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LANDAU–LIFSHITZ–SLONCZEWSKI EQUATIONS: GLOBAL WEAK AND CLASSICAL SOLUTIONS*

CHRISTOF MELCHER† AND MARIYA PTASHNYK‡

Abstract. We consider magnetization dynamics under the influence of a spin-polarized current, given in terms of a spin-velocity field $\boldsymbol{v}$, governed by the following modification of the Landau–Lifshitz–Gilbert equation $\frac{\partial \boldsymbol{m}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{m} = \boldsymbol{m} \times (\alpha \frac{\partial \boldsymbol{m}}{\partial t} + \beta\, \boldsymbol{v} \cdot \nabla \boldsymbol{m} - \Delta \boldsymbol{m})$, called the Landau–Lifshitz–Slonczewski equation. We focus on the situation of magnetizations defined on the entire Euclidean space $\boldsymbol{m}(t) : \mathbb{R}^3 \to \mathbb{S}^2$. Our construction of global weak solutions relies on a discrete lattice approximation in the spirit of Slonczevski's spin-transfer-torque model and provides a rigorous justification of the continuous model. Using the method of moving frames, we show global existence and uniqueness of classical solutions under smallness conditions on the initial data in terms of the $\dot{W}^{1,3}$ norm and on the spin velocity $\boldsymbol{v}$ in terms of weighted-in-time Lebesgue norms, both optimal with respect to the natural scaling of the equation.

Key words. Landau–Lifshitz–Gilbert equations, spin-transfer torque, continuum limit of lattice approximations, moving-frame method

AMS subject classifications. 35K45, 35Q60, 35Q56, 35K55, 35D30

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1. Introduction.

1.1. The Landau–Lifshitz–Slonczewski equation. Spin-polarized currents interact with ferromagnetic structures via the so-called spin-transfer torque. This effect can be used to excite magnetization oscillations or to switch magnetization orientation, and it gave rise to a number of new trends for future storage technologies and spintronic applications. A well-accepted continuum equation for the magnetization

$$\boldsymbol{m} = \boldsymbol{m}(x,t) : \mathbb{R}^3 \times (0,\infty) \to \mathbb{S}^2 \quad \text{(unit sphere in } \mathbb{R}^3\text{)}$$

in the presence of a spin-polarized current, represented by a spin-velocity field

$$\boldsymbol{v} : \mathbb{R}^3 \times (0,\infty) \to \mathbb{R}^3,$$

is the following modification of the classical Landau–Lifshitz–Gilbert equation

$$\frac{\partial \boldsymbol{m}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{m} = \boldsymbol{m} \times \left(\alpha \frac{\partial \boldsymbol{m}}{\partial t} + \beta\, \boldsymbol{v} \cdot \nabla \boldsymbol{m} - \gamma\, \boldsymbol{h}_{\text{eff}} \right); \tag{1}$$

see, e.g., [40, 41, 49] and [15, 25]. Here, the parameters $\alpha > 0$ and $\gamma > 0$ are, as usual, the Gilbert damping and gyromagnetic constant, respectively, while $\beta > 0$ is a new parameter, the so-called ratio of nonadiabaticity of spin transfer which we shall explain below. The constant γ is a frequency scale that we shall set to $\gamma = 1$ upon rescaling time and renormalizing $\boldsymbol{v}$. Finally, $\boldsymbol{h}_{\text{eff}} = -\nabla E(\boldsymbol{m})$ is the effective field

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†Lehrstuhl I für Mathematik & JARA - Fundamentals of Future Information Technology, RWTH Aachen University, 52056 Aachen, Germany (melcher@rwth-aachen.de).
‡Division of Mathematics, University of Dundee, Dundee DD1 4HN, Scotland, UK (mptashnyk@maths.dundee.ac.uk).

given by minus the L^2 gradient of the underlying interaction energy $E = E(\boldsymbol{m})$. Note that the effective field corresponding to the Dirichlet energy

$$E(\boldsymbol{m}) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \boldsymbol{m}|^2 dx,$$

as a model for exchange interaction, see [17], is $\boldsymbol{h}_{\text{eff}} = \Delta \boldsymbol{m}$. In the context of micromagnetics, the magnetization is usually defined on a finite domain and satisfies homogeneous Neumann boundary conditions. Moreover, the effective field includes further components of lower order stemming from demagnetization, crystalline anisotropy, and external fields; see, e.g., [17, 45]. In this paper we shall only consider the exchange interaction, i.e., after rescaling space if necessary, $\boldsymbol{h}_{\text{eff}} = \Delta \boldsymbol{m}$.

Existence and regularity questions for the classical Landau–Lifshitz–Gilbert equation (LLG) without spin-torque interaction (i.e., $\boldsymbol{v} = 0$) have been the subject of intense research; see, e.g., [1, 9, 16, 28, 31, 45, 46]. A particularly interesting and challenging feature is the combination of dissipative and symplectic ingredients. In the case $\boldsymbol{h}_{\text{eff}} = \Delta \boldsymbol{m}$, LLG can be understood as a hybrid of heat and Schrödinger flow for harmonic maps. It has been shown that LLG inherits many analytical features of the harmonic map heat flow equation (HMHF), at least in small space dimensions ($d \leq 4$); see [28, 31, 46]. However, unlike the HMHF, which is semilinear, LLG has the structure of a quasilinear parabolic system. The natural a priori estimate, carrying over from the HMHF, is the energy inequality

$$\alpha \int_0^T \left\| \frac{\partial \boldsymbol{m}}{\partial t} \right\|_{L^2}^2 dt + E(\boldsymbol{m}(T)) \leq E(\boldsymbol{m}(0)) \quad \text{for all} \quad T > 0,$$

which is used in the construction and is valid for suitable weak solutions $\boldsymbol{m}$ in the energy space $L_t^\infty \dot{H}_x^1 \cap \dot{H}_t^1 L_x^2$. In dimensions $d = 3$ and higher, LLG is supercritical with respect to this energy law, i.e., concentrations cannot be ruled out by uniform energy bounds, as suggested by a simple scaling argument. In fact, finite time singularities have been shown to occur for smooth initial data of arbitrarily small initial energy; see [12]. Global regularity requires a control of adapted quantities, such as the *scaling-invariant* $\dot{W}^{1,d}$ norms (with $d \geq 3$ the space dimension) used in [30, 47]. As $\alpha \searrow 0$, the parabolicity of LLG degenerates, and solutions converge weakly to so-called Schrödinger maps; see [1, 34]. Striking results on the well-posedness of Schrödingier flows in spaces $\dot{H}^{d/2}$ have recently been obtained in [6, 5]. Another field of intense mathematical research is special solutions of LLG in connection with magnetic pattern formation, in particular, the gyrotropic motion of magnetic domain walls and vortices in thin film structures; see, e.g., [8, 10, 23, 27, 29, 43].

In the presence of spin-polarized currents $\boldsymbol{v} \neq 0$, (1) has been proposed and considered in connection with current-induced domain-wall motion [40, 48, 49] and magnetic vortex excitation [2, 20, 22, 24]. Observe that (1) is obtained from the original LLG, written in Gilbert form (see [15, 17]) by adding two nonvariational terms

$$\boldsymbol{v} \cdot \nabla \boldsymbol{m} \quad \text{and} \quad \beta\, \boldsymbol{m} \times (\boldsymbol{v} \cdot \nabla \boldsymbol{m})\,.$$

These terms are referred to as the adiabatic and nonadiabatic spin-transfer torque, respectively. After some simple algebra, using the fact $-\boldsymbol{m} \times \boldsymbol{m} \times \xi$ is the orthogonal projection of $\xi \in \mathbb{R}^3$ on the tangent space at $\boldsymbol{m}$, and writing

$$\nabla_{\boldsymbol{v}} \boldsymbol{m} = (\boldsymbol{v} \cdot \nabla) \boldsymbol{m},$$

we obtain the Landau–Lifshitz form of (1), i.e.,

$$(2)\quad \begin{aligned}(1+\alpha^2)\,\frac{\partial \boldsymbol{m}}{\partial t} + (1+\alpha\beta)\nabla_{\boldsymbol{v}}\boldsymbol{m} + (\alpha-\beta)\boldsymbol{m}\times\nabla_{\boldsymbol{v}}\boldsymbol{m} \\ = -\boldsymbol{m}\times\Big(\Delta\boldsymbol{m} + \alpha\,\boldsymbol{m}\times\Delta\boldsymbol{m}\Big).\end{aligned}$$

Rescaling time by $1/(1+\alpha^2)$ and defining individual velocity fields

$$(3)\quad \boldsymbol{v}_1 = (1+\alpha\beta)\boldsymbol{v} \quad \text{and} \quad \boldsymbol{v}_2 = (\alpha-\beta)\boldsymbol{v},$$

(2) can conveniently be written as

$$(4)\quad \frac{\partial \boldsymbol{m}}{\partial t} + \nabla_{\boldsymbol{v}_1}\boldsymbol{m} + \boldsymbol{m}\times\nabla_{\boldsymbol{v}_2}\boldsymbol{m} = -\,\boldsymbol{m}\times\Delta\boldsymbol{m} - \alpha\,\boldsymbol{m}\times\boldsymbol{m}\times\Delta\boldsymbol{m}.$$

The case $\alpha=\beta$, originally suggested in [4, 3] on thermodynamic grounds, is special. In this case we have $\boldsymbol{v}_2 = 0$, and after redefining $\boldsymbol{v} = \boldsymbol{v}_1$ and introducing the particle derivative

$$D_{\boldsymbol{v}}\boldsymbol{m} = \frac{\partial \boldsymbol{m}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{m},$$

(4) exhibits the particularly simple form

$$(5)\quad D_{\boldsymbol{v}}\boldsymbol{m} = -\boldsymbol{m}\times\Delta\boldsymbol{m} - \alpha\,\boldsymbol{m}\times\boldsymbol{m}\times\Delta\boldsymbol{m}.$$

In particular, (5) is obtained from the original Landau–Lifshitz equation by adding a single term $(\boldsymbol{v}\cdot\nabla)\boldsymbol{m}$ that gives rise to an adiabatic term and a nonadiabatic term when written in Gilbert form. As it turns out, the convective term in (5) can be understood as a continuum limit of discrete spin-transfer terms. But in order to obtain a nonadiabatic contribution as in (1), it clearly matters to which form of the LLG (Gilbert or Landau–Lifshitz form) it is added or, in other words, how internal damping is expressed.

1.2. Slonczewski's spin-transfer torque. The spin-transfer terms (at least the adiabatic one) in (1) are typically interpreted as homogenized versions of Slonczewski's spin-transfer torque (STT); see, e.g., [20, 40]. The term refers to fundamental work by Slonczewski [36] and Berger [7] on spin transfer between two homogeneously magnetized layers separated by paramagnetic spacers within a five-layer system. If no additional interaction or damping mechanisms are active, Slonczewski's semiclassical approach has led, in suitable units, to a dynamical system

$$(6)\quad \begin{cases} M\,\dot{\boldsymbol{m}}_1 = I\,g\;\boldsymbol{m}_1\times(\boldsymbol{m}_1\times\boldsymbol{m}_2), \\ M\,\dot{\boldsymbol{m}}_2 = I\,g\;\boldsymbol{m}_2\times(\boldsymbol{m}_1\times\boldsymbol{m}_2) \end{cases}$$

in terms of two director fields $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ that represent the homogeneous spins of the two ferromagnetic layers. The scalars M and I are the magnitude of magnetic moment and the current density (per unit charge), respectively. The function g is given by

$$g(\boldsymbol{m}_1\cdot\boldsymbol{m}_2) = \left[\frac{(1+P)^3(3+\boldsymbol{m}_1\cdot\boldsymbol{m}_2)}{4P^{\frac{3}{2}}} - 4\right]^{-1},$$

where $P \in (0,1)$ is called the spin-polarization factor. Extending Slonczewski's model to an infinite spin chain $\{\boldsymbol{m}_k\}_{k\in\mathbb{Z}}$ of lattice spacing $h = M$ equal to the magnitude of magnetic moments gives rise to an infinite system such that

$$\dot{\boldsymbol{m}}_k = \pm \frac{I\, g(\boldsymbol{m}_k \cdot \boldsymbol{m}_{k\pm1})}{h}\, \boldsymbol{m}_k \times (\boldsymbol{m}_k \times \boldsymbol{m}_{k\pm1}) \tag{7}$$

for all $k \in \mathbb{Z}$. Hence, writing $g_k = g(\boldsymbol{m}_k \cdot \boldsymbol{m}_{k+1})$, we obtain

$$\begin{aligned}\dot{\boldsymbol{m}}_k = I g_k\, \boldsymbol{m}_k \times \boldsymbol{m}_k \times \frac{(\boldsymbol{m}_{k+1} - \boldsymbol{m}_{k-1})}{2h}\\ + \frac{I}{2h}\,(g_k - g_{k-1})\;\boldsymbol{m}_k \times (\boldsymbol{m}_k \times \boldsymbol{m}_{k-1})\,.\end{aligned}$$

Observe that the first term is the symmetric difference quotient at $\boldsymbol{m}_k$. By antisymmetry of the cross product, the absolute value of the second term on the right is bounded by

$$\frac{IL}{2h}\, |\boldsymbol{m}_k - \boldsymbol{m}_{k+1}|\, |\boldsymbol{m}_k - \boldsymbol{m}_{k-1}|,$$

where L is the Lipschitz constant of g. Including exchange-interaction-preferring alignment of neighboring spins, we expect in an appropriate scaling regime $h \ll 1$

$$\boldsymbol{m}_k \cdot \boldsymbol{m}_{k+1} = 1 + \mathcal{O}(h) \quad \text{hence} \quad g\,(\boldsymbol{m}_k \cdot \boldsymbol{m}_{k+1}) = g(1) + \mathcal{O}(h),$$

and we are led to define the modified Slonczewski spin-transfer term

$$\boldsymbol{\tau}_h = I g_k\, \boldsymbol{m}_k \times \boldsymbol{m}_k \times \frac{(\boldsymbol{m}_{k+1} - \boldsymbol{m}_{k-1})}{2h}. \tag{8}$$

In a discrete-to-continuum limit where $\{\boldsymbol{m}_k\}_{k\in\mathbb{Z}}$ approximates $\boldsymbol{m}$, we expect

$$\boldsymbol{\tau}_h \to v\, \boldsymbol{m} \times \boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial x} = -v\, \frac{\partial \boldsymbol{m}}{\partial x} = \boldsymbol{\tau}$$

as $h \searrow 0$ with a (macroscopic) spin velocity $v = I\, g(1)$, which is the one-dimensional version of the spin-transfer term in (5).

1.3. Outline and results. In section 2 we shall consider Slonczewski's STT model extended to a cubic lattice of fixed lattice size h. For spin velocities $\boldsymbol{v}$ and magnetizations $\boldsymbol{m} = \boldsymbol{m}^h$ defined on this lattice, the Slonczewski term reads

$$\boldsymbol{\tau}^h = -(\boldsymbol{v} \cdot \nabla^h)\boldsymbol{m},$$

where ∇^h is the symmetric discrete gradient. In this context we shall interpret the discrete Landau–Lifshitz–Slonczewski equation (LLS)

$$\frac{d\boldsymbol{m}^h}{dt} = \boldsymbol{m}^h \times \left(\alpha \frac{d\boldsymbol{m}^h}{dt} - \Delta^h \boldsymbol{m}^h + \beta\, \nabla^h_{\boldsymbol{v}} \boldsymbol{m}^h + \boldsymbol{m}^h \times \nabla^h_{\boldsymbol{v}} \boldsymbol{m}^h\right) \tag{9}$$

as follows: It is well-known and easy to show that the original version of LLG, the Landau–Lifshitz form $\dot{\boldsymbol{m}} = -\boldsymbol{m} \times (\boldsymbol{h}_{\text{eff}} + \alpha\, \boldsymbol{m} \times \boldsymbol{h}_{\text{eff}})$ introduced in [25], is, up to a time scale, equivalent to the Gilbert form introduced in [15]. This equivalence fails to be valid, however, when including nonvariational terms. Adding $\boldsymbol{\tau}^h$ to the Gilbert form $\dot{\boldsymbol{m}} = \boldsymbol{m} \times (\alpha\,\dot{\boldsymbol{m}} + \boldsymbol{h}_{\text{eff}})$ of LLG (as done in [36]) leads to (9) with $\beta = 0$, while adding $\boldsymbol{\tau}^h$ to the Landau–Lifshitz form of LLG leads to (9) with $\beta = \alpha$.

In section 3 we shall pass to the limit $h \searrow 0$ in (9) for a family of initial data with uniformly bounded discrete Dirichlet energy and obtain, in the spirit of [1] and [37], a weak solution

$$\boldsymbol{m} \in L^\infty_t \dot{H}^1_x \cap \dot{H}^1_t L^2_x((0,\infty) \times \mathbb{R}^3; \mathbb{S}^2)$$

of the continuous LLS equation equation

$$\frac{\partial \boldsymbol{m}}{\partial t} = \boldsymbol{m} \times \left(\alpha \frac{\partial \boldsymbol{m}}{\partial t} - \Delta \boldsymbol{m} + \beta\, \nabla_{\boldsymbol{v}} \boldsymbol{m} + \boldsymbol{m} \times \nabla_{\boldsymbol{v}} \boldsymbol{m} \right). \tag{10}$$

As a byproduct we obtain, upon discretizing initial data, an existence result for global weak solutions of (10). In the case of a bounded regular domain and in the absence of the adiabatic spin-transfer torque, global weak solutions have been obtained in [42] by a Galerkin method.

In section 4 we shall study local solvability of the Cauchy problem for the generalized LLS equation with two individual velocity fields $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$:

$$\frac{\partial \boldsymbol{m}}{\partial t} + \nabla_{\boldsymbol{v}_1} \boldsymbol{m} + \boldsymbol{m} \times \nabla_{\boldsymbol{v}_2} \boldsymbol{m} = -\,\boldsymbol{m} \times \Delta \boldsymbol{m} - \alpha\, \boldsymbol{m} \times \boldsymbol{m} \times \Delta \boldsymbol{m} \tag{11}$$

in Sobolev spaces of large enough differentiability exponent $\sigma \geq 3$ ($\sigma \in \mathbb{N}$), i.e.,

$$\boldsymbol{m} \in C^0\left([0,T]; H^\sigma(\mathbb{R}^3; \mathbb{S}^2)\right) \quad \text{assuming} \quad \boldsymbol{v} \in C^0\left([0,\infty); H^{\sigma-1}(\mathbb{R}^3; \mathbb{R}^3 \times \mathbb{R}^3)\right).$$

The results are based on higher-order Sobolev estimates using the fact that under the above conditions $\nabla \boldsymbol{m}(t)$ and $\boldsymbol{v}(t) \in H^{\sigma-1}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$. As a consequence, a singular time T^* is characterized by a blow-up of the gradient, i.e.,

$$\limsup_{t \nearrow T^*} \|\nabla \boldsymbol{m}(t)\|_{L^\infty} = \infty,$$

and hence global strong solvability follows from a global gradient bound.

In section 5 we shall investigate global strong solvability of (11) under suitable smallness conditions on the initial data and the velocity fields in the framework of [30]. Projecting $\frac{\partial \boldsymbol{m}}{\partial t}$ and $\nabla \boldsymbol{m}$ onto a moving orthonormal frame along $\boldsymbol{m}$ gives rise to complex-valued derivative functions $u_0 = u_t$ and $u = (u_1, u_2, u_3)$, respectively. Combining the velocity fields $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ into a complex velocity field $v = \boldsymbol{v}_1 + i\boldsymbol{v}_2$, (11) turns into a covariant convective Ginzburg–Landau equation

$$\mathcal{D}_t u + \mathcal{D}_x (v \cdot u) = (\alpha - i)\left(\mathcal{D}_x^2 u + \mathcal{R}\, u\right)$$

with covariant derivatives $\mathcal{D}_x = \nabla + i\,a$, $\mathcal{D}_t = \frac{\partial}{\partial t} + i\,a_0$, and curvature $\mathcal{R}_{k\ell} = [\mathcal{D}_k, \mathcal{D}_\ell]$, which is a quadratic function of u. Fixing the Coulomb gauge $\nabla \cdot a = 0$, the above equation exhibits global well-posedness properties of the standard cubic Ginzburg–Landau equation with a convection term, i.e., global existence, uniqueness, and regularity under a smallness condition on the initial data $u(x,0)$ in L^3_x, on the spin-velocity $v(x,t)$ in $L^\infty_t L^3_x$, and on the weighted-in-time spin velocity $\sqrt{t}\, v(x,t)$ in $L^\infty_{x,t}$. These spaces are optimal with respect to the natural scaling of the equation. As $|\nabla \boldsymbol{m}_0| = |u(0)|$, we infer a global gradient bound and hence global solvability of the Cauchy problem for (11) if

$$\|\nabla \boldsymbol{m}_0\|_{L^3} + \sup_{t>0} \|\boldsymbol{v}(t)\|_{L^3} + \sup_{t>0} \sqrt{t}\, \|\boldsymbol{v}(t)\|_{L^\infty}$$

is sufficiently small. Moreover, $\boldsymbol{m}(t) \to$ const. converges uniformly as $t \to \infty$. Finally we remark that the local and global existence results carry over with adapted exponents to higher space dimensions $d > 3$.

2. From Slonczevski's STT model to the discrete LLS.

2.1. The discrete equation. We wish to include a spin-transfer torque in the three-dimensional LLG equation for an isotropic ferromagnet consistent with Slonczewski's one-dimensional approach. To this end we follow the approach in [1, 37] and consider a cubic lattice of mesh size $h > 0$, i.e.,

$$(h\,\mathbb{Z})^3 = \left\{ x_\nu = \sum_{k=1}^{3} h\nu_k \hat{e}_k \ : \ \nu \in \mathbb{Z}^3 \right\},$$

and magnetization fields $\boldsymbol{m}^h : (h\,\mathbb{Z})^3 \to \mathbb{S}^2$. For continuous spin-velocity fields

$$\boldsymbol{v} = \sum_{k=1}^{3} v_k\, \hat{e}_k \quad \text{with} \quad v_k = v_k(x,t),$$

we consider, as an analogue to (8), spin-transfer terms of the form

$$\sum_{k=1}^{3} v_k(x_\nu, t)\, \boldsymbol{m}^h(x_\nu) \times \boldsymbol{m}^h(x_\nu) \times \frac{\boldsymbol{m}^h(x_\nu + h\hat{e}_k) - \boldsymbol{m}^h(x_\nu - h\hat{e}_k)}{2h}.$$

We shall also take into account exchange interaction created by the nearest neighbors; see, e.g., [37]. Neglecting demagnetization effects and in the absense of external magnetic fields this amounts to an effective field

$$\boldsymbol{h}_{\text{eff}} = \left(\frac{d}{h}\right)^2 \sum_{k=1}^{3} \Big(\boldsymbol{m}^h(x_\nu + h\hat{e}_k) + \boldsymbol{m}^h(x_\nu - h\hat{e}_k) \Big)$$

with exchange length $d > 0$. Rescaling $\boldsymbol{v}$ and t we can assume without loss of generality $d = 1$. While the commonly used Gilbert form of LLG is, after adapting parameters and time scale, equivalent to the original Landau–Lifshitz equation, adding a spin-torque term to one form or the other matters and leads to different results. We shall see that including the Slonczewski term in the Landau–Lifshitz form, i.e.,

$$\dot{\boldsymbol{m}} = -\boldsymbol{m} \times \boldsymbol{h}_{\text{eff}} - \alpha\, \boldsymbol{m} \times \boldsymbol{m} \times \boldsymbol{h}_{\text{eff}} + \text{Slonczewski term}$$

results in two terms when written in Gilbert form. In a continuum limit these two terms can be identified as the adiabatic and nonadiabatic spin-transfer torque. Accordingly, we shall consider the following modified Landau–Lifshitz equation

$$\begin{aligned} \frac{d}{dt}\boldsymbol{m}^h(x_\nu) = &- \sum_{k=1}^{3} \boldsymbol{m}^h(x_\nu) \times \frac{\boldsymbol{m}^h(x_\nu + h\hat{e}_k) + \boldsymbol{m}^h(x_\nu - h\hat{e}_k)}{h^2} \\ &- \alpha \sum_{k=1}^{3} \boldsymbol{m}^h(x_\nu) \times \boldsymbol{m}^h(x_\nu) \times \frac{\boldsymbol{m}^h(x_\nu + h\hat{e}_k) + \boldsymbol{m}^h(x_\nu - h\hat{e}_k)}{h^2} \\ &+ \sum_{k=1}^{3} v_k(x_\nu, t) \boldsymbol{m}^h(x_\nu) \times \boldsymbol{m}^h(x_\nu) \times \left(\frac{\boldsymbol{m}^h(x_\nu + h\hat{e}_k) - \boldsymbol{m}^h(x_\nu - h\hat{e}_k)}{2h} \right) \end{aligned} \tag{12}$$

with initial conditions

$$\boldsymbol{m}^h(x_\nu, 0) = \boldsymbol{m}^h_0(x_\nu) \ \text{ with } \ |\boldsymbol{m}^h_0(x_\nu)| = 1 \ \text{ for } \ \nu \in \mathbb{Z}^3. \tag{13}$$

2.2. Notations of discrete calculus. Let us first fix appropriate notation. For vector fields $u, w : (h\,\mathbb{Z})^3 \to \mathbb{R}^3$ we define corresponding forward ($+h$), backward ($-h$), and symmetric (unsigned h) difference quotients

$$\partial_k^{\pm h} u(x_\nu) = \pm\frac{u(x_\nu \pm h\hat{e}_k) - u(x_\nu)}{h},$$
$$\partial_k^{h} u(x_\nu) = \frac{u(x_\nu + h\hat{e}_k) - u(x_\nu - h\hat{e}_k)}{2h}$$

for all $x_\nu = \sum_{k=1}^3 h\nu_k\hat{e}_k$ and $\nu \in \mathbb{Z}^3$. Clearly $\partial_k^h u = \frac{1}{2}(\partial_k^{+h}u + \partial_k^{-h}u)$.

The discrete gradients (forward, backward, and symmetric) and the discrete Laplace operator are defined as

$$\nabla^{\pm h} u = \sum_{k=1}^{3} \hat{e}_k \otimes \left(\partial_k^{\pm h} u\right), \quad \nabla^{h} u = \sum_{k=1}^{3} \hat{e}_k \otimes \left(\partial_k^{h} u\right),$$
$$\Delta^h u = \sum_{k=1}^{3} \partial_k^{+h}\partial_k^{-h} u = \sum_{k=1}^{3} \partial_k^{-h}\partial_k^{+h} u,$$

i.e., $\Delta^h = \nabla^{+h} \cdot \nabla^{-h} = \nabla^{-h} \cdot \nabla^{+h}$. The discrete L_h^2 scalar product is given by

$$(u, w)_h = h^3 \sum_{\nu \in \mathbb{Z}^3} u(x_\nu) w(x_\nu). \tag{14}$$

We also introduce the following discrete norms,

$$|u|_{L_h^\infty} = \sup_{\nu\in\mathbb{Z}^3} |u(x_\nu)|, \quad |u|_{L_h^2}^2 = (u,u)_h, \quad |u|_{\dot{H}_h^1}^2 = |\nabla^{+h}u|_{L_h^2}^2$$

defining the discrete function spaces L_h^∞, L_h^2, $\dot{H}_h^1$, and $H_h^1 = \{u \in L_h^2 : |u|_{\dot{H}_h^1} < \infty\}$. The discrete Dirichlet energy E_h corresponds to the homogeneous $\dot{H}_h^1$ norm, i.e.,

$$E_h(u) = \frac{h^3}{2} \sum_{\nu\in\mathbb{Z}^3} |\nabla^{+h} u(x_\nu)|^2.$$

For $u, w \in H_h^1$ or with appropriate decay properties, we have a discrete version of the integration by parts formula

$$\sum_{\nu\in\mathbb{Z}^3} u(x_\nu)\partial_k^{+h} w(x_\nu) = -\sum_{\nu\in\mathbb{Z}^3} \partial_k^{-h} u(x_\nu) w(x_\nu), \tag{15}$$

i.e., $(u, \partial_k^{+h} w)_h = (\partial_k^{-h} u, w)_h$.

Now, using this notation and the properties of the cross product, we rewrite (12)

$$\frac{d\boldsymbol{m}^h}{dt} = -\boldsymbol{m}^h \times \Delta^h \boldsymbol{m}^h - \alpha\, \boldsymbol{m}^h \times \boldsymbol{m}^h \times \Delta^h \boldsymbol{m}^h + \boldsymbol{m}^h \times \boldsymbol{m}^h \times \nabla_{\boldsymbol{v}}^h \boldsymbol{m}^h, \tag{16}$$

where we used the notation

$$\nabla_{\boldsymbol{v}}^h \boldsymbol{m}^h = (\boldsymbol{v} \cdot \nabla^h)\boldsymbol{m}^h.$$

Renormalizing $1/t$ and $\boldsymbol{v}$ by a factor $1+\alpha^2$, (16) can equally be written as

$$\frac{d\boldsymbol{m}^h}{dt} = \boldsymbol{m}^h \times \left(\alpha\frac{d\boldsymbol{m}^h}{dt} - \Delta^h \boldsymbol{m}^h + \alpha\, \nabla_{\boldsymbol{v}}^h \boldsymbol{m}^h + \boldsymbol{m}^h \times \nabla_{\boldsymbol{v}}^h \boldsymbol{m}^h\right)$$

which, for $\alpha = \beta$, is the discretized LLS equation

$$\frac{d\boldsymbol{m}^h}{dt} = \boldsymbol{m}^h \times \left(\alpha \frac{d\boldsymbol{m}^h}{dt} - \Delta^h \boldsymbol{m}^h + \beta\, \nabla^h_{\boldsymbol{v}} \boldsymbol{m}^h + \boldsymbol{m}^h \times \nabla^h_{\boldsymbol{v}} \boldsymbol{m}^h \right). \tag{17}$$

Equation (17) is, from the point of view of partial differential equations and compactness, more convenient due to the divergence structure of the highest-order term, i.e.,

$$\boldsymbol{m}^h \times \Delta^h \boldsymbol{m}^h = \nabla^{-h} \cdot \left(\boldsymbol{m}^h \times \nabla^{+h} \boldsymbol{m} \right). \tag{18}$$

Global existence and uniqueness for the discrete LLS. We notice that, for $h > 0$ fixed and $\boldsymbol{v} : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ bounded and continuous, the right-hand side of (17) is a Lipschitz continuous function on the bounded sets of the Banach space

$$\mathbb{E}_h = L^\infty_h \cap \dot{H}^1_h.$$

Thus, for initial data

$$\boldsymbol{m}^h_0 : (h\mathbb{Z})^3 \to \mathbb{S}^2 \in \mathbb{E}_h$$

there exists a unique local solution $\boldsymbol{m}^h$ of (17) on a time interval $[0,\tau]$, for some $0 < \tau \le T$. But then the estimates in Lemma 1 ensure $\|\boldsymbol{m}^h(t)\|_{\mathbb{E}_h} \le C$ for all $0 < t \le \tau$, and hence the solutions $\boldsymbol{m}^h$ extend uniquely to $[0,T]$. Observe that only $L^2_t L^\infty_x$ bounds on the spin velocity $\boldsymbol{v}$ enter the a priori estimate (19). In view of Caratheodory's existence theorem, this suggests an existence result under weaker conditions on $\boldsymbol{v}$. We shall, however, not stress this possible extension in our further discussions.

LEMMA 1. *Suppose $\boldsymbol{v} : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ is bounded and continuous. Given initial data $\boldsymbol{m}^h_0 \in \dot{H}^1_h$ such that $|\boldsymbol{m}^h_0| = 1$, then the corresponding solution $\boldsymbol{m}^h$ of* (17) *exists on* $[0,T]$,

$$\boldsymbol{m}^h : [0,T] \times (h\mathbb{Z})^3 \to \mathbb{S}^2,$$

and satisfies

$$\frac{\alpha}{2} \int_0^\tau \left| \frac{d\boldsymbol{m}^h(t)}{dt} \right|^2_{L^2_h} dt + E_h(\boldsymbol{m}^h(\tau)) \le \exp\left(c \int_0^\tau \|\boldsymbol{v}(t)\|^2_{L^\infty} dt \right) E_h(\boldsymbol{m}^h_0) \tag{19}$$

for all $\tau \in (0,T]$ and a universal constant $c > 0$ that only depends on α.

Proof. Multiplying (16) by $\boldsymbol{m}^h$ gives, for all $t > 0$ and $\nu \in \mathbb{Z}^3$,

$$\frac{d}{dt} \frac{|\boldsymbol{m}^h(x_\nu,t)|^2}{2} = \frac{d\boldsymbol{m}^h(x_\nu,t)}{dt} \cdot \boldsymbol{m}^h(x_\nu,t) = 0.$$

As $|\boldsymbol{m}_0(x_\nu)| = 1$ for $\nu \in \mathbb{Z}^3$, we obtain $|\boldsymbol{m}^h| = 1$. To show (19) we use

$$\boldsymbol{m}^h \times \frac{d\boldsymbol{m}^h}{dt}$$

as a test function in (17). Observe that $(\boldsymbol{m}^h \times \mathbf{a}) \cdot (\boldsymbol{m}^h \times \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$ for all $\mathbf{a}, \mathbf{b} \perp \boldsymbol{m}^h$. Hence, after discrete integration by parts, we obtain

$$\alpha \left| \frac{d\boldsymbol{m}^h}{dt} \right|^2_{L^2_h} + \frac{d}{dt} E_h(\boldsymbol{m}^h) = \left(\beta \boldsymbol{v} \cdot \nabla^h \boldsymbol{m}^h + \boldsymbol{m}^h \times (\boldsymbol{v} \cdot \nabla^h) \boldsymbol{m}^h, \frac{d\boldsymbol{m}^h}{dt} \right)_h. \tag{20}$$

By definition of $\nabla^h \boldsymbol{m}^h$ we have

$$\left|\nabla^h \boldsymbol{m}^h\right|^2_{L^2_h} \le |\boldsymbol{m}^h|^2_{\dot{H}^1_h} = 2E_h(\boldsymbol{m}^h).$$

Thus, by Young's inequality, the terms on the right-hand side of (20) are bounded by

$$\frac{(1+\beta)^2}{\alpha} E_h(\boldsymbol{m}^h)\|\boldsymbol{v}(t)\|^2_{L^\infty} + \frac{\alpha}{2}\left|\frac{d\boldsymbol{m}^h}{dt}\right|^2_{L^2_h}. \tag{21}$$

Absorbing the second term in (21) into the left-hand side of (20) and integrating in time, (19) follows from Gronwall's inequality. ∎

3. From discrete to continuous LLS.

3.1. Discretization and interpolation. Locally integrable maps $u : \mathbb{R}^3 \to \mathbb{R}^3$ are discretized by letting, for $h > 0$ fixed and $x_\nu \in (h\mathbb{Z})^3$,

$$u^h(x_\nu) = h^{-3}\int_{C^h_\nu} u(y)\, dy,$$

where

$$C^h_\nu = \left\{x \in \mathbb{R}^3 : x^k_\nu \le x^k < x^k_\nu + h \text{ for } k = 1, 2, 3\right\}.$$

We shall denote the piecewise constant extension of a discrete map $u^h : (h\mathbb{Z})^3 \to \mathbb{R}^3$ to a map on the entire $\mathbb{R}^3$ by the same symbol, i.e.,

$$u^h(x) = u^h(x_\nu) \quad \text{for every} \quad x \in C^h_\nu.$$

Observe that

$$(u^h \times w^h)(x) = u^h(x_\nu) \times w^h(x_\nu) \quad \text{and} \quad \partial^{\pm h}_k u^h(x) = \partial^{\pm h}_k u^h(x_\nu) \tag{22}$$

for all $x \in C^h_\nu$. Moreover, $|u^h|_{L^2_h} = \|u^h\|_{L^2}$.

We denote by $\overline{u}^h$ the cubic interpolation of u^h with $\overline{u}^h(x_\nu) = u^h(x_\nu)$ and such that

$$\begin{aligned} \overline{u}^h(x) = u^h(x_\nu) &+ \sum_{j=1}^{3} \partial^{+h}_j u^h(x_\nu)\cdot(x^j - x^j_\nu) \\ &+ \sum_{k<j} \partial^{+h}_k \partial^{+h}_j u^h(x_\nu)(x^k - x^k_\nu)(x^j - x^j_\nu) \\ &+ \partial^{+h}_1 \partial^{+h}_2 \partial^{+h}_3 u^h(x_\nu)(x^1 - x^1_\nu)(x^2 - x^2_\nu)(x^3 - x^3_\nu) \end{aligned} \tag{23}$$

for $x \in C^h_\nu$; see [1, 32, 37]. We observe that the cubic interpolation is linear in every variable. It is easy to verify that u^h and $\overline{u}^h$ are norm equivalent in the sense that

$$\|u^h\|_{L^2} \sim \|\overline{u}^h\|_{L^2} \quad \text{and} \quad E_h(u^h) \sim E(\overline{u}^h) \tag{24}$$

with uniform bounds from above and below. Moreover, we have estimates

$$\|u^h - \overline{u}^h\|^2_{L^2} + h^2\|u^h - \overline{u}^h\|^2_{L^\infty} \le C\, h^2\, E_h(u^h) \tag{25}$$

for a constant $C > 0$ independent of u and h; see, e.g., [33].

3.2. The continuum limit. For $\boldsymbol{v} : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ bounded and continuous and $h > 0$ we consider a family $\boldsymbol{m}^h$ of solutions of the discrete LLS,

$$\frac{d\boldsymbol{m}^h}{dt} = \boldsymbol{m}^h \times \left(\alpha \frac{d\boldsymbol{m}^h}{dt} - \Delta^h \boldsymbol{m}^h + \beta \, \nabla^h_{\boldsymbol{v}} \boldsymbol{m}^h + \boldsymbol{m}^h \times \nabla^h_{\boldsymbol{v}} \boldsymbol{m}^h \right)$$

on $(h\mathbb{Z})^3$ with initial data $\boldsymbol{m}^h_0 : (h\mathbb{Z})^3 \to \mathbb{S}^2$ such that

$$\sup_{h>0} E_h \left(\boldsymbol{m}^h_0 \right) < \infty.$$

We wish to investigate the discrete-to-continuum limit to the continuous LLS

$$\frac{\partial \boldsymbol{m}}{\partial t} + \nabla_{\boldsymbol{v}} \boldsymbol{m} = \boldsymbol{m} \times \left(\alpha \, \frac{\partial \boldsymbol{m}}{\partial t} + \beta \, \nabla_{\boldsymbol{v}} \boldsymbol{m} - \Delta \boldsymbol{m} \right) \tag{26}$$

in its weak formulation

$$\int_0^T \left\langle \frac{\partial \boldsymbol{m}}{\partial t} + \nabla_{\boldsymbol{v}} \boldsymbol{m}, \phi \right\rangle = \int_0^T \left\langle \boldsymbol{m} \times \left(\alpha \frac{\partial \boldsymbol{m}}{\partial t} + \beta \, \nabla_{\boldsymbol{v}} \boldsymbol{m} \right), \phi \right\rangle + \int_0^T \langle \boldsymbol{m} \times \nabla \boldsymbol{m}, \nabla \phi \rangle$$

for all $\phi \in C^\infty_0(\mathbb{R}^3 \times (0,T); \mathbb{R}^3)$.

THEOREM 1. *There exists a subsequence $h_k \searrow 0$ and a vector field*

$$\boldsymbol{m} \in L^\infty(0,T; \dot{H}^1(\mathbb{R}^3; \mathbb{S}^2)) \cap \dot{H}^1(0,T; L^2(\mathbb{R}^3; \mathbb{R}^3))$$

such that

$$\boldsymbol{m}^{h_k} \to \boldsymbol{m} \quad \textit{strongly in} \quad L^2_{\mathrm{loc}}(\mathbb{R}^3 \times (0,T); \mathbb{R}^3)$$

and $\boldsymbol{m}$ is a weak solution of (26) *with $\boldsymbol{m}(0) = \lim_{k\to\infty} \boldsymbol{m}^{h_k}_0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{R}^3)$.*

Proof. From the discrete energy estimate in Lemma 1 we deduce uniform bounds

$$\sup_{h>0} \left\| \nabla^{\pm h} \boldsymbol{m}^h \right\|_{L^\infty(0,T; L^2(\mathbb{R}^3))} + \sup_{h>0} \left\| \frac{d\boldsymbol{m}^h}{dt} \right\|_{L^2(0,T; L^2(\mathbb{R}^3))} < \infty. \tag{27}$$

Hence, by (24), the interpolants $\overline{\boldsymbol{m}}^h$ have uniform bounds and are weakly* compact in the energy space

$$\mathbb{E} = L^\infty(0,T; \dot{H}^1(\mathbb{R}^3; \mathbb{R}^3)) \cap \dot{H}^1(0,T; L^2(\mathbb{R}^3; \mathbb{R}^3)).$$

Passing to a subsequence $h_k \searrow 0$ we may assume that, for some $\boldsymbol{m} \in \mathbb{E}$,

$$\overline{\boldsymbol{m}}^{h_k} \rightharpoonup \boldsymbol{m} \quad \text{weakly in } \mathbb{E} \text{ and strongly in } L^2_{\mathrm{loc}}(\mathbb{R}^3 \times (0,T); \mathbb{R}^3) \tag{28}$$

and, by Sobolev embedding and Rellich compact embedding,

$$\overline{\boldsymbol{m}}^{h_k}_0 \to \boldsymbol{m}(0) \quad \text{strongly in } L^2_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{R}^3). \tag{29}$$

Taking into account (25) we can assume that the extensions converge

$$\boldsymbol{m}^{h_k} \to \boldsymbol{m} \quad \text{strongly in } L^2_{\mathrm{loc}}(\mathbb{R}^3 \times (0,T); \mathbb{R}^3) \tag{30}$$

and almost everywhere in $\mathbb{R}^3 \times (0,T)$. From $|\boldsymbol{m}^h| = 1$ it follows that $|\boldsymbol{m}| = 1$ almost everywhere in $\mathbb{R}^3 \times (0,T)$. Hence $\boldsymbol{m} \in L^\infty(0,T; \dot{H}^1(\mathbb{R}^3; \mathbb{S}^2)) \cap \dot{H}^1(0,T; L^2(\mathbb{R}^3; \mathbb{R}^3))$, the first claim of Theorem 1. From (27), (30), the uniqueness of distributional limits (in fact, $\nabla^{\pm h}\varphi \to \nabla\varphi$ uniformly as $h \searrow 0$ for every smooth function φ), and the continuity of $\boldsymbol{v}$ we deduce that

$$\frac{\partial \boldsymbol{m}^{h_k}}{\partial t} \rightharpoonup \frac{\partial \boldsymbol{m}}{\partial t}, \quad \nabla^{\pm h_k}\boldsymbol{m}^{h_k} \rightharpoonup \nabla\boldsymbol{m}, \quad \text{and} \quad \nabla_{\boldsymbol{v}}^{h_k}\boldsymbol{m}^{h_k} \rightharpoonup \nabla_{\boldsymbol{v}}\boldsymbol{m} \tag{31}$$

converge weakly in $L^2_{\mathrm{loc}}(\mathbb{R}^3 \times (0,T))$. Hence, again using (30) and Lebesgue's convergence theorem,

$$\begin{aligned} \nabla^{-h_k} \cdot \left(\boldsymbol{m}^{h_k} \times \nabla^{+h_k}\boldsymbol{m}^{h_k}\right) &\to \nabla \cdot (\boldsymbol{m} \times \nabla\boldsymbol{m}), \\ \boldsymbol{m}^{h_k} \times \boldsymbol{m}^{h_k} \times \nabla_{\boldsymbol{v}}^{h_k}\boldsymbol{m}^{h_k} &\to \boldsymbol{m} \times \boldsymbol{m} \times \nabla_{\boldsymbol{v}}\boldsymbol{m}, \\ \boldsymbol{m}^{h_k} \times \nabla_{\boldsymbol{v}}^{h_k}\boldsymbol{m}^{h_k} &\to \boldsymbol{m} \times \nabla_{\boldsymbol{v}}\boldsymbol{m}, \\ \boldsymbol{m}^{h_k} \times \frac{\partial \boldsymbol{m}^{h_k}}{\partial t} &\to \boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial t} \end{aligned}$$

converge in the sense of distributions. In view of (18) and the fact that

$$\boldsymbol{m} \times \boldsymbol{m} \times \nabla_{\boldsymbol{v}}\boldsymbol{m} = -\nabla_{\boldsymbol{v}}\boldsymbol{m}$$

the latter convergences prove that $\boldsymbol{m}$ is a weak solution of (26).

Finally, in view of Sobolev embedding, evaluation at $t = 0$ is a bounded linear operator from $H^1((0,T); L^2_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^3))$ into $L^2_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^3)$, and from (29) and (25) it follows that $\boldsymbol{m}_0^{h_k} \to \boldsymbol{m}(0)$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^3)$. □

3.3. Possible extensions. The argument carried out to construct weak solutions can easily be extended to include additional effective anisotropy fields

$$\boldsymbol{h}_{\mathrm{an}} = -(\nabla\Phi)(\boldsymbol{m})$$

stemming from densities $\Phi(\boldsymbol{m}) \geq 0$ which are even polynomials in the components of $\boldsymbol{m}$. Observe that by the saturation condition $|\boldsymbol{m}^h| = 1$, the local L^2 compactness of $(\boldsymbol{m}^h)$ improves to local L^p compactness for every $p < \infty$.

A characteristic feature of micromagnetic theory is dipolar stray-field interaction giving rise to a nonlocal contribution to the effective field. Recall that on the continuum level the stray field $\boldsymbol{h}_{\mathrm{stray}} = -\nabla\zeta$ is determined by the magnetostatic Maxwell equation

$$div(-\nabla\zeta + \boldsymbol{m}) = 0 \quad \text{in} \quad \mathbb{R}^3;$$

see [17, 21]. Starting from the continuous version of LLS coupled to Maxwell's equations suggests a complementary approach to weak solutions which is rather based on a continuous Galerkin approximation as used, e.g., in [42, 45]. On the discrete level, the stray field may be determined by summing over all dipole fields of the individual magnetic moments, i.e.,

$$\boldsymbol{h}_{\mathrm{stray}}(x) = \sum_{\nu\in\mathbb{Z}^3} \boldsymbol{K}(x - x_\nu)\boldsymbol{m}(x_\nu), \quad \text{where} \quad \boldsymbol{K}_{ij}(y) = \frac{\partial^2}{\partial y^i \partial y^j}\left(\frac{1}{4\pi}|y|^{-1}\right);$$

see, e.g., [21]. Observe that this discrete convolution corresponds to a discretization of the singular integral for the double Riesz transform applied to $\boldsymbol{m}$, which is the solution operator $\boldsymbol{m} \mapsto \boldsymbol{h}_{\mathrm{stray}}$ for the macroscopic Maxwell equation above. In this setting, a discrete-to-continuum limit for LLS including dipolar stray-field interactions seems possible but requires additional arguments beyond the scope of this paper.

4. Local solvability of the Cauchy problem. Having constructed global weak solutions, we shall now investigate local existence and uniqueness of classical solutions of the generalized LLS equation with two individual velocity fields $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$:

$$\frac{\partial \boldsymbol{m}}{\partial t} + (\boldsymbol{v}_1 \cdot \nabla)\, \boldsymbol{m} + \boldsymbol{m} \times (\boldsymbol{v}_2 \cdot \nabla)\, \boldsymbol{m} = -\boldsymbol{m} \times \Delta \boldsymbol{m} - \alpha\, \boldsymbol{m} \times \boldsymbol{m} \times \Delta \boldsymbol{m}. \tag{32}$$

For the unperturbed LLG ($\boldsymbol{v}_1, \boldsymbol{v}_2 = 0$), local solvability in appropriate Sobolev spaces has been investigated in [13, 19, 30]. Hence we shall focus on the new ingredients and necessary regularity conditions on the spin velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ in order to obtain analogous results. Let us fix the magnetization at infinity $\boldsymbol{m}_\infty \in \mathbb{S}^2$ and set

$$H^\sigma(\mathbb{R}^3; \mathbb{S}^2) = \{\boldsymbol{m} : \mathbb{R}^3 \to \mathbb{S}^2 : \boldsymbol{m} - \boldsymbol{m}_\infty \in H^\sigma(\mathbb{R}^3; \mathbb{R}^3)\},$$

where $H^\sigma(\mathbb{R}^3) = (1-\Delta)^{-\frac{\sigma}{2}} L^2(\mathbb{R}^3)$ is the usual Sobolev space. For some integer $\sigma \geq 3$ we assume initial data $\boldsymbol{m}_0 : \mathbb{R}^3 \to \mathbb{S}^2$ to satisfy

$$\boldsymbol{m}_0 \in H^\sigma(\mathbb{R}^3; \mathbb{S}^2) \tag{33}$$

and spin velocities $\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2) : \mathbb{R}^3 \times (0, \infty) \to \mathbb{R}^3 \times \mathbb{R}^3$ to be included in

$$\boldsymbol{v} \in C^0([0,\infty); H^{\sigma-1}(\mathbb{R}^3; \mathbb{R}^3 \times \mathbb{R}^3)). \tag{34}$$

THEOREM 2. *Suppose* (33) *and* (34) *for some integer $\sigma \geq 3$. Then there exists a terminal time $T^* > 0$ and a unique solution*

$$\boldsymbol{m} \in C^0\left([0,T]; H^\sigma(\mathbb{R}^3; \mathbb{S}^2)\right) \quad \textit{and} \quad \frac{\partial \boldsymbol{m}}{\partial t} \in C^0\left([0,T]; H^{\sigma-2}(\mathbb{R}^3; \mathbb{R}^3)\right)$$

for all $T < T^*$ of (32) *with $\boldsymbol{m}(0) = \boldsymbol{m}_0$ and $\boldsymbol{v}$ as above. If $T^* < \infty$, then*

$$\limsup_{t \nearrow T^*} \|\nabla \boldsymbol{m}(t)\|_{L^\infty} = \infty.$$

Remark. If $\boldsymbol{v}$ is in addition smooth in $\mathbb{R}^3 \times (0, \infty)$, then the solution $\boldsymbol{m}$ is in fact a classical solution and

$$\boldsymbol{m} \in C^0\left((0,T^*); H^\infty(\mathbb{R}^3; \mathbb{S}^2)\right), \quad \text{where} \quad H^\infty(\mathbb{R}^3; \mathbb{S}^2) = \bigcap_{\sigma \in \mathbb{Z}} H^\sigma(\mathbb{R}^3; \mathbb{S}^2).$$

Recalling the identities

$$\boldsymbol{m} \times \Delta \boldsymbol{m} = \nabla \cdot (\boldsymbol{m} \times \nabla \boldsymbol{m}) \quad \text{and} \quad -\boldsymbol{m} \times \boldsymbol{m} \times \Delta \boldsymbol{m} = \Delta \boldsymbol{m} + |\nabla \boldsymbol{m}|^2 \boldsymbol{m},$$

we observe that (32) is a quasilinear parabolic system in divergence form. Using the ansatz $\boldsymbol{u} = \boldsymbol{m} - \boldsymbol{m}_\infty$, we obtain a Cauchy problem of the form

$$\frac{\partial \boldsymbol{u}}{\partial t} = \nabla \cdot \Big(A(\boldsymbol{u}) \nabla \boldsymbol{u} \Big) + B(x, t, \boldsymbol{u}, \nabla \boldsymbol{u}) \quad \text{with} \quad \boldsymbol{u}(0) \in H^\sigma(\mathbb{R}^3; \mathbb{R}^3),$$

where $\langle \boldsymbol{\xi}, A(\boldsymbol{u})\, \boldsymbol{\xi} \rangle = \alpha\, |\boldsymbol{\xi}|^2$ for every $\boldsymbol{u} \in \mathbb{R}^3$ and $\boldsymbol{\xi} \in \mathbb{R}^{3\times 3}$. Such problems can be approximated, e.g., by a modified Galerkin method based on spectral truncations as presented in great detail in [38, 39]. Uniform higher-order Sobolev estimates as discussed in the next paragraph, provide the requisite continuation and compactness properties. Having constructed the solution $\boldsymbol{u}$, it can be verified by a Gronwall argument as in [30] that $\boldsymbol{m} = \boldsymbol{u} + \boldsymbol{m}_\infty$ satisfies $|\boldsymbol{m}| = 1$.

4.1. Higher-order Sobolev estimates and stability. As pointed out earlier, a crucial ingredient to the proof of Theorem 2 is higher-order Sobolev estimates

$$\|\nabla \boldsymbol{m}(T)\|_{H^{\sigma-1}}^2 + \frac{\alpha}{2}\int_0^T \|\nabla \boldsymbol{m}(t)\|_{H^\sigma}^2 \, dt \le e^{C(T)}\|\nabla \boldsymbol{m}(0)\|_{H^{\sigma-1}}^2 \tag{35}$$

for integers $\sigma \ge 3$, where

$$C(t) = c\int_0^t \Big(\|\nabla \boldsymbol{m}(s)\|_{L^\infty}^2 + \big(1+\|\nabla \boldsymbol{m}(s)\|_{L^\infty}^2\big)\, \|\boldsymbol{v}(s)\|_{H^{\sigma-1}}^2\Big)\, ds \tag{36}$$

for a universal constant $c > 0$ that only depends on α and σ. Such an estimate implies that a breakdown of strong solvability is necessarily accompanied by a blow-up of $\nabla \boldsymbol{m}$. A second ingredient is the stability estimate

$$\|(\boldsymbol{m}_1 - \boldsymbol{m}_2)(t)\|_{L^2}^2 \le \exp\left(C_1(t) + C_2(t)\right) \|(\boldsymbol{m}_1 - \boldsymbol{m}_2)(0)\|_{L^2}^2 \tag{37}$$

for solutions $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$, where $C_i(t)$ can be taken to be a multiple of the constant in (36) for $\boldsymbol{m} = \boldsymbol{m}_i$, which yields uniqueness. In the case $\boldsymbol{v} = 0$, these inequalities have been shown in [30, Lemmas 1 and 2], respectively. Here we shall briefly comment on how to modify the argument in order to obtain (35) and (37) in the presence of spin velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

Estimate (35) is obtained by successive differentiation of (32), i.e., applying $\partial^{\boldsymbol{k}}$ to (32) for a multiindex $1 \le |\boldsymbol{k}| \le \sigma$, multiplication by corresponding derivatives $\partial^{\boldsymbol{k}}\boldsymbol{m}$, and integration in space. Summing over all possible $\boldsymbol{k}$ yields

$$\begin{aligned}\frac{d}{dt}\, &\|\nabla \boldsymbol{m}(t)\|_{H^{\sigma-1}}^2 + \alpha\, \|\nabla \boldsymbol{m}(t)\|_{H^\sigma}^2 \\ &\le c\left(\|\nabla \boldsymbol{m}(t)\|_{L^\infty}^2 + \big(1+\|\nabla \boldsymbol{m}(t)\|_{L^\infty}^2\big)\, \|\boldsymbol{v}(t)\|_{H^{\sigma-1}}^2\right) \|\nabla \boldsymbol{m}(t)\|_{H^{\sigma-1}}^2\end{aligned}$$

from which (35) follows by Gronwall's inequality. For the derivation of this inequality, we focus on the new terms stemming from the spin velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, respectively. Concerning the term including $\boldsymbol{v}_1$, integration by parts and taking into account that $H^{\sigma-1}(\mathbb{R}^3)$ $(\sigma \ge 3)$ is an algebra yields

$$\left|\langle \partial^{\boldsymbol{k}} \nabla_{\boldsymbol{v}_1} \boldsymbol{m}, \partial^{\boldsymbol{k}} \boldsymbol{m}\rangle\right| \le c\, \|\boldsymbol{v}_1\|_{H^{\sigma-1}} \|\nabla \boldsymbol{m}\|_{H^{\sigma-1}} \|\nabla \boldsymbol{m}\|_{H^\sigma}. \tag{38}$$

Here and in what follows, $c > 0$ is a generic constant. Regarding the term including $\boldsymbol{v}_2$, we first integrate by parts and estimate

$$\left|\langle \partial^{\boldsymbol{k}} (\boldsymbol{m} \times \nabla_{\boldsymbol{v}_2} \boldsymbol{m}), \partial^{\boldsymbol{k}} \boldsymbol{m}\rangle\right| \le \|\nabla^{|\boldsymbol{k}|-1} (\boldsymbol{m} \times \nabla_{\boldsymbol{v}_2} \boldsymbol{m})\|_{L^2} \|\nabla \boldsymbol{m}\|_{H^\sigma}. \tag{39}$$

For $|\boldsymbol{k}| > 1$ we further estimate, using $|\boldsymbol{m}| = 1$,

$$\left\|\nabla^{|\boldsymbol{k}|-1} (\boldsymbol{m} \times \nabla_{\boldsymbol{v}_2} \boldsymbol{m})\right\|_{L^2} \le c\, \Big(\|\nabla_{\boldsymbol{v}_2} \boldsymbol{m}\|_{H^{\sigma-1}} + \|\nabla \boldsymbol{m} \times \nabla_{\boldsymbol{v}_2} \boldsymbol{m}\|_{H^{\sigma-2}}\Big). \tag{40}$$

As above $\|\nabla_{\boldsymbol{v}_2} \boldsymbol{m}\|_{H^{\sigma-1}} \le c\, \|\boldsymbol{v}_2\|_{H^{\sigma-1}} \|\nabla \boldsymbol{m}\|_{H^{\sigma-1}}$, while by Moser's estimate and Sobolev embedding

$$\begin{aligned}\|\nabla \boldsymbol{m} \times \nabla_{\boldsymbol{v}_2} \boldsymbol{m}\|_{H^{\sigma-2}} &\le c\Big(\|\nabla \boldsymbol{m}\|_{L^\infty} \|\nabla_{\boldsymbol{v}_2} \boldsymbol{m}\|_{H^{\sigma-2}} + \|\nabla_{\boldsymbol{v}_2} \boldsymbol{m}\|_{L^\infty} \|\nabla \boldsymbol{m}\|_{H^{\sigma-2}}\Big) \\ &\le c\, \|\boldsymbol{v}_2\|_{H^{\sigma-1}} \|\nabla \boldsymbol{m}\|_{L^\infty} \|\nabla \boldsymbol{m}\|_{H^{\sigma-1}}.\end{aligned}$$

Hence, by (38), (39), (40), and by virtue of Young's inequality

$$\begin{aligned}
&\left|\langle \partial^{\boldsymbol{k}} \nabla_{\boldsymbol{v}_1} \boldsymbol{m}, \partial^{\boldsymbol{k}} \boldsymbol{m}\rangle\right| + \left|\langle \partial^{\boldsymbol{k}} (\boldsymbol{m} \times \nabla_{\boldsymbol{v}_2} \boldsymbol{m}), \partial^{\boldsymbol{k}} \boldsymbol{m}\rangle\right| \\
&\qquad \le c \left(\|\nabla \boldsymbol{m}\|_{L^\infty}^2 + \left(1 + \|\nabla \boldsymbol{m}\|_{L^\infty}^2\right) \|\boldsymbol{v}\|_{H^{\sigma-1}}^2 \right) \|\nabla \boldsymbol{m}\|_{H^{\sigma-1}}^2 + \delta\, \alpha\, \|\nabla \boldsymbol{m}\|_{H^\sigma}^2,
\end{aligned}$$

so that the second term on the right can be absorbed for sufficiently small $\delta > 0$.

The stability estimate is obtained by subtracting equations for $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$, multiplying by $\Phi = \boldsymbol{m}_1 - \boldsymbol{m}_2$, and integrating in space and time, leading to

$$\begin{aligned}
&\|\Phi(t)\|_{L^2}^2 + 2 \int_0^t \left(\alpha \|\nabla \Phi\|_{L^2}^2 + \langle \nabla_{\boldsymbol{v}_1} \Phi, \Phi\rangle + \langle \boldsymbol{m}_2 \times \nabla_{\boldsymbol{v}_2} \Phi, \Phi\rangle \right) ds \\
&\quad \le \|\Phi(0)\|_{L^2}^2 + 2\alpha \int_0^t \|\Phi \nabla \boldsymbol{m}_1\|_{L^2}^2 ds \\
&\qquad + 2 \int_0^t \left(\alpha\, \|\Phi\, \nabla (\boldsymbol{m}_1 + \boldsymbol{m}_2)\|_{L^2} + \|\Phi\, \times \nabla \boldsymbol{m}_1\|_{L^2} \right) \|\nabla \Phi\|_{L^2}\, ds.
\end{aligned}$$

As in [30], the terms on the right can be estimated using L^∞ bounds of $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$. Concerning the new terms on the left, stemming from the spin velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, we have by Cauchy–Schwarz and Sobolev embedding

$$|\langle \nabla_{\boldsymbol{v}_1} \Phi, \Phi\rangle| \le c\, \|\boldsymbol{v}_1\|_{H^{\sigma-1}} \|\Phi\|_{L^2} \|\nabla \Phi\|_{L^2}$$

and

$$|\langle \boldsymbol{m}_2 \times \nabla_{\boldsymbol{v}_2} \Phi, \Phi\rangle| \le c\, \|\boldsymbol{v}_2\|_{H^{\sigma-1}} \|\Phi\|_{L^2} \|\nabla \Phi\|_{L^2},$$

where again the last factor on the right can be absorbed by a corresponding term stemming from the elliptic part.

5. Global solvability of the Cauchy problem for LLS. The estimates in the previous section also motivate a global existence result under a smallness condition on initial gradients $\nabla \boldsymbol{m}_0$ in $H^2(\mathbb{R}^3; \mathbb{R}^{3\times 3})$ and spin velocities $\boldsymbol{v}$ in $C^0([0,\infty); H^2(\mathbb{R}^3; \mathbb{R}^3 \times \mathbb{R}^3))$, respectively. In this final section we shall see that global solvability of the initial value problem holds true under smallness conditions which do not contain any derivatives of initial gradients and the spin velocities. Moreover, these conditions are critical with respect to the natural scaling of the equation. To this end, the methods available for generic quasilinear parabolic systems as used in section 4 are insufficient, and a suitable reformulation of the equation becomes necessary. The argument is based on the method of moving frames, which transforms (32) into a gauged complex Ginzburg–Landau equation with convection, and a priori estimates for suitable weighted-in-time Lebesgue and Sobolev norms. The result is an extension of [30], where global solvability of the free LLG (with $\boldsymbol{v} = 0$) has been investigated. The novelty is the appearance of certain convection-type terms in the transformed equation, which we shall estimate appropriately by using critical space-time norms of $\boldsymbol{v}$, compatible with the critical bounds for $\nabla \boldsymbol{m}$. Our main motivation is qualitative conditions under which singularities may or may not form under the influence of STT interactions. For simplicity we shall assume in addition

$$\boldsymbol{m}_0 - \boldsymbol{m}_\infty \in H^\infty(\mathbb{R}^3; \mathbb{R}^3) \quad \text{and} \quad \boldsymbol{v}_1, \boldsymbol{v}_2 \in C^0([0,\infty); H^\infty(\mathbb{R}^3; \mathbb{R}^3)),$$

where H^∞ is equipped with the usual Fréchet metric, so that, by virtue of Theorem 2,

$$\boldsymbol{m} \in C^0\left([0,T]; H^\infty(\mathbb{R}^3;\mathbb{S}^2)\right) \quad \text{for all} \quad T < T^*.$$

THEOREM 3. *There exist constants $\rho > 0$ and $c > 0$ with the following property: If*

$$\|\nabla \boldsymbol{m}_0\|_{L^3} + \sup_{t>0} \|\boldsymbol{v}(t)\|_{L^3} + \sup_{t>0} \sqrt{t}\, \|\boldsymbol{v}(t)\|_{L^\infty} < \rho,$$

then the solution in Theorem 2 extends to a global one such that, for all $t > 0$,

$$\sup_{t>0} \|\nabla \boldsymbol{m}(t)\|_{L^3} + \sup_{t>0} \sqrt{t}\; \|\nabla \boldsymbol{m}(t)\|_{L^\infty} \le c\, \|\nabla \boldsymbol{m}_0\|_{L^3}$$

and $\lim_{t\to\infty} \boldsymbol{m}(t) = \boldsymbol{m}_\infty$ uniformly. Moreover, $\boldsymbol{m}$ is unique in its class.

Remark. The smallness condition on the velocities $\boldsymbol{v}$ is given in terms of a space-time norm that corresponds to the natural space-time bounds for the gradient $\nabla \boldsymbol{m}$ of the solution $\boldsymbol{m}$, as predicted by the natural scaling of the equation.

Remark. Using an idea of Schoen and Uhlenbeck [35], it can be shown that maps $\boldsymbol{m} : \mathbb{R}^3 \to \mathbb{S}^2$ such that $\boldsymbol{m} - \boldsymbol{m}_\infty \in H^1 \cap W^{1,3}(\mathbb{R}^3;\mathbb{R}^3)$ can be approximated strongly by smooth maps $\boldsymbol{m}^h \in H^\infty(\mathbb{R}^3;\mathbb{S}^2)$. If $\boldsymbol{m} \in H^\sigma(\mathbb{R}^3;\mathbb{S}^2)$ for $\sigma > 1$, then the approximating sequence is bounded in H^σ; see [30, Lemma 8]. Therefore the results obtained for smooth initial data carry over to maps $\boldsymbol{m}_0 \in H^\sigma$ and $\boldsymbol{v} \in H^{\sigma-1}$.

5.1. Moving frames and the covariant Landau–Lifshitz system. We briefly discuss the method of moving frames that leads to a reformulation of (32) as a covariant complex Ginzburg–Landau equation. For a detailed discussion about the existence and regularity of a moving frame we refer to [6, 34]. Suppose

$$\boldsymbol{m} \in C^0\left([0,T]; H^\infty(\mathbb{R}^3;\mathbb{S}^2)\right) \quad \text{and} \quad \frac{\partial \boldsymbol{m}}{\partial t} \in C^0\left([0,T]; H^\infty(\mathbb{R}^3;\mathbb{R}^3)\right)$$

and the pair $\{X, Y\}$ forms an orthonormal tangent frame along $\boldsymbol{m}$, i.e.,

$$X, Y : [0,T] \times \mathbb{R}^3 \to \mathbb{S}^2 \quad \text{with} \quad \boldsymbol{m} \times X = Y$$

and has the same space-time regularity as $\boldsymbol{m}$. For space-time indices $\nu = 0, \dots, 3$ the coefficients a_ν of the associate connection form are given by

$$a_\nu = \langle \partial_\nu X, Y \rangle = -\langle \partial_\nu Y, X \rangle,$$

and induce a covariant derivative

$$\mathcal{D}_\nu = \partial_\nu + i a_\nu,$$

where $\partial_0 = \frac{\partial}{\partial t}$ and $\partial_k = \frac{\partial}{\partial x_k}$ for $k = 1, 2, 3$. The main task is to express (32) in terms of the complex derivative functions u_ν ($\nu = 0, \dots, 3$) defined by

$$u_\nu = \langle \partial_\nu \boldsymbol{m}, X \rangle + i \langle \partial_\nu \boldsymbol{m}, Y \rangle. \tag{41}$$

For the spatial parts of the connection form and the derivative functions we write $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ and $u = (u_1, u_2, u_3) \in \mathbb{C}^3$, respectively, and for $u, v \in \mathbb{C}^3$

$$u \cdot v = \sum_{k=1}^3 u_k v_k, \ \text{where } u_k v_k \text{ is the complex product.}$$

Second derivatives projected onto X and Y are expressed in terms of covariant derivatives applied to the derivative functions, i.e.,

$$\mathcal{D}_\mu u_\nu = \langle \partial_\mu \partial_\nu \boldsymbol{m}, X\rangle + i\,\langle \partial_\mu \partial_\nu \boldsymbol{m}, Y\rangle\,. \tag{42}$$

The fundamental relations between u_ν, a_ν, and $\mathcal{D}_\nu$ are the zero torsion condition

$$\mathcal{D}_\nu u_\mu = \mathcal{D}_\mu u_\nu \tag{43}$$

and the curvature identity

$$\mathcal{R}_{\nu\mu} := [\mathcal{D}_\nu, \mathcal{D}_\mu] = i\,(\partial_\nu a_\mu - \partial_\mu a_\nu) = i\,\mathrm{Im}(u_\nu \bar{u}_\mu). \tag{44}$$

Suppose $T \in (0, T^*)$ and $\boldsymbol{m}$ is a solution of (32) with $\boldsymbol{m}(0) = \boldsymbol{m}_0 \in H^\infty(\mathbb{R}^3; \mathbb{S}^2)$ as in Theorem 2. On $[0, T]$ the construction of a moving frame $\{X, Y\}$ along $\boldsymbol{m}$ as before can be carried out; see [6]. It is convenient to write $\{X, Y\}$ and $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ as complex vector fields

$$Z = X + iY \quad \text{and} \quad v = \boldsymbol{v}_1 + i\boldsymbol{v}_2.$$

Taking into account (41), (42), and the identities

$$\langle Z, \boldsymbol{m}\rangle = 0 \quad \text{and} \quad \langle Z, \boldsymbol{m} \times \xi\rangle = i\,\langle Z, \xi\rangle \quad \text{for all} \quad \xi \in \mathbb{R}^3,$$

(32) turns into the following system that we shall call the covariant LLS system.

Proposition 1. *For every $T \in (0, T^*)$ and $\nu = 0, \dots, 3$ we have*

$$u_\nu \in C^0([0, T] \times \mathbb{R}^3; \mathbb{C}) \quad \text{and} \quad a_\nu \in C^0([0, T] \times \mathbb{R}^3; \mathbb{R})$$

satisfying, for $v = \boldsymbol{v}_1 + i\boldsymbol{v}_2$, the covariant system

$$\begin{cases} u_0 + \sum_{k=1}^3 v_k u_k = (\alpha - i) \sum_{k=1}^3 \mathcal{D}_k u_k\,, \\ \mathcal{D}_\mu u_\nu = \mathcal{D}_\nu u_\mu\,, \\ \partial_\nu a_\mu - \partial_\mu a_\nu = \mathrm{Im}(u_\nu \bar{u}_\mu)\,. \end{cases} \tag{45}$$

Covariant differentiation of the first equation in (45) and using the zero torsion condition and curvature identity we find that $u = (u_1, u_2, u_3)$ solves the covariant complex Ginzburg–Landau equation

$$\mathcal{D}_t u_\ell + \mathcal{D}_\ell \sum_{k=1}^3 (v_k u_k) = (\alpha - i) \sum_{k=1}^3 (\mathcal{D}_k \mathcal{D}_k u_\ell + \mathcal{R}_{\ell k} u_k) \quad \text{for} \quad \ell \in \{1, 2, 3\} \tag{46}$$

and attains initial values

$$u(0) = \langle \nabla \boldsymbol{m}_0, X\rangle + i\langle \nabla \boldsymbol{m}_0, Y\rangle \quad \text{in} \quad H^\infty(\mathbb{R}^3; \mathbb{C}^3).$$

The covariant LLS system (45) is invariant with respect to the individual choice of the moving orthonormal frame $\{X, Y\}$. A local rotation of $\{X, Y\}$ expressed in terms of a rotation angle $\theta : \mathbb{R}^3 \times [0, T] \to \mathbb{R}$ gives rise to a gauge transformation

$$u_\nu \mapsto e^{-i\theta} u_\nu \quad \text{and} \quad a_\nu \mapsto a_\nu + \partial_\nu \theta. \tag{47}$$

Selecting a suitable gauge makes (45) a closed system and enables us in section 5.2 to find appropriate bounds for the nonlinear terms in (46). From these bounds, we shall obtain in section 5.3 a priori bounds for $u = (u_1, u_2, u_3)$ in weighted-in-time Lebesgue and Sobolev norms

$$\sup_{t\in(0,T)} t^{\frac{1-\delta}{2}} \|u(t)\|_{L^{\frac{3}{\delta}}} \quad \text{and} \quad \sup_{t\in(0,T)} \sqrt{t}\,\|\nabla u(t)\|_{L^3} \tag{48}$$

for suitable $\delta \in (\frac{1}{2}, 1)$, as a consequence of linear estimates and Duhamel's formula. These bounds will serve to obtain our final a priori estimate for

$$\sup_{t\in(0,T)} \|u(t)\|_{L^3} \quad \text{and} \quad \sup_{t\in(0,T)} \sqrt{t}\,\|u(t)\|_{L^\infty}.$$

5.2. Coulomb gauge and nonlinear estimates. A canonical choice of gauge for the covariant LLS system (45) is the Coulomb gauge [34, 44], which is characterized by the equation

$$\operatorname{div} a = \sum_{k=1}^{3} \frac{\partial a_k}{\partial x_k} = 0. \tag{49}$$

The Coulomb gauge can be obtained from our original frame $Z = X + iY$ by the local rotation $\theta \in C^0([0,T]; \dot{H}^1(\mathbb{R}^3))$ solving the elliptic equation

$$\Delta\theta + \operatorname{div} a = 0 \quad \text{in} \quad \mathbb{R}^3 \quad \text{for all} \quad t \in [0,T];$$

see [6, 30]. It follows from elliptic regularity that the new frame $e^{i\theta}Z$ inherits the regularity of the original frame $Z = X + iY$.

Having fixed the Coulomb gauge, we shall now obtain bounds for $a = (a_1, a_2, a_3)$ and a_0 in terms of (48). In the case $v = 0$, this has been done in [30, Lemma 3]. For $v \neq 0$ the bound on temporal component a_0 of the connection form will also depend on v. An appropriate choice of norm is

$$\sup_{t>0} t^{1-\delta} \|v(t)\|_{L^{\frac{3}{2\delta-1}}}$$

which is, by interpolation and according to the assumption in Theorem 3, a small quantity. We also remark that this norm corresponds to the bound we obtain for the spatial components $a = (a_1, a_2, a_3)$ of the connection form.

LEMMA 2. *Under the Coulomb gauge* (49) *we have, for* $\delta \in (\frac{1}{2}, 1)$,

$$\|a(t)\|_{L^{\frac{3}{2\delta-1}}} \le c\,\|u(t)\|^2_{L^{\frac{3}{\delta}}} \quad \textit{for all} \quad t \in [0,T]. \tag{50}$$

Moreover, there exists a decomposition $a_0 = a_0^{(1)} + a_0^{(2)}$ *such that for all* $t \in [0,T]$

$$\|a_0^{(1)}(t)\|_{L^{\frac{3}{\delta}}} \le c\,\|u(t)\|_{L^{\frac{3}{\delta}}} \|\nabla u(t)\|_{L^3} \tag{51}$$

and

$$\|a_0^{(2)}(t)\|_{L^{\frac{3}{4\delta-2}}} \le c\,\left(\|u(t)\|^2_{L^{\frac{3}{\delta}}} + \|v(t)\|_{L^{\frac{3}{2\delta-1}}}\right) \|u(t)\|^2_{L^{\frac{3}{\delta}}}. \tag{52}$$

The constants $c > 0$ *only depend on* α *and* δ.

Proof. The Coulomb gauge results in an elliptic equation for the connection terms

$$-\Delta a_\nu = \sum_{k=1}^{3} \frac{\partial}{\partial x_k} \operatorname{Im}(u_\nu \bar{u}_k) \quad \text{for} \quad \nu = 0, \ldots, 3. \tag{53}$$

Hence we fix t and suppress time dependence. Estimate (50) follows from standard elliptic theory, in particular the Hardy–Littlewood–Sobolev inequality. Estimates (51) and (52) also follow from elliptic theory upon substituting

$$u_0 = \sum_{k=1}^{3} (\alpha - i)\mathcal{D}_k u_k - v_k u_k,$$

which gives rise to a decomposition

$$-\Delta a_0^{(1)} = \operatorname{div}\Big(\alpha \operatorname{Im}\big(\bar u \operatorname{div} u\big) - \operatorname{Re}\big(\bar u \operatorname{div} u\big)\Big)$$

and

$$-\Delta a_0^{(2)} = \operatorname{div}\Big(\alpha \operatorname{Re}\big(\bar u\,(a\cdot u)\big) + \operatorname{Im}\big(\bar u\,(a\cdot u)\big) - \operatorname{Im}\big(\bar u\,(v\cdot u)\big)\Big).$$

We only consider the second equation involving v. By virtue of the Hölder and the Hardy–Littlewood–Sobolev inequality we have

$$\|a_0^{(2)}\|_{L^{\frac{3}{4\delta-2}}} \le c\, \big\||a||u|^2 + |v||u|^2\big\|_{L^{\frac{3}{4\delta-1}}} \le c\left(\|a\|_{L^{\frac{3}{2\delta-1}}} + \|v\|_{L^{\frac{3}{2\delta-1}}}\right)\|u\|^2_{L^{\frac{3}{\delta}}}$$

for a constant $c > 0$. Now the estimate follows from (50). ∎

Now we turn to the covariant Ginzburg–Landau system (46) which, under the Coulomb gauge (49), can be written as

$$\frac{\partial u}{\partial t} + (i-\alpha)\Delta u = F(a_0, a, u) \tag{54}$$

with the nonlinearity given by

$$F(a_0,a,u) = (\alpha - i)\left\{\mathcal{R}\,u + 2\,i\,(a\cdot\nabla)u - |a|^2 u\right\} - i\,a_0 u - (ia + \nabla)(v\cdot u),$$

where $\mathcal{R}_{\ell k} = i\operatorname{Im}(u_k \overline{u}_\ell)$ is the spatial part of the curvature tensor. Taking into account the decomposition of $a_0 = a_0^{(1)} + a_0^{(2)}$ from Lemma 2 we find that F splits into seven terms that we shall merge into four terms according to the type of bounds they satisfy, respectively, i.e.,

$$F(a_0,a,u) = f^{(1)} + f^{(2)} + f^{(3)} - \nabla g,$$

where

$$\begin{aligned}
f^{(1)} &:= (\alpha - i)\mathcal{R}\,u,\\
f^{(2)} &:= (\alpha - i)\; 2i\,(a\cdot\nabla)u - i\,a_0^{(1)}\,u,\\
f^{(3)} &:= -(\alpha - i)\;|a|^2 u - i\,a_0^{(2)}\,u - i\,a\,(v\cdot u),
\end{aligned}$$

and $g := v\cdot u$. In fact, we obtain the following from Lemma 2 and Hölder's inequality.

Lemma 3. *Suppose $\delta \in (\frac12, 1)$. There exists a constant c such that for all $t \in [0,T]$*

$$\|f^{(1)}(t)\|_{L^{\frac{1}{\delta}}} \le c\,\|u(t)\|^3_{L^{\frac{3}{\delta}}}, \tag{55}$$

$$\|f^{(2)}(t)\|_{L^{\frac{3}{2\delta}}} \le c\,\|u(t)\|^2_{L^{\frac{3}{\delta}}}\|\nabla u(t)\|_{L^3}, \tag{56}$$

$$\|f^{(3)}(t)\|_{L^{\frac{3}{5\delta-2}}} \le c\,\|u(t)\|^5_{L^{\frac{3}{\delta}}} + c\,\|v(t)\|_{L^{\frac{3}{2\delta-1}}}\|u(t)\|^3_{L^{\frac{3}{\delta}}}, \tag{57}$$

$$\|g(t)\|_{L^{\frac{3}{3\delta-1}}} \le c\,\|v(t)\|_{L^{\frac{3}{2\delta-1}}}\;\|u(t)\|_{L^{\frac{3}{\delta}}}. \tag{58}$$

Thus we are led to consider

$$\frac{\partial u}{\partial t} + (i-\alpha)\Delta u = f - \nabla g \tag{59}$$

with $f = f^{(1)} + f^{(2)} + f^{(3)}$ and $g = v \cdot u$ satisfying the above L^p bounds.

5.3. Linear estimates of Fujita–Kato type. In this section we finish the proof of Theorem 3. The following estimates, based on mapping properties of the linear semigroup and weighted-in time Lebesgue spaces, are inspired by the fundamental work by Fujita and Kato [14] and Kato [18] on the Navier–Stokes equation.

Let us first recall crucial mapping properties of the dissipative Schrödinger semigroup $S = S(t)$ which is generated by $(\alpha - i)\Delta$ for $\alpha > 0$. As $S(t)f = S_t * f$, i.e., convolution in space with $S_t(x) = t^{-\frac{n}{2}} S(x/\sqrt{t})$ for some Schwartz function S on $\mathbb{R}^3$, the following estimates can be obtained from Young's convolution inequality and a scaling argument.

LEMMA 4. *Suppose $\alpha > 0$, $1 \le p \le q \le \infty$, and $k \in \mathbb{N}_0$. Then there exists a constant $c > 0$ such that (on $\mathbb{R}^3$)*

$$\|\nabla^k S(t)\|_{\mathcal{L}(L^p;L^q)} \le c\, t^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{k}{2}} \quad \textit{for all} \quad t > 0.$$

Duhamel's representation formula applied to (59) yields

$$u(t) = S(t)u(0) + (S * f)(t) + (\nabla S * g)(t), \tag{60}$$

where

$$(S * f)(t) := \int_0^t S(t-s) f(s)\, ds, \tag{61}$$

i.e., convolution in space and time. Based on this representation we shall derive a priori estimates in suitable (scaling invariant) weighted-in-time Lebesgue and Sobolev spaces; see (48). For this purpose we set, for $\delta \in (\frac{1}{2}, 1)$ and $t \in [0, T]$,

$$R(t) = \max\left\{ \sup_{\tau \in (0,t)} \tau^{\frac{1-\delta}{2}} \|u(\tau)\|_{L^{\frac{3}{\delta}}},\ \sup_{\tau \in (0,t)} \sqrt{\tau}\, \|\nabla u(\tau)\|_{L^3} \right\}$$

and

$$V(t) = \sup_{\tau \in (0,t)} \tau^{1-\delta} \|v(\tau)\|_{L^{\frac{3}{2\delta-1}}}.$$

As $u, v \in C^0([0,T]; H^{\sigma-1}(\mathbb{R}^3; \mathbb{C}^3))$, it follows from Sobolev embedding that

$$R : [0,T] \to [0,\infty) \quad \text{and} \quad V : [0,T] \to [0,\infty)$$

are continuous nondecreasing functions with

$$\lim_{t \searrow 0} R(t) = \lim_{t \searrow 0} V(t) = 0.$$

By Hölder's inequality we have

$$V(t) \le \max\left\{ \sup_{\tau \in (0,t)} \|v(\tau)\|_{L^3},\ \sup_{\tau \in (0,t)} \sqrt{\tau}\, \|v(\tau)\|_{L^\infty} \right\} \tag{62}$$

corresponding to the bounds on v required in Theorem 3. We also introduce

$$R_0(t) = \max\left\{ \sup_{\tau\in(0,t)} \tau^{\frac{1-\delta}{2}} \|S(\tau)u(0)\|_{L^{\frac{3}{\delta}}} \,,\ \sup_{\tau\in(0,t)} \sqrt{\tau}\, \|\nabla S(\tau)u(0)\|_{L^3} \right\}.$$

According to Lemma 4 we have

$$R_0(t) \le c\,\|u(0)\|_{L^3} \tag{63}$$

independently of $t > 0$ and for a constant c that only depends on δ and α.

LEMMA 5. *Suppose* $\gamma \in (0, \frac{1}{5})$ *and* $\delta \in (\frac{3}{5}, \frac{2-\gamma}{3})$. *Then there exists a constant* $c > 0$ *such that for every* $t \in (0, T]$

$$R(t) \le R_0(t) + c\left(V(t) + R(t)^2\right)\left(1 + R(t)^2\right) R(t), \tag{64}$$

$$\|u(t)\|_{L^3} \le c\Big(\|u(0)\|_{L^3} + \left(V(t) + R(t)^2\right)\left(1 + R(t)^2\right) R(t)\Big), \tag{65}$$

$$\|\nabla u(t)\|_{L^{\frac{3}{1-\gamma}}} \le \frac{c}{t^{\frac{1+\gamma}{2}}}\Big(\|u(0)\|_{L^3} + \left(V(t) + R(t)^2\right)\left(1 + R(t)^2\right) R(t)\Big). \tag{66}$$

Moreover, there exists a constant $r_0 > 0$ *with the following property:*

$$\text{If } \sup_{t\in(0,T)} \left(R_0(t) + V(t)\right) < r_0, \quad \text{then} \quad R(t) \le 2\,R_0(t) \quad \text{for every} \quad t \in [0, T]. \tag{67}$$

Proof. Taking gradients and L^p-norms in Duhamel's formula (60) we obtain

$$\left\|\nabla^k u(t)\right\|_{L^p} \le \left\|\nabla^k S(t)u(0)\right\|_{L^p} + \int_0^t \left\|\nabla^k S(t-s)f(s)\right\|_{L^p} ds + \int_0^t \left\|\nabla^{k+1} S(t-s)g(s)\right\|_{L^p} ds$$

valid for all $k \in \mathbb{N}_0$, $p \ge 1$, and $t \in (0, T]$. Then the desired bounds follow easily from Lemmas 3 and 4. The estimates concerning the first and second term on the right were obtained (with $V(t) \equiv 0$) in [30, Lemma 6]. The only difference in the result is a new term $V(t)R(t)^3$ coming from the contribution of v in $f^{(3)}$. Hence we focus on the new estimates involving g, which result in bounds in terms of $V(t)R(t)$. For all $t \in (0, T]$ and a generic constant c that only depends on γ and δ we have

$$\|(\nabla S * g)(t)\|_{L^{\frac{3}{\delta}}} \le c\,V(t)R(t) \int_0^t (t-s)^{-\delta} s^{\frac{3}{2}(\delta-1)}\, ds = c\,V(t)R(t)\, t^{\frac{\delta-1}{2}}$$

and

$$\left\|\left(\nabla^2 S * g\right)(t)\right\|_{L^3} \le c\,V(t)R(t) \int_0^t (t-s)^{-\frac{3\delta}{2}} s^{\frac{3}{2}(\delta-1)}\, ds = c\,V(t)R(t)\, t^{-\frac{1}{2}}$$

which imply (64). Similarly (65) and (66) follow, respectively, from

$$\|(\nabla S * g)(t)\|_{L^3} \le c\,V(t)R(t) \int_0^t (t-s)^{\frac{1-3\delta}{2}} s^{\frac{3}{2}(\delta-1)}\, ds = c\,V(t)R(t)$$

and

$$\left\|\left(\nabla^2 S * g\right)(t)\right\|_{L^{\frac{3}{1-\gamma}}} \le c\,V(t)R(t) \int_0^t (t-s)^{-\frac{1}{2}(3\delta+\gamma)} s^{\frac{3}{2}(\delta-1)}\, ds = c\,V(t)R(t)\, t^{-\frac{1+\gamma}{2}}.$$

Note that the restriction on γ and δ in Lemma 5 guarantees that all time integrals are finite.

To prove (67) we follow the continuity argument in [30]. Assuming there exists $t_0 \in (0,T)$ such that $R(t_0) = 2R_0(t_0) \neq 0$, we would have, by virtue of (64),

$$2R_0(t_0) \leq R_0(t_0) + 2c\left(V(t_0) + 4R_0(t_0)^2\right)\left(1 + 4R_0(t_0)^2\right) R_0(t_0).$$

Thus, for $0 < R_0(t_0) + V(t_0) < r_0$ with r_0 sufficiently small,

$$1/c \leq 2\left(V(t_0) + 4R_0(t_0)^2\right)\left(1 + 4R_0(t_0)^2\right) < 1/c,$$

a contradiction. Hence the continuous curves $t \mapsto R(t)$ and $t \mapsto 2R_0(t)$ cannot intersect in $(0,T)$. Now suppose that $R(t) > 2R_0(t)$ for all $t \in (0,T)$. Then again by (64) we would have

$$1 < \frac{1}{2} + c\left(V(t) + R(t)^2\right)\left(1 + R(t)^2\right) \quad \text{for all} \quad t \in (0,T).$$

Since $\lim_{t\searrow 0} R(t) = \lim_{t\searrow 0} V(t) = 0$, this is impossible, and the claim follows from the continuity of R and R_0. ◻

Proposition 2. *There exist positive constants ρ and c depending only on α with the following property: If*

$$\|u(0)\|_{L^3} + \sup_{t>0} \|v(t)\|_{L^3} + \sup_{t>0} \sqrt{t}\, \|v(t)\|_{L^\infty} < \rho, \tag{68}$$

then the following estimate holds true for every $t \in [0,T]$,

$$\sqrt{t}\, \|u(t)\|_{L^\infty} + \|u(t)\|_{L^3} \leq c\, \|u(0)\|_{L^3}.$$

Proof. Recall that by (62) and (63) the smallness condition in (67) can be expressed in terms of (68), defining the constant ρ. Then the bound on the L^3 norm follows immediately from (63), (65), and (67). The L^∞ bound follows from Morrey's inequality, (66), (67), and a scaling argument. In fact,

$$\|u(t)\|_{L^\infty} \leq \|u(t)\|_{C^\gamma} \leq c\left(\|u(t)\|_{L^3} + \|\nabla u(t)\|_{L^{\frac{3}{1-\gamma}}}\right),$$

hence

$$\|u(t)\|_{L^\infty} \leq c\ \|u(t)\|_{L^3}^{\frac{\gamma}{1+\gamma}}\, \|\nabla u(t)\|_{L^{\frac{3}{1-\gamma}}}^{\frac{1}{1+\gamma}} \leq \frac{c}{\sqrt{t}} \|u(0)\|_{L^3},$$

(see [39, p. 9]), for all $t \in (0,T]$. The proof is complete. ◻

Proof of Theorem 3. Since $|u| = |\nabla \boldsymbol{m}|$ and $|v| = |\boldsymbol{v}|$, the claim of Proposition 2 holds true with u replaced by $\nabla \boldsymbol{m}$ and v replaced by $\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2)$. By Theorem 2, $\boldsymbol{m}$ extends to a global solution, and the bounds on $\nabla \boldsymbol{m}$ hold true for every $T > 0$ with a universal constant, which is the assertion of Theorem 3.

Remark. The method of moving frames as used here is tailored to variants of harmonic flows into $\mathbb{S}^2$ which are formulated in terms of space-time derivatives $\partial_\alpha \boldsymbol{m}$ and $\boldsymbol{m} \times \partial_\alpha \boldsymbol{m}$, respectively. Taking into account further contributions to the effective field requires additional perturbation arguments (possibly including commutator estimates as in [11, 26]), suitable for the nonlocal nature of micromagnetic equations, and will be considered elsewhere.

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