q_T} f^2(x,t) e^{2\lambda t}\, dx\, dt, \quad k \in \mathbb{N}, \quad \delta_1 > 0, \tag{15}$$

Denote

$$F_\lambda = \int_{Q_T} f^2(x,t)e^{2\lambda t}\,dx\,dt.$$

It is easy to verify that, under the conditions of the theorem, relation (15) yields the following estimates:

$$\int_\Omega \left[b_0 \sum_{i=1}^n \left(u_{x_i}^{k,\varepsilon}\right)^2 + \left(u^{k,\varepsilon}\right)^2 \right] e^{2\lambda t}\,dx \le \mu_0 F_\lambda,$$

$$\int_{Q_T} \left[\left(u^{k,\varepsilon}\right)^2 + b_0 \sum_{i=1}^n \left(u_{x_i}^{k,\varepsilon}\right)^2 \right] e^{2\lambda t}\,dx\,dt \le \mu_0 F_\lambda,$$

$$\int_{Q_T} \mathcal{B}(u^{k,\varepsilon})u^{k,\varepsilon} e^{2\lambda t}\,dx\,dt \le \varepsilon\mu_0 F_\lambda, \quad k \in \mathbb{N}, \tag{16}$$

where μ_0 does not depend on ε and k.

Estimates (16) imply that there exists a subsequence $\{u^{k_m,\varepsilon}\}$ of the sequence $\{u^{k,\varepsilon}\}$ such that

$$\begin{aligned} &e^{\lambda t}u^{k_m,\varepsilon} \to e^{\lambda t}u^\varepsilon \quad *\text{-weakly in} \quad L^\infty((-\infty;T],V), \\ &e^{\lambda t}u^{k_m,\varepsilon} \to e^{\lambda t}u^\varepsilon \quad \text{weakly in} \quad L^\infty((-\infty;T],V) \quad \text{as} \quad k_m \to \infty. \end{aligned} \tag{17}$$

It follows from (17) that $u^\varepsilon(x,t)$ is a solution of the equation

$$u_t + \sum_{i,j=1}^n \left(b_{ij}(x,t)u_{x_i,t}\right)_{x_j} + A(t)u + \frac{1}{\varepsilon}\mathcal{B}(u) = f(x,t). \tag{18}$$

Furthermore,

$$e^{\lambda t}u^\varepsilon \in L^\infty((-\infty;T],V) \cap L^2((-\infty;T],V), \qquad e^{\lambda t}u_t^\varepsilon \in L^2\left((-\infty;T],V^*\right),$$

$$e^{\lambda t}u_{x_i}^\varepsilon \in L^2((-\infty;T],V), \qquad e^{\lambda t}u_{t,x_i}^\varepsilon \in L^2\left((-\infty;T],V^*\right), \quad i = 1,\ldots, \quad n,$$

and the function $u^\varepsilon(x,t)$ satisfies estimates (16).

Let a function $v(x,t)$ be such that $v, v_x \in W$ and $v \in K$ for almost all $t \in (-\infty;T]$. Since $\mathcal{B}(v) = 0$, by virtue of (18) we get

$$\int_{Q_{t_1,t_2}} \Bigg(v_t(v-u^\varepsilon) + \sum_{i,j=1}^n b_{ij}(x,t)v_{t,x_i}(v_{x_j} - u_{x_j}^\varepsilon)$$

$$+ \sum_{i,j=1}^n a_{ij}(x,t)u_{x_i}^\varepsilon(v_{x_j} - u_{x_j}^\varepsilon) + \sum_{i,j=1}^n \left(\frac{1}{2}b_{ijt}(x,t) + \lambda b_{ij}(x,t) \right)$$

$$\times (v_{x_i} - u^{\varepsilon}_{x_i})(v_{x_j} - u^{\varepsilon}_{x_j}) + \sum_{i=1}^{n} c_i u^{\varepsilon}_{x_i}(v - u^{\varepsilon}) + c_0 u^{\varepsilon}(v - u^{\varepsilon})$$

$$+ \lambda(v - u^{\varepsilon})^2 - f(x,t)(v - u^{\varepsilon})\Bigg) e^{2\lambda t} dx\, dt$$

$$= \frac{1}{\varepsilon} \int_{Q_{t_1,t_2}} \big(\mathcal{B}(v) - \mathcal{B}(u^{\varepsilon})\big)(v - u^{\varepsilon}) e^{2\lambda t} dx\, dt$$

$$+ \frac{1}{2} \int_{\Omega} \Bigg(\sum_{i,j=1}^{n} b_{ij}(x,t_2)\big(v_{x_i}(x,t_2) - u^{\varepsilon}_{x_i}(x,t_2)\big)\big(v_{x_j}(x,t_2) - u^{\varepsilon}_{x_j}(x,t_2)\big)$$

$$+ \big(v(x,t_2) - u^{\varepsilon}(x,t_2)\big)^2 \Bigg) e^{2\lambda t_2} dx - \frac{1}{2} \int_{\Omega} \Bigg(\sum_{i,j=1}^{n} b_{ij}(x,t_1)$$

$$\times \big(v_{x_i}(x,t_1) - u^{\varepsilon}_{x_i}(x,t_1)\big)\big(v_{x_j}(x,t_1) - u^{\varepsilon}_{x_j}(x,t_1)\big) + \big(v(x,t_1) - u^{\varepsilon}(x,t_1)\big)^2 \Bigg) e^{2\lambda t_1} dx$$

$$\geq \frac{1}{2} \int_{\Omega} \Bigg(\sum_{i,j=1}^{n} b_{ij}(x,t_2)\big(v_{x_i}(x,t_2) - u^{\varepsilon}_{x_i}(x,t_2)\big)\big(v_{x_j}(x,t_2) - u^{\varepsilon}_{x_j}(x,t_2)\big)$$

$$+ \big(v(x,t_2) - u^{\varepsilon}(x,t_2)\big)^2 \Bigg) e^{2\lambda t_2} dx - \frac{1}{2} \int_{\Omega} \Bigg(\sum_{i,j=1}^{n} b_{ij}(x,t_1)$$

$$\times \big(v_{x_i}(x,t_1) - u^{\varepsilon}_{x_i}(x,t_1)\big)\big(v_{x_j}(x,t_1) - u^{\varepsilon}_{x_j}(x,t_1)\big) + \big(v(x,t_1) - u^{\varepsilon}(x,t_1)\big)^2 \Bigg) e^{2\lambda t_1} dx. \quad (19)$$

Let us show that there exists a sequence $\{e^{\lambda t} u^{\varepsilon_m}(x,t)\} \subset \{e^{\lambda t} u^{\varepsilon}(x,t)\}$ of functions that are defined for $t \in (-\infty; T]$, take values in V, and are jointly continuous on any interval $[T_1, T_2] \subset (-\infty; T]$. Since the second estimate in (16) is valid for the functions $e^{\lambda t} u^{\varepsilon}(x,t)$, $\varepsilon > 0$, by virtue of the Fatou lemma we obtain

$$\int_{T_1 - 1}^{T_1} e^{2\lambda t} \liminf \left\| u^{\varepsilon}(x,t) \right\|_V^2 dt \leq \mu_0 F_{\lambda}.$$

This implies that, for almost all $t \in [T_1 - 1, T_1]$,

$$e^{2\lambda t} \liminf \left\| u^{\varepsilon}(x,t) \right\|_V^2 < \infty.$$

Then there exists $\tilde{T} \in [T_1 - 1, T]$ such that

$$e^{2\lambda} \liminf \left\| u^{\varepsilon}(x,\tilde{T}) \right\|_V^2 \leq \mu_1.$$

Let $\tilde{T} = T_1$ and let $\{u^{\varepsilon_m}(x,t)\}$ be a sequence for which

$$\liminf \left\| u^{\varepsilon_m}(x,T_1)\right\|_V^2 = \lim_{m\to\infty} \left\| u^{\varepsilon_m}(x,T_1)\right\|_V^2.$$

Then $\left\| u^{\varepsilon_m}(x,T_1)\right\|_V^2 \le \mu_2$ for all $m \in \mathbb{N}$. We set $t_1 = T_1$, $t_2 = T_2 + \delta$, and $v(x,t) = u^{\varepsilon_m}(x,T_1)e^{2\lambda(T_1 - t)}$ in (19). By simple calculation, we obtain the following estimate:

$$\frac{1}{2}\int_\Omega \left(\sum_{i=1}^n b_{ij}(x,T_1+\delta)\left(u_{x_i}^{\varepsilon_m}(x,T_1+\delta)e^{\lambda(T_1+\delta)} - u_{x_i}^{\varepsilon_m}(x,T_1)e^{\lambda T_1}\right)\right.$$
$$\times \left(u_{x_j}^{\varepsilon_m}(x,T_1+\delta)e^{\lambda(T_1+\delta)} - u_{x_j}^{\varepsilon_m}(x,T_1)e^{\lambda T_1}\right)$$
$$\left. + \left(u^{\varepsilon_m}(x,T_1+\delta)e^{\lambda(T_1+\delta)} - u^{\varepsilon_m}(x,T_1)e^{\lambda T_1}\right)^2\right)dx \le \delta\mu_3, \tag{20}$$

where the constant $\mu_3 > 0$ does not depend on m.

By using estimate (6) with

$$t_1 = T_1, \quad t_2 = t, \quad f_1(x,t) = f(x,t) - A(t)u^{\varepsilon_m}(x,t),$$
$$f_2(x,t) = f(x,t+\delta)e^{\lambda\delta} - A(t+\delta)u^{\varepsilon_m}(x,t+\delta)e^{\lambda\delta},$$
$$u_1(x,t) = u^{\varepsilon_m}(x,t), \qquad u_2(x,t) = u^{\varepsilon_m}(x,t+\delta)e^{\lambda\delta},$$

we get

$$\frac{1}{2}\int_\Omega \left(u^{\varepsilon_m}(x,t) - u^{\varepsilon_m}(x,t+\delta)e^{\lambda\delta}\right)^2 e^{2\lambda t}dx$$
$$-\frac{1}{2}\int_\Omega \left(u^{\varepsilon_m}(x,T_1) - u^{\varepsilon_m}(x,T_1+\delta)e^{\lambda\delta}\right)^2 e^{2\lambda T_1}dx$$
$$+\frac{1}{2}\int_\Omega \sum_{i,j=1}^n b_{ij}(x,t)\left(u_{x_i}^{\varepsilon_m}(x,t) - u_{x_i}^{\varepsilon_m}(x,t+\delta)e^{\lambda\delta}\right)$$
$$\times \left(u_{x_j}^{\varepsilon_m}(x,t) - u_{x_j}^{\varepsilon_m}(x,t+\delta)e^{\lambda\delta}\right)e^{2\lambda t}\,dx$$
$$-\frac{1}{2}\int_\Omega \sum_{i,j=1}^n b_{ij}(x,T_1)\left(u_{x_i}^{\varepsilon_m}(x,T_1) - u_{x_i}^{\varepsilon_m}(x,T_1+\delta)e^{\lambda\delta}\right)$$
$$\times \left(u_{x_j}^{\varepsilon_m}(x,T_1) - u_{x_j}^{\varepsilon_m}(x,T_1+\delta)e^{\lambda\delta}\right)e^{2\lambda T_1}dx$$

$$\le \mu_4 \int\limits_{Q_{T_1,t}} \left(u^{\varepsilon_m}(x,t) - u^{\varepsilon_m}(x,t+\delta)e^{\lambda\delta}\right)^2 e^{2\lambda t} dx\,dt$$

$$+ \int\limits_{Q_{T_1,t}} \sum_{i,j=1}^{n} \left(\lambda b_{ij} + \frac{1}{2} b_{ij,t}\right)\left(u_{x_i}^{\varepsilon_m}(x,t) - u_{x_i}^{\varepsilon_m}(x,t+\delta)e^{\lambda\delta}\right)$$

$$\times \left(u_{x_j}^{\varepsilon_m}(x,t) - u_{x_j}^{\varepsilon_m}(x,t+\delta)\right) e^{2\lambda t}\, dx\,dt$$

$$+ \int\limits_{Q_{T_1,t}} \left[f(x,t) - A(t)u^{\varepsilon_m}(x,t) - f(x,t+\delta) + A(t+\delta)u^{\varepsilon_m}(x,t+\delta)\right]$$

$$\times \left(u^{\varepsilon_m}(x,t) - u^{\varepsilon_m}(x,t+\delta)e^{\lambda\delta}\right) e^{2\lambda t} dx\,dt. \tag{21}$$

Note that the continuity of $f(x, t)$ in the variable t implies that the function

$$\int\limits_{\Omega} f(x,t)\,dx$$

is uniformly continuous on the interval $[T_1, T_2]$. Therefore,

$$\int\limits_{Q_{T,t}} [f(x,t+\delta) - f(x,t)]^2 e^{2\lambda t} \le \delta\mu_5. \tag{22}$$

By virtue of (20)–(22) and the Gronwall–Bellman lemma, we have

$$\int\limits_{\Omega} \left[\sum_{i=1}^{n} \left|u_{x_i}^{\varepsilon_m}(x,t+\delta)e^{\lambda(t+\delta)} - u_{x_i}^{\varepsilon_m}(x,t)e^{\lambda t}\right|^2 + \left|u^{\varepsilon_m}(x,t+\delta)e^{\lambda(t+\delta)} - u^{\varepsilon_m}(x,t)e^{\lambda t}\right|^2\right] dx \le \delta\mu_6. \tag{23}$$

It follows from inequality (23) that the sequence $\left\{u^{\varepsilon_m}(x,t)e^{\lambda t}\right\}$ is jointly continuous in the variable t on the interval $[T_1, T_2]$. Since the terms of the sequence $\left\{u^{\varepsilon_m}(x,t)e^{\lambda t}\right\}$ satisfy estimates (16), one can select a subsequence $\left\{u^{\varepsilon_{m_k}}(x,t)e^{\lambda t}\right\}$ of this sequence such that

$$u^{\varepsilon_{m_k}}(x,t)e^{\lambda t} \to u(x,t)e^{\lambda t} \quad \text{*-weakly in} \quad L^\infty((-\infty;T], V),$$

$$u^{\varepsilon_{m_k}}(x,t)e^{\lambda t} \to u(x,t)e^{\lambda t} \quad \text{weakly in} \quad L^\infty((-\infty;T], V),$$

$$u^{\varepsilon_{m_k}}(x,t)e^{\lambda t} \to u(x,t)e^{\lambda t} \quad \text{uniformly in} \quad C([T_1;T_2], V)$$

as $m_k \to \infty$.

Now consider the intervals $[T-k, T]$, $k \in \mathbb{N}$. According to the Arzelá–Ascoli theorem, we can construct a diagonal subsequence such that, for any $T_1 \in (-\infty; T]$,

$$u^{m,m}(x,t)e^{\lambda t} \to u(x,t)e^{\lambda t} \quad *\text{-weakly in } L^\infty((-\infty;T],V),$$

$$u^{m,m}(x,t)e^{\lambda t} \to u(x,t)e^{\lambda t} \quad \text{weakly in } L^\infty((-\infty;T],V),$$

$$u^{\varepsilon_{m_k}}(x,t)e^{\lambda t} \to u(x,t)e^{\lambda t} \quad \text{uniformly in } C\big([T_1;T_2],L^2(\Omega)\big)$$

as $m\to\infty$.

It is easy to show that, for any $t_1, t_2 \in (-\infty;T]$, $t_1 < t_2$, the functions $\{u^{m,m}(x,t)\}$ satisfy the inequality

$$\int_{Q_{t_1,t_2}} \Bigg(v_t(v-u^{m,m}) + \sum_{i,j=1}^n b_{ij} v_{x_i,t}(v_{x_j} - u^{m,m}_{x_j})$$

$$+ \sum_{i,j=1}^n a_{ij} u^{m,m}_{x_i}(v_{x_j} - u^{m,m}_{x_j}) + \frac{1}{2}\sum_{i,j=1}^n b_{ij,t}(v_{x_i} - u^{m,m}_{x_i})(v_{x_j} - u^{m,m}_{x_j})$$

$$+ \lambda \sum_{i,j=1}^n b_{ij}(v_{x_i} - u^{m,m}_{x_i})(v_{x_j} - u^{m,m}_{x_j}) + \sum_{i=1}^n c_i u^{m,m}_{x_i}(v-u^{m,m})$$

$$+ \lambda(v-u^{m,m})^2 + c_0 u^{m,m}(v-u^{m,m}) - f(v-u^{m,m}) \Bigg) e^{2\lambda t}\,dx\,dt$$

$$\geq \frac{1}{2}\int_\Omega \Bigg(\sum_{i,j=1}^n b_{ij}(x,t_2)\big(v_{x_i}(x,t_2) - u^{m,m}_{x_i}(x,t_2)\big)\big(v_{x_j}(x,t_2) - u^{m,m}_{x_j}(x,t_2)\big)$$

$$+ \big(v(x,t_2) - u^{m,m}(x,t_2)\big)^2 \Bigg) e^{2\lambda t_2}\,dx - \frac{1}{2}\int_\Omega \Bigg(\sum_{i,j=1}^n b_{ij}(x,t_1)$$

$$\times \big(v_{x_i}(x,t_1) - u^{m,m}_{x_i}(x,t_1)\big)\big(v_{x_j}(x,t_1) - u^{m,m}_{x_j}(x,t_1)\big) + \big(v(x,t_1) - u^{m,m}(x,t_1)\big)^2 \Bigg) e^{2\lambda t_1}\,dx, \quad (24)$$

where v is an arbitrary function such that $v, v_{x_i} \in W$, $i = 1, \ldots, n$, $v \in K$ for almost all $t \in (-\infty;T]$.

Passing to the limit as $m\to\infty$ in (24) and using estimates (16), we establish that $u(x,t)$ is a solution of inequality (1) in the sense of Definition 1 and satisfies condition (11). The theorem is proved.

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