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ON CERTAIN NONLINEAR PSEUDOPARABOLIC VARIATIONAL INEQUALITIES WITHOUT INITIAL CONDITIONS

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We consider a nonlinear pseudoparabolic variational inequality in a tube domain semibounded in variable t. Under certain conditions imposed on coefficients of the inequality, we prove the theorems of existence and uniqueness of a solution without any restriction on its behavior as $t \to -\infty$.

It is known that fluid filtration in media with double porosity [1], heat transfer in a heterogeneous medium [2], and moisture transfer in soil [3] are modelled by boundary-value problems for pseudoparabolic equations. The general theory of such equations and boundary conditions for them were a subject of investigations of many authors [3–15]. For example, problems without initial conditions for some pseudoparabolic systems were investigated in [14, 15].

Pseudoparabolic variational inequalities make it possible to obtain the conditions for correct solvability of some other boundary-value problems for pseudoparabolic equations.

In the present paper, we prove the correctness of a nonlinear pseudoparabolic variational inequality without initial conditions in the class of functions with arbitrary behavior as $t \to -\infty$.

Note that certain parabolic variational inequalities without initial conditions were investigated in [16–18]. Moreover, the conditions for unique solvability of a pseudoparabolic inequality generated by a linear pseudoparabolic operator were obtained in [19]. In this case, the behavior of a solution was restricted by the condition that it increase not faster than $e^{-\lambda t}$ as $t \to -\infty$, where λ is determined by coefficients of the inequality. The results presented in [19] cannot be obtained from the present paper.

Let Ω be a bounded region of the space $\mathbb{R}^n$ with the boundary $\partial\Omega \subset C^1$, $Q_T = \Omega \times (-\infty, T)$, $T < \infty$, $\Omega_{t_1, t_2} = \Omega \times (t_1, t_2)$, $t_1 < t_2 < T$, $\Omega_\tau = Q_T \cap \{t = \tau\}$, let V be a closed subspace continuously and compactly imbedded in $L^2(\Omega)$, $\mathring{H}^1(\Omega) \cap \mathring{W}^{1,p}(\Omega) \subset V \subset H^1(\Omega) \cap W^{1,p}(\Omega)$, $p > 2$, and let K be a closed convex subset in V which contains the zero element.

We define a norm in the space V as the norm of the space $H^1(\Omega) \cap W^{1,p}(\Omega)$.

Consider the following variational inequality:

$$\int\limits_{Q_{t_1,t_2}} \Bigg[v_t (v-u) + \sum_{i,j=1}^{n} b_{ij}(x,t)\, v_{x_i,t} (v_{x_j} - u_{x_j})$$

$$+ \sum_{i,j=1}^{n} a_{ij}(x,t)\, u_{x_i} (v_{x_j} - u_{x_j}) + \frac{1}{2} \sum_{i,j=1}^{n} b_{ij,t}(x,t)(v_{x_i} - u_{x_i})(v_{x_j} - u_{x_j})$$

$$+ \sum_{i=1}^{n} c_i(x,t)\, u_{x_i} (v-u) + c_0(x,t)\, u (v-u)$$

$$+ \sum_{i=1}^{n} \alpha_i(x) |u_{x_i}|^{p-2} u_{x_i} (v_{x_i} - u_{x_i}) + g(x)|u|^{p-2} u (v-u) - f_0(x,t)(v-u)$$

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$$- \sum_{i=1}^{n} f_i(x,t)(v_{x_i} - u_{x_i}) \Bigg] dx\,dt$$

$$\geq \frac{1}{2} \int_{Q_{t_2}} \left[\sum_{i,j=1}^{n} b_{ij}(x,t_2)(v_{x_i} - u_{x_i})(v_{x_j} - u_{x_j}) + (v-u)^2 \right] dx$$

$$- \frac{1}{2} \int_{Q_{t_1}} \left[\sum_{i,j=1}^{n} b_{ij}(x,t_1)(v_{x_i} - u_{x_i})(v_{x_j} - u_{x_j}) + (v-u)^2 \right] dx. \qquad (1)$$

Definition 1. *A solution of inequality (1) is a function* $u(x,t)$ *such that:* $u \in L^2_{\text{loc}}((-\infty,T]; H^1(\Omega))$, $u \in L^p_{\text{loc}}((-\infty,T]; W^{1,p}(\Omega))$ *and* $u_t \in L^2_{\text{loc}}((-\infty,T]; H^1(\Omega))$, $u \in K$ *for almost all* $t \in (-\infty, T]$, $u(x,t)$ *satisfies (1) for all* $t_1, t_2 \in (-\infty, T]$, $t_1 < t_2$, *and for all functions* $v(x,t)$ *such that* $v \in H^2_{\text{loc}}((-\infty,T]; H^1(\Omega)) \cap L^p_{\text{loc}}((-\infty,T]; W^{1,p}(\Omega))$ *and* $v \in K$ *for almost all* $t \in (-\infty, T]$.

Let the coefficients of inequality (1) satisfy, respectively, the following conditions:

condition A_1:

$$a_{ij} \in L^\infty(Q_T), \quad i,j = 1,\dots,n, \qquad \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq a_0 \sum_{i=1}^{n} \xi_i^2, \quad a_0 > 0,$$

for almost all $\xi \in \mathbb{R}^n$ and almost all $(x,t) \in Q_T$,

condition A_2:

$$\alpha_i \in L^\infty(\Omega), \quad \alpha_i(x) \geq \alpha_0 > 0, \quad i = 1,\dots,n,$$

condition B:

$$b_{ij}(x,t) = b_{ji}(x,t), \qquad b_{ij},\ b_{ijt} \in L^\infty(Q_T), \quad i,j = 1,\dots,n,$$

$$b_k \sum_{i=1}^{n} \xi_i^2 \leq \sum_{i,j=1}^{n} \frac{\partial^k b_{ij}(x,t)}{\partial t^k} \xi_i \xi_j \leq b^k \sum_{i=1}^{n} \xi_i^2,$$

$$b_0 > 0, \quad k = 0, 1; \quad b = \min\{b_1, -b^1\}$$

for all $\xi \in \mathbb{R}^n$ and almost all $(x,t) \in Q_T$,

condition C:

$$c_i \in L^\infty(Q_T), \quad i = 1,\dots,n, \qquad \sup_{Q_T} \sum_{i=1}^{n} c_i^2(x,t) = c^1,$$

$$c_0 \in L^2_{\text{loc}}((-\infty,T]; L^\infty(\Omega)), \quad c_0(x,t) \geq c^0 > 0, \quad (x,t) \in Q_T,$$

condition G:

$$g \in L^{\infty}(\Omega), \quad g(x) \geq g_0 > 0, \quad x \in \Omega.$$

Theorem 1. *Let the coefficients of inequality (1) satisfy conditions* A_1, A_2, B, C, *and* G, *and, moreover, let* $(4a_0 - 2b^1)c^0 > c^1$. *Then inequality (1) cannot have more than one solution.*

Proof. Define operators A and B_1 according to the following formulas:

$$\langle A w^1, w^2 \rangle(t) = \int_{\Omega} \Bigg(\sum_{i,j=1}^{n} a_{ij}(x,t)\, w^1_{x_i} w^2_{x_j} + \sum_{i=1}^{n} \alpha_i(x) |w^1_{x_i}|^{p-2} w^1_{x_i} w^2_{x_i}$$

$$+ \sum_{i=1}^{n} c_i(x,t)\, w^1_{x_i} w^2 + c_0(x,t)\, w^1 w^2 + g(x)|w^1|^{p-2} w^1 w^2 \Bigg) dx,$$

$$\langle B_1 w^1, w^2 \rangle = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} b_{ijt}(x,t)\, w^1_{x_i} w^2_{x_j}\, dx,$$

where w^1, w^2 are arbitrary functions from V.

It is easy to verify that, under the conditions of the theorem, the operator $A - B_1$ is uniformly monotone. Indeed,

$$\langle (A - B_1) w^1 - (A - B_1) w^2, w^1 - w^2 \rangle$$

$$\geq \int_{\Omega} \Bigg[\left(a_0 - \frac{b^1 + c^1 \delta_0}{2} \right) \sum_{i=1}^{n} |w^1_{x_i} - w^2_{x_i}|^2 + \left(c^0 - \frac{1}{2\delta_0} \right) |w^1 - w^2|^2$$

$$+ 2^{2-p} \alpha_0 \sum_{i=1}^{n} |w^1_{x_i} - w^2_{x_i}|^p + 2^{2-p} g_0 |w^1 - w^2|^p \Bigg] dx$$

$$\geq \beta_0 \int_{\Omega} \Bigg(\sum_{i,j=1}^{n} b_{ij}(x,t) |w^1_{x_i} - w^2_{x_i}| (w^1_{x_j} - w^2_{x_j}) + (w^1 - w^2)^2 \Bigg) dx, \tag{2}$$

where $\beta_0 = (n+1)^{(2-p)/2} 2^{2-p} \min\{\alpha_0, g_0\} \left(\min\left\{ 1, \frac{1}{b_0} \right\} \right)^{p/2}$.

Let $u^1(x,t)$ and $u^2(x,t)$ be two solutions of inequality (1). Then, for the function $v(x,t) = \frac{1}{2}(u^1(x,t) + u^2(x,t))$, the following inequalities are valid:

$$\int_{Q_{t_1,t_2}} \Bigg[(v_t - f^k)(v - u^k) + \sum_{i,j=1}^{n} b_{ij} v_{x_i,t} (v_{x_j} - u^k_{x_j}) + \frac{1}{2} \sum_{i,j=1}^{n} b_{ijt} (v_{x_i} - u^k_{x_i})(v_{x_j} - u^k_{x_j}) \Bigg] dx\, dt$$

$$\geq \frac{1}{2}\int\limits_{Q_{t_2}}\left[\sum_{i,j=1}^{n} b_{ij}(v_{x_i}-u^k_{x_i})(v_{x_j}-u^k_{x_j})+(v-u^k)^2\right]dx$$

$$-\frac{1}{2}\int\limits_{Q_{t_1}}\left[\sum_{i,j=1}^{n} b_{ij}(v_{x_i}-u^k_{x_i})(v_{x_j}-u^k_{x_j})+(v-u^k)^2\right]dx,$$

where

$$f^k = f_0 - \sum_{i=1}^{n} f_{i,x_i} - Au^k, \qquad k=1,2.$$

By summing these two inequalities, we obtain

$$\int\limits_{Q_{t_1,t_2}}\left[(f^1-f^2)(u^1-u^2)+\frac{1}{2}\sum_{i,j=1}^{n} b_{ijt}(u^1_{x_i}-u^2_{x_i})(u^1_{x_j}-u^2_{x_j})\right]dx\,dt$$

$$\geq \frac{1}{2}\int\limits_{Q_{t_2}}\left[\sum_{i,j=1}^{n} b_{ij}(u^1_{x_i}-u^2_{x_i})(u^1_{x_j}-u^2_{x_j})+(u^1-u^2)^2\right]dx$$

$$-\frac{1}{2}\int\limits_{Q_{t_1}}\left[\sum_{i,j=1}^{n} b_{ij}(u^1_{x_i}-u^2_{x_i})(u^1_{x_j}-u^2_{x_j})+(u^1-u^2)^2\right]dx. \tag{3}$$

In view of the expressions of the functions f^k, $k=1,2$, we can rewrite estimate (3) in the form

$$\frac{1}{2}\int\limits_{t_1}^{t_2}\frac{d}{dt}\left(\int\limits_{\Omega}\left(\sum_{i,j=1}^{n} b_{ij}(x,t)(u^1_{x_i}-u^2_{x_i})(u^1_{x_j}-u^2_{x_j})+(u^1-u^2)^2\right)dx\right)dt$$

$$+\int\limits_{t_1}^{t_2}\langle (A-B_1)u^1-(A-B_1)u^2,\,u^1-u^2\rangle\,dt \leq 0.$$

Hence, by using estimate (2), we obtain the inequality

$$\int\limits_{t_1}^{t_2}(y'(t)+\beta_1(y(t))^{p/2})\,dt \leq 0, \tag{4}$$

where

$$y(t) = \int\limits_{\Omega_t}\left[\sum_{i,j=1}^{n} b_{ij}(u^1_{x_i}-u^2_{x_i})(u^1_{x_j}-u^2_{x_j})+(u^1-u^2)^2\right]dx,$$

$$\beta_1 = \frac{2\beta_0}{(\text{mes}\,\Omega)^{(p-2)/2}}.$$

Since the numbers t_1 and t_2 are arbitrary, we obtain from estimate (4) the inequality

$$y'(t) + \beta_1 (y(t))^{p/2} \le 0$$

for almost all $t \in (-\infty, T]$.

Then, by virtue of Lemma 2 in [20], we have $y(t) = 0$ for almost all $t \in (-\infty, T]$, i.e., $u^1(x,t) = u^2(x,t)$ almost everywhere in Q_T. Theorem 1 is proved.

Theorem 2. *Let the coefficients of inequality (1) satisfy conditions* A_1, A_2, B, C, *and* G, *and, moreover, let* $a_{ij,t} \in L^\infty(Q_T)$, $i,j = 1,\dots,n$, $c_{i,t} \in L^\infty(Q_T)$, $i = 0, 1, \dots, n$. *Let there exist a positive number* γ *such that*

$$p_0 = 2a_0 - 2b^0\gamma - b > 0, \quad 2(g_0 - \gamma)p_0 > h_1,$$

$$\int_{Q_T} \sum_{i=0}^{n} (f_i^2(x,t) + f_{i,t}^2(x,t)) e^{2\gamma t}\, dx\, dt < \infty.$$

Then there exists a solution $u(x,t)$ *of inequality (1).*

Proof. Consider a sequence of functions $\{\varphi^i\}$ which possess the following properties: $\varphi^i \in W^{1,p}(\Omega)$, $i = 1, 2, \dots$, the functions $\varphi^1, \dots, \varphi^k$ are linearly independent for arbitrary k, and linear combinations of φ^i are dense in $W^{1,p}(\Omega)$.

Let

$$u^N(x,t) = \sum_{k=1}^{N} c_k^N(t)\, \varphi^k(x), \qquad N = 1, 2, \dots,$$

where $c_1^N, \dots, c_N^N$ is a solution of the Cauchy problem

$$\int_{\Omega_t} \Bigg(u_t^N \varphi^k + \sum_{i,j=1}^{n} b_{ij} u_{x_i,t}^N \varphi_{x_j}^k + \sum_{i,j=1}^{n} a_{ij} u_{x_i}^N \varphi_{x_j}^k + \sum_{i=1}^{n} c_i u_{x_i}^N \varphi^k + c_0 u^N \varphi^k$$

$$+ \sum_{i=1}^{n} \alpha_i |u_{x_i}^N|^{p-2} u_{x_i}^N \varphi_{x_i}^k + g|u^N|^{p-2} u^N \varphi^k - f_0^{t_0} \varphi^k - \sum_{i=1}^{n} f_i^{t_0} \varphi_{x_i}^k \Bigg) dx$$

$$+ \frac{1}{\varepsilon} \langle \mathcal{B}(u^N e^{\gamma t}), \varphi^k \rangle = 0, \quad t \in [t_0, T], \tag{5}$$

$$c_k^N(t_0) = 0, \quad k = 1, 2, \dots, N. \tag{6}$$

Here, $\varepsilon > 0$, $\mathcal{B}$ is the penalty operator [16, p. 384], $\mathcal{B}(u) = J(u - P_K u)$, J is the operator of duality between the spaces $H^1(\Omega)$ and $(H^1(\Omega))^*$, P is the operator of projection on the set K, and

$$f_i^{t_0}(x,t) = \begin{cases} f_i(x,t), & \text{if } (x,t)\in Q_{t_0,T}; \\ 0, & \text{if } (x,t)\in Q_{t_0}. \end{cases}$$

The existence of a solution of problem (5), (6) stems from the following *a priori* estimates. Continue the functions $c_k^N(t)$ by zero to the interval $(-\infty, t_0]$ and perform the substitution $u^N(x,t) = v^N(x,t)e^{-\gamma t}$ in system (5). Then $u_t^N(x,t) = v_t^N(x,t)e^{-\gamma t} - \gamma v_t^N(x,t)e^{-\gamma t}$, and problem (5), (6) acquires the form

$$\int_{\Omega_t}\Bigg[v_t^N\varphi^k + \sum_{i,j=1}^n b_{ij}v_{x_i,t}^N\varphi_{x_j}^k + \sum_{i,j=1}^n (a_{ij}-\gamma b_{ij})v_{x_i}^N\varphi_{x_j}^k + \sum_{i=1}^n c_i v_{x_i}^N\varphi^k + (c_0-\gamma)v^N\varphi^k + e^{-\gamma(p-2)t}\sum_{i=1}^n \alpha_i|v_{x_i}^N|^{p-2}v_{x_i}^N\varphi_{x_i}^k + e^{-\gamma(p-2)t}g|v^N|^{p-2}v^N\varphi^k - \left(f_0^{t_0}\varphi^k + \sum_{i=1}^n f_i^{t_0}\varphi_{x_i}^k\right)e^{\gamma t}\Bigg]dx + \frac{1}{\varepsilon}\langle\mathcal{B}(v^N),\varphi^k\rangle = 0, \tag{7}$$

$$v^N(t_0) = 0. \tag{8}$$

Multiplying each equation of system (7), respectively, by the function $c_k^N(t)e^{\gamma t}$, summing them over the index k from 1 to N, and integrating over the segment $[t_1,\tau]\subset[t_0,T]$, we obtain

$$\int_{\Omega_{t_1,\tau}}\Bigg[v_t^N v^N + \sum_{i,j=1}^n b_{ij}v_{x_i,t}^N v_{x_j}^N + \sum_{i,j=1}^n (a_{ij}-\gamma b_{ij})v_{x_i}^N v_{x_j}^N + \sum_{i=1}^n c_i v_{x_i}^N v^N + (c_0-\gamma)(v^N)^2 + e^{-\gamma(p-2)t}\left(\sum_{i=1}^n \alpha_i|v_{x_i}^N|^p + g|v^N|^p\right) - \left(f_0^{t_k}v^N + \sum_{i=1}^n f_i^{t_0}v_{x_i}^N\right)e^{\gamma t}\Bigg]dx\,dt + \frac{1}{\varepsilon}\int_{t_1}^{T}\langle\mathcal{B}(v^N),v^N\rangle\,dt = 0. \tag{9}$$

On the basis of the conditions of the theorem, one easily obtains the following estimates from equality (9):

$$\int_{\Omega_\tau}\left(|v^N|^2 + \sum_{i=1}^n |v_{x_i}^N|^2\right)dx \le \mu_1 F_{0,\gamma}, \tag{10}$$

$$\int_{Q_{t_1,\tau}}\left(|v^N|^2 + \sum_{i=1}^n |v_{x_i}^N|^2\right)dx\,dt \le \mu_1 F_{0,\gamma}, \tag{11}$$

$$\int_{Q_{t_1,\tau}} e^{-\gamma(p-2)t}\left(|v^N|^p + \sum_{i=1}^n |v_{x_i}^N|^{p-2}\right)dx\,dt \le \mu_1 F_{0,\gamma}, \tag{12}$$

$$\int_{t_1}^{T} \langle B(v^N), v^N \rangle \, dt \;\le\; \mu_1 \varepsilon F_{0,\gamma}, \tag{13}$$

where $\tau \in [t_1, T]$, the constant μ_1 is independent of ε, n, and t_1, and

$$F_{0,\gamma} = \int_{Q_T} \sum_{i=0}^{n} |f_i(x,t)|^2 e^{2\gamma t} \, dx\, dt.$$

We differentiate equation (7) with respect to the variable t, multiply each equation, respectively, by the function $(c_{k,t}^N(t) + \gamma c_k^N(t))e^{\gamma t}$, then sum them over k from 1 to N and integrate over the segment $[t_1, \tau]$. As a result, we obtain the equality

$$\begin{aligned}
&\int_{Q_{t_1,\tau}} \Bigg[v_{tt}^N v_t^N + \sum_{i,j=1}^{n} b_{ij} v_{x_i,tt}^N v_{x_j,t}^N + \sum_{i,j=1}^{n} (a_{ij} - \gamma b_{ij} + b_{ij,t}) v_{x_i,t}^N v_{x_j,t}^N \\
&\quad + \sum_{i=1}^{n} c_i v_{x_i,t}^N v_t^N + (c_0 - \gamma)(v_t^N)^2 \Bigg] dx\, dt \\
&\quad + (p-1) \int_{Q_{t_1,\tau}} e^{-\gamma(p-2)t} \left(\sum_{i=1}^{n} \alpha_i |v_{x_i}^N|^{p-2} (v_{x_i,t}^N)^2 + g |v^N|^{p-2} (v_t^N)^2 \right) dx\, dt \\
&\quad - \gamma(p-2) \int_{Q_{t_1,\tau}} e^{-\gamma(p-2)t} \left[\sum_{i=1}^{n} \alpha_i |v_{x_i}^N|^{p-2} v_{x_i}^N v_{x_i,t}^N + g |v^N|^{p-2} v^N v_t^N \right] dx\, dt \\
&\quad + \int_{Q_{t_1,\tau}} \Bigg[\sum_{i,j=1}^{n} (a_{ij,t} - \gamma b_{ij,t}) v_{x_i}^N v_{x_j,t}^N + \sum_{i=1}^{n} c_{i,t} v_{x_i}^N v_t^N + c_{0,t} v^N v_t^N \\
&\quad - \left((f_{0,t}^{t_0} + \gamma f_0^{t_0}) v_t^N + \sum_{i=1}^{n} (f_{i,t}^{t_0} + \gamma f_i^{t_0}) v_{x_i,t}^N \right) e^{\gamma t} \Bigg] dx\, dt + \frac{1}{\varepsilon} \int_{t_1}^{T} \langle \mathcal{B}_t(v^N), v_t^N \rangle \, dt = 0.
\end{aligned} \tag{14}$$

Taking into account the inequality $\langle \mathcal{B}_t(v^N), v_t^N \rangle \ge 0$ [16, p. 413], conditions of the theorem, and estimates (10)–(12), we easily obtain the inequality

$$\begin{aligned}
&\int_{\Omega_\tau} \left(|v_t^N|^2 + \sum_{i=1}^{n} |v_{x_i,t}^N|^2 \right) dx + \int_{Q_{t_1,\tau}} \Bigg[\left(a_0 - \gamma b^0 + \frac{1}{2} b_1 - \frac{1}{2}\delta_0 c^1 - \delta_1 \right) \sum_{i=1}^{n} |v_{x_i,t}^N|^2 + \\
&\qquad + \left(c_0 - \gamma - \frac{1}{2\delta_0} - \delta_2 \right) |v_t^N|^2 \Bigg] dx\, dt \\
&\le \mu_2 e^{-\gamma(p-2)\tau} \int_{\Omega_\tau} \left(|v^N|^p + \sum_{i=1}^{n} |v_{x_i}^N|^p \right) dx + \mu_2 F_{0,\gamma} + \mu_2 F_{1,\gamma}
\end{aligned} \tag{15}$$

from equality (14). In (15), the constant μ_2 is independent of ε, N, and t_1, $\delta_0 > 0$, $\delta_1 > 0$, $\delta_2 > 0$, and

$$F_{1,\gamma} = \int\limits_{\Omega_T} \sum_{i=1}^{n} |f_{i,t}(x,t)|^2 e^{2\gamma t}\, dx\, dt.$$

On the basis of estimate (12) and the Fatou lemma, we have

$$\int\limits_{t_1}^{T} \liminf \|v^N\|^p_{W^{1,p}(\Omega)} e^{-\gamma(p-2)t}\, dt \le \mu_1 F_{0,\gamma}.$$

Hence,

$$e^{-\gamma(p-2)t} \liminf \|v^N\|^p_{W^{1,p}(\Omega)} < \infty$$

for almost all $t \in [t_1, \tau]$. Then there exists $\bar{\tau} \in [T-1, T]$ such that the specified lower boundary is finite for $\tau = \bar{\tau}$. By replacing τ by $\bar{\tau}$ and passing, if necessary, to a subsequence, we can consider that

$$e^{-\gamma(p-2)\bar{\tau}} \|v^N(\bar{\tau})\|^p_{W^{1,p}(\Omega)} < \mu_3, \tag{16}$$

where the constant μ_3 is independent of ε, N, and t_1.

On the basis of the conditions of the theorem, one can choose the numbers δ_0, δ_1, and δ_2 such that inequality (15) will imply [in view of estimate (16)] the estimate

$$\int\limits_{Q_{t_1,\tau}} \left(|v_t^N|^2 + \sum_{i=1}^{n} |v_{x_i,t}^N|^2 \right) dx\, dt \le \mu_4 (F_{0,\gamma} + F_{1,\gamma}), \tag{17}$$

where the constant μ_4 is independent of ε, N, and t_1.

Taking into account estimates (10)–(12), (17) and the monotonicity of the operators A_0, $\mathcal{B}$, where the operator A_0 is defined by the formula

$$\langle A_0 w^1, w^2 \rangle = \int\limits_{\Omega_t} \left(\sum_{i=1}^{n} \alpha_i |w^1_{x_i}|^{p-2} w^1_{x_i} w^2_{x_i} + g |w^1|^{p-2} w^1 w^2 \right) dx,$$

$$w^1,\ w^2 \in W^{1,p}(\Omega),$$

we can assert the existence of a boundary point $v^{t_0}(x,t)$ of the sequence $\{v^N(x,t)\}$ which satisfies the equality

$$\int\limits_{\Omega_{t_1,\tau}} \left[v_t^{t_0} w + \sum_{i,j=1}^{n} b_{ij} v^{t_0}_{x_i,t} w_{x_j} + \sum_{i,j=1}^{n} (a_{ij} - \gamma b_{ij}) v^{t_0}_{x_i} w_{x_j} + \sum_{i=1}^{n} c_i v^{t_0}_{x_i} w + (c_0 - \gamma) v^{t_0} w \right] dx\, dt$$

$$+ \int\limits_{t_1}^{\tau} \left(e^{-\gamma(p-2)t} \langle A_0 v^{t_0}, w \rangle + \frac{1}{\varepsilon} \langle \mathcal{B}(v^{t_0}), w \rangle \right) dt$$

$$= \int\limits_{Q_{t_1,\tau}} \left(f_0 w + \sum_{i=1}^{n} f_i w_{x_i} \right) e^{\gamma t}\, dx\, dt \tag{18}$$

for an arbitrary function $w \in L^p_{\text{loc}}((-\infty, T]; W^{1,p}(\Omega))$, where t_1 is an arbitrary number from $[t_0, T]$.

Moreover, estimates (10)–(12) and (17) are valid for the function $v^{t_0}(x,t)$. If one successively sets $t_0 = T - 1$, $t_0 = T-2, \ldots$, $t_0 = T-k, \ldots$, then one obtains a new sequence of functions $\{v^k(x,t)\}$, each of which is a solution of Eq. (18) and satisfies estimates (10)–(12), (17). Therefore, the given sequence also has a boundary point $v^{\varepsilon}(x,t)$ that satisfies Eq. (18) and estimates (10)–(12), (17). Thus, one can select a subsequence $\{v^{\varepsilon_k}(x,t)\} \subset \{v^{\varepsilon}(x,t)\}$ such that

$$v^{\varepsilon_k} \to v \quad \text{weakly in} \quad L^p((t_1,t_2), W^{1,p}(\Omega)),$$

$$v^{\varepsilon_k} \to v \quad \text{weakly in} \quad L^2((t_1,t_2), H^1(\Omega)),$$

$$v_t^{\varepsilon_k} \to v_t \quad \text{weakly in} \quad L^2((t_1,t_2), H^1(\Omega)),$$

$$v^{\varepsilon_k} \to v \quad \text{uniformly in} \quad C([t_1,t_2], H^1(\Omega))$$

as $\varepsilon \to 0$ for arbitrary $t_1, t_2 \in (-\infty, T]$, $t_1 < t_2$.

Taking into account the monotonicity of the operators A_0, $\mathcal{B}$, we have

$$A_0 v^{\varepsilon_k} \to A_0 v \quad \text{weakly in} \quad L^p((t_1,t_2), (W^{1,p}(\Omega))^*),$$

$$\mathcal{B}(v^{\varepsilon_k}) \to \mathcal{B}(v) \quad \text{weakly in} \quad L^2((t_1,t_2), (H^1(\Omega))^*)$$

as $\varepsilon \to 0$. By using equality (18), which is satisfied by the functions $v^{\varepsilon_k}(x,t)$ for $\tau = t_2$, we obtain

$$\mathcal{B}(v^{\varepsilon_k}) \to 0 \quad \text{weakly in} \quad L^2((t_1,t_2), (H^1(\Omega))^*).$$

Hence, $\mathcal{B}(v) = 0$, i.e., $v \in K$ for almost all $t \in (-\infty, T]$. Now consider equality (18) for the functions $v^{\varepsilon_k}(x,t)$ and $\tau = t_2$ by setting $w = (z - u^k)e^{-\gamma t}$, $u^k = v^{\varepsilon_k} e^{-\gamma t}$, where $t \in K$ for almost all $t \in (-\infty, T]$, $z \in H^1_{\text{loc}}((-\infty, T]; H^1(\Omega)) \cup L^p_{\text{loc}}((-\infty, T]; W^{1,p}(\Omega))$:

$$\int\limits_{Q_{t_1,t_2}} \Bigg[u_t^k(z-u^k) + \sum_{i,j=1}^n b_{ij} u^k_{x_i,t}(z_{x_j} - u^k_{x_j}) + \sum_{i,j=1}^n a_{ij} u^k_{x_i}(z_{x_j} - u^k_{x_j})$$

$$+ \sum_{i=1}^n \alpha_i |u^k_{x_i}|^{p-2} u^k_{x_i}(z_{x_i} - u^k_{x_i}) + \sum_{i=1}^n c_i u^k_{x_i}(z-u) + c_0 u^k(z-u^k)$$

$$+ g|u^k|^{p-2}u^k(z-u^k) - f_0(z-u^k) - \sum_{i=1}^n f_i(z_{x_i} - u^k_{x_i}) \Bigg] dx\,dt$$

$$= \frac{1}{2}\int\limits_{t_1}^{t_2} \langle B(ze^{-\gamma t}) - B(u^k e^{-\gamma t}), z - u^k \rangle \, dt \geq 0. \tag{19}$$

After elementary transformations of the integral

$$\int_{\Omega_{t_1,t_2}} \left[u_t^k (z - u^k) + \sum_{i,j=1}^{n} b_{ij} u_{x_i,t}^k (z_{x_j} - u_{x_j}^k) \right] dx\,dt,$$

inequality (19) takes the form

$$\int_{\Omega_{t_1,t_2}} \Bigg[z_t (z - u^k) + \frac{1}{2} \sum_{i,j=1}^{n} b_{ij,t} (z_{x_i} - u_{x_i}^k)(z_{x_j} - u_{x_j}^k)$$

$$+ \sum_{i,j=1}^{n} b_{ij} z_{x_i,t} (z_{x_j} - u_{x_j}^k) + \sum_{i,j=1}^{n} a_{ij} u_{x_i}^k (z_{x_j} - u_{x_j}^k)$$

$$+ \sum_{i=1}^{n} \alpha_i |u_{x_i}^k|^{p-2} u_{x_i}^k (z_{x_i} - u_{x_i}^k) + \sum_{i=1}^{n} c_i u_{x_i}^k (z - u^k) + c_0 u^k (z - u^k)$$

$$+ g |u^k|^{p-2} u^k (z - u^k) - f_0 (z - u^k) - \sum_{i=1}^{n} f_i (z_{x_i} - u_{x_i}^k) \Bigg] dx\,dt$$

$$\geq \frac{1}{2} \int_{Q_{t_2}} \left[\sum_{i,j=1}^{n} b_{ij} (z_{x_i} - u_{x_i}^k)(z_{x_j} - u_{x_j}^k) + |z - u^k|^2 \right] dx$$

$$- \frac{1}{2} \int_{Q_{t_1}} \left[\sum_{i,j=1}^{n} b_{ij} (z_{x_i} - u_{x_i}^k)(z_{x_j} - u_{x_j}^k) + |z - u^k|^2 \right] dx. \qquad (20)$$

By setting $z = u$ in (20), we obtain strong convergence of the sequence $\{u^k(x,t)\}$ to the function $u(x,t)$ in the space $W^{1,p}(\Omega)$. Therefore, one can pass to the limit in inequality (20) as $k \to \infty$. In this case, we obtain inequality (1), i.e., the function $u(x,t)$ is the required one. Theorem 2 is proved.

REFERENCES

1. G. I. Barenblat, Yu. P. Zheltov, and I. N. Kochina, "On basic representations of the theory of filtration of homogeneous fluids in cracked rocks," *Prikl. Mat. Mekh.*, **24**, Issue 3, 852–864 (1960).
2. L. I. Rubinshtein, "On the process of heat transfer in heterogeneous media," *Izv. Akad. Nauk SSSR. Ser. Geograf. Geofiz.*, **12**, No. 1, 27–45 (1948).
3. A. F. Chudnovskii, *Thermal Physics of Soils* [in Russian], Nauka, Moscow (1976).
4. M. Majchrovski, "On inverse problems with nonlocal conditions for parabolic systems of partial differential equations and pseudoparabolic equations," *Demonstr. Math.*, **26**, No. 1, 255–275 (1993).
5. T. W. Ting, "Parabolic and pseudoparabolic partial differential equations," *J. Math. Soc. Jpn.*, **21**, No. 3, 440–453 (1969).
6. H. Gajewski, K. Gröger, and K. Zacharias, *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*, Akademie-Verlag, Berlin (1974).
7. V. R. Gopala Rao and T. W. Ting, "Initial-boundary value problems for pseudoparabolic partial differential equations," *Indiana Univ. Math. J.*, **23**, No. 2, 131–153 (1973).
8. R. E. Showalter, "Partial differential equations of Sobolev-Galpern type," *Pacif. J. Math.*, **31**, No. 3, 787–793 (1969).

9. W. Rundel, "The solution of initial-boundary value problem for pseudoparabolic partial differential equations," *Proc. Roy. Soc. Edinburgh,* **A 74**, 311–326 (1976).
10. D. Colton, "Pseudoparabolic equations in one space variable," *J. Different. Equat.,* **12**, No. 3, 559–565 (1972).
11. I. V. Suveika, "Mixed problems for a nonstationary equation," *Mat. Issled.,* No. 58, 99–123 (1980).
12. M. Kh. Shkhanukov, "On some boundary-value problems for third-order equations arising on the modelling of fluid filtration in porous media," *Differents. Uravn.,* **18**, No. 10, 689–699 (1982).
13. J. P. Cannon and Lin Jamping, "Classical and weak solution for one-dimensional pseudoparabolic equations with typical boundary data," *Ann. Mat. Pure Appl.,* No. 152, 375–389 (1988).
14. M. O. Bas and S. P. Lavrenyuk, "On the uniqueness of a solution of the Fourier problem for a system of Sobolev–Galpern type," *Ukr. Mat. Zh.,* **48**, No. 1, 124–128 (1996).
15. M. O. Bas and S. P. Lavrenyuk, *The Fourier Problem for a Nonlinear Pseudoparabolic System* [in Russian], Dep. at GPNTB of Ukraine No. 2017-Uk.95, Kiev (1995).
16. J.-L. Lions, *Quelques Méthodes de Resolution des Problémes aux Limites Non Linéaires* [Russian translation], Mir, Moscow (1972).
17. A. A. Pankov, *Bounded and Almost Periodic Solutions of Nonlinear Differential–Operator Equations* [in Russian], Naukova Dumka, Kiev (1985).
18. S. P. Lavrenyuk, "Parabolic variational inequalities without initial conditions," *Differents. Uravn.,* **32**, No. 10, 1–5 (1996).
19. S. P. Lavrenyuk and M. B. Ptashnyk, "Pseudoparabolic variational inequalities without initial conditions," Ukr. Mat. Zh., **50**, No. 7, 919–929 (1998).
20. N. M. Bokalo, "On the problem without initial conditions for some classes of nonlinear parabolic equations," *Tr. Sem. Petrovskogo,* Issue 14, 3–44 (1989).