Research Article

Mathematical Methods in the Applied Sciences

Received 7 May 2014 Published online in Wiley Online Library

(wileyonlinelibrary.com) DOI: 10.1002/mma.3330
MOS subject classification: 35Q30

Integral representation of a solution to the Stokes–Darcy problem

Dagmar Medková[a*†], Mariya Ptashnyk[b] and Werner Varnhorn[c]

Communicated by W. L. Wendland

With methods of potential theory, we develop a representation of a solution of the coupled Stokes–Darcy model in a Lipschitz domain for boundary data in $H^{-1/2}$. Copyright © 2014 John Wiley & Sons, Ltd.

Keywords: Stokes–Darcy problem; boundary layer potential

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, that is, a bounded open connected set, with Lipschitz boundary $\partial\Omega$, and suppose that Ω_S is a nonempty subdomain of Ω with Lipschitz boundary $\partial\Omega_S$ such that $\Omega_S \neq \Omega$. Then, $\Omega_D := \Omega \setminus \overline{\Omega_S}$ is a bounded open set, not necessarily connected. We suppose that Ω_D has Lipschitz boundary. Remark that $\partial\Omega_S \cap \partial\Omega_D \cap \Omega$ is always nonempty, and it is locally a graph of Lipschitz function. Let Γ be a nonempty closed subset of $\partial\Omega_S \cap \partial\Omega_D$. Then, Γ might reach $\partial\Omega$ or not (See Figure 1).

We want to study the following problem

$$-\eta\Delta \boldsymbol{v}^S + \nabla q^S = 0, \quad \text{div}\, \boldsymbol{v}^S = 0 \qquad \text{in } \Omega_S,$$

$$\boldsymbol{v}^D + k\nabla q^D = 0, \quad \text{div}\, \boldsymbol{v}^D = 0 \qquad \text{in } \Omega_D,$$

$$\boldsymbol{v}^S = \boldsymbol{f} \quad \text{on } \partial\Omega_S \setminus \Gamma,$$

$$\boldsymbol{v}^D \cdot \boldsymbol{n} = h^D \quad \text{on } \partial\Omega_D \setminus \Gamma,$$

$$\boldsymbol{v}^D \cdot \boldsymbol{n} - \boldsymbol{v}^S \cdot \boldsymbol{n} = h^\Gamma, \quad \boldsymbol{v}^S_{\boldsymbol{\tau}} = \boldsymbol{f}_{\boldsymbol{\tau}} \quad \text{on } \Gamma,$$

$$\left[\left(-2\eta\, \mathbf{D}\, \boldsymbol{v}^S + q^S I\right)\boldsymbol{n}\right] \cdot \boldsymbol{n} = \quad q^D + \boldsymbol{v}^D \cdot \boldsymbol{n} - \tilde{\boldsymbol{g}} \cdot \boldsymbol{n} \quad \text{on } \Gamma,$$

where I is the identity matrix and

$$\mathbf{D}\boldsymbol{v} = \frac{1}{2}[\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T]$$

is the symmetric gradient of $\boldsymbol{v}$ (i.e., also a matrix). Putting $q^S = \eta p^S$ and $q^D = p^D/k$, we obtain the following coupled Stokes–Darcy problem:

$$-\Delta \boldsymbol{v}^S + \nabla p^S = 0, \quad \text{div}\, \boldsymbol{v}^S = 0 \qquad \text{in } \Omega_S, \tag{1}$$

[a] Mathematical Institute, Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Prague, Czech Republic
[b] Division of Mathematics, University of Dundee, Dundee, DD1 4HN, Scotland, UK
[c] Institute of Mathematics, University of Kassel, Kassel, 34109, Germany
* Correspondence to: Dagmar Medková, Mathematical Institute, Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Prague, Czech Republic.
† E-mail: medkova@math.cas.cz

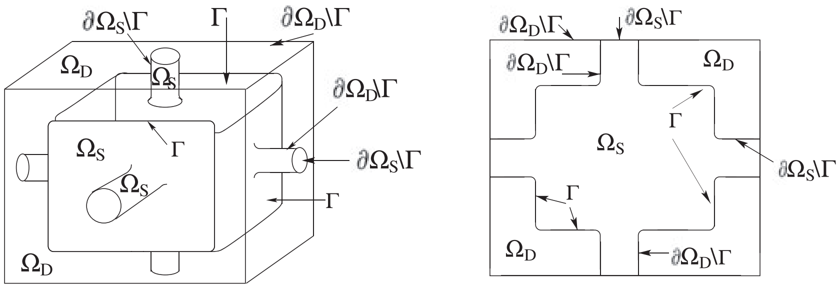

Figure 1. Representative geometry of Ω.

$$\boldsymbol{v}^D + \nabla p^D = 0, \quad \operatorname{div} \boldsymbol{v}^D = 0 \qquad \text{in } \Omega_D, \tag{2}$$

$$\boldsymbol{v}^S = \boldsymbol{f} \quad \text{on } \partial\Omega_S \setminus \Gamma, \tag{3}$$

$$\boldsymbol{v}^D \cdot \boldsymbol{n} = h^D \quad \text{on } \partial\Omega_D \setminus \Gamma, \tag{4}$$

$$\boldsymbol{v}^D \cdot \boldsymbol{n} - \boldsymbol{v}^S \cdot \boldsymbol{n} = h^\Gamma, \quad \boldsymbol{v}^S_{\boldsymbol{\tau}} = \boldsymbol{f}_{\boldsymbol{\tau}} \quad \text{on } \Gamma, \tag{5}$$

$$\eta \left[T\left(\boldsymbol{v}^S, p^S\right) \boldsymbol{n} \right] \cdot \boldsymbol{n} + p^D/k + \boldsymbol{v}^D \cdot \boldsymbol{n} = \boldsymbol{g} \cdot \boldsymbol{n} \quad \text{on } \Gamma. \tag{6}$$

Here, η and k are positive constants, $\boldsymbol{v}^D = \left(v_1^D, v_2^D, v_3^D\right)$ denotes the Darcy velocity vector, and $\boldsymbol{v}^S = \left(v_1^S, v_2^S, v_3^S\right)$ represents the Stokes flow, whereas

$$T(\boldsymbol{v}, p) = 2\mathbf{D}\boldsymbol{v} - pI,$$

is the stress tensor. By $\boldsymbol{n} = \boldsymbol{n}^S$, we mean the exterior unit normal vector of Ω_S. If $\boldsymbol{v}$ is a vector function on $\partial\Omega_S$, then $\boldsymbol{v} \cdot \boldsymbol{n}$ denotes the scalar product of $\boldsymbol{v}$ and $\boldsymbol{n}$, that is, $\boldsymbol{v} \cdot \boldsymbol{n}$ is a scalar function. Denote by $\boldsymbol{v}_{\boldsymbol{n}}$ the normal part of $\boldsymbol{v}$ and by $\boldsymbol{v}_{\boldsymbol{\tau}}$ the tangential part of $\boldsymbol{v}$, that is, $\boldsymbol{v}_{\boldsymbol{n}}$ and $\boldsymbol{v}_{\boldsymbol{\tau}}$ are vectors, $\boldsymbol{v} = \boldsymbol{v}_{\boldsymbol{n}} + \boldsymbol{v}_{\boldsymbol{\tau}}$, $\boldsymbol{v}_{\boldsymbol{n}} = (\boldsymbol{v} \cdot \boldsymbol{n})\, \boldsymbol{n}$, $\boldsymbol{v}_{\boldsymbol{\tau}} = \boldsymbol{v} - \boldsymbol{v}_{\boldsymbol{n}}$.

If $\overline{\Omega_S} \subset \Omega$ and $\partial\Omega_S = \Gamma$, then the condition (3) disappears. (For example, if $\Omega = \{\boldsymbol{x} \in \mathbb{R}^3; |\boldsymbol{x}| < 2\}$, $\Omega_S = \{\boldsymbol{x} \in \mathbb{R}^3; |\boldsymbol{x}| < 1\}$, $\Omega_D = \{\boldsymbol{x} \in \mathbb{R}^3; 1 < |\boldsymbol{x}| < 2\}$, $\Gamma = \{\boldsymbol{x} \in \mathbb{R}^3; |\boldsymbol{x}| = 1\}$.) If $\overline{\Omega_D} \subset \Omega$ and $\partial\Omega_D = \Gamma$, then the condition (1) disappears. (For example, if $\Omega = \{\boldsymbol{x} \in \mathbb{R}^3; |\boldsymbol{x}| < 2\}$, $\Omega_D = \{\boldsymbol{x} \in \mathbb{R}^3; |\boldsymbol{x}| < 1\}$, $\Omega_S = \{\boldsymbol{x} \in \mathbb{R}^3; 1 < |\boldsymbol{x}| < 2\}$, $\Gamma = \{\boldsymbol{x} \in \mathbb{R}^3; |\boldsymbol{x}| = 1\}$.) In all other cases, we have all conditions (3)–(6). The interface Γ might reach the boundary ($\Omega = \{\boldsymbol{x} \in \mathbb{R}^3; -1 < x_j < 1\}$, $\Omega_S = \{\boldsymbol{x} \in \Omega; x_1 < 0\}$, $\Omega_D = \{\boldsymbol{x} \in \Omega; 0 < x_1\}$, $\Gamma = \{\boldsymbol{x} \in \mathbb{R}^m; x_1 = 0, |x_2| \le 1, |x_3| \le 1\}$) or might not reach the boundary ($\Omega = \{\boldsymbol{x} \in \mathbb{R}^3; 1 < |\boldsymbol{x}| < 3\}$, $\Omega_S = \{\boldsymbol{x} \in \mathbb{R}^3; 2 < |\boldsymbol{x}| < 3\}$, $\Omega_D = \{\boldsymbol{x} \in \mathbb{R}^3; 1 < |\boldsymbol{x}| < 2\}$, $\Gamma = \{\boldsymbol{x} \in \mathbb{R}^3; |\boldsymbol{x}| = 2\}$, $\partial\Omega_D \setminus \Gamma = \{\boldsymbol{x} \in \mathbb{R}^3; |\boldsymbol{x}| = 1\}$, $\partial\Omega_S \setminus \Gamma = \{\boldsymbol{x} \in \mathbb{R}^3; |\boldsymbol{x}| = 3\}$).

The aforementioned problem arises from the modeling of water flow through a tissue of plant cells [1]. Water flow in plant tissues takes place in two different physical domains separated by semipermeable membranes, denoted as *symplast* and *apoplast* [2]. The apoplast is composed of cell walls and intercellular spaces, while the symplast is constituted by cell insides, which can be connected by plasmodesmata. The complex microstructure of the cell walls, composed of polymers and microfibrils, can in simplified form be represented as a porous medium. The water flow in the cell walls can be modeled by Darcy's law. The Stokes equations can be used to describe viscous flow in the cell cytoplasm.

Coupled free fluid and porous media problems have received an increasing attention during the last years both from the mathematical and the numerical point of view. Well-posedness analysis and numerical algorithms for coupled Stokes–Darcy and Navier–Stokes–Darcy problems with Beavers–Joseph–Saffman transmission conditions between the free fluid and the porous medium have been investigated in [3–5] and the references therein. Multiscale analysis for the Stokes–Darcy system modeling water flow in a vuggy porous media with Beavers–Joseph–Saffman transmission condition was considered in [6].

The main difference of our problem to the well-known models coupling free fluid and porous media [6, 7] is that the free fluid and the porous media domains do not interact directly, as the membrane separates the domains and controls actively and passively the fluxes of the water and the solutes. Thus, the continuity of the normal forces and the Beavers–Joseph–Saffman transmission condition between the free fluid and the porous medium does not apply. The regulation of the water flow from the cell symplast into the cell wall apoplast is represented via the transmission conditions on the boundary Γ, comprising the normal component of the Darcy velocity $\boldsymbol{v}^D \cdot \boldsymbol{n}$ and a given function $\boldsymbol{g} \cdot \boldsymbol{n}$, which corresponds to the difference between the solute concentrations in the symplast and the apoplast, respectively [1]. The transmission conditions at the cell-membrane-cell wall interface and the coupling between the flow velocity and the solute concentrations via transmission conditions reflect the osmotic nature of the water flow through a semipermeable membrane.

The aim of the paper is to study the solvability of the coupled Stokes–Darcy model problem (1)–(6) and to develop an integral representation of the solution of this problem. It is important for calculation of a solution using the boundary element method [8, 9]. At first, we determine necessary and sufficient conditions for the existence of a solution $\boldsymbol{v}^S \in [H^1(\Omega_S)]^3$, $p^S \in L^2(\Omega_S)$, $p^D \in H^1(\Omega_D)$, $\boldsymbol{v}^D \in [L^2(\Omega_D)]^3$ of (1)–(6) for $\boldsymbol{g} \in [H^{-1/2}(\partial\Omega_S)]^3$, $\boldsymbol{f} \in [H^{1/2}(\partial\Omega_S)]^3$, and $h \in H^{-1/2}(\partial\Omega_D)$. We prove the existence of the problem (1)–(6) by the integral equation method. We show that the velocity fields and the pressures of a solution of the problem (1)–(6) can be represented in terms of boundary single layer potentials, such that the Darcy pressure $p^D = \mathcal{S}_{\Omega_D}\psi$ is a harmonic single layer potential with density $\psi \in H^{-1/2}(\partial\Omega_D)$, while the velocity field for the Darcy flow is defined by $\boldsymbol{v}^D = \nabla\mathcal{S}_{\Omega_D}\psi$. For the Stokes flow, we obtain that $[\boldsymbol{v}^S, p^S] = \tilde{E}_{\Omega_S}\Psi$ is a modified hydrodynamical single layer potential with density $\Psi \in [H^{-1/2}(\partial\Omega_S)]^3$.

To derive integral representations for the solutions of the model (1)–(6), we study two auxiliary problems: the Robin problem for the Laplace equation and the mixed Navier–Dirichlet problem for the Stokes system. It is a tradition to study the Dirichlet and the Neumann problems for the Laplace equation in different spaces by the integral equation method [10–12]. Later, a solution of the Robin problem for the Laplace equation has been looked for in the form of a harmonic single layer potential for boundary conditions given by real measures [13–15] or p-integrable functions on the boundary [16–18]. The classical result of the theory of partial differential equations says that the Robin problem for the Laplace equation is uniquely solvable in $H^1(\Omega)$ [19]. It was shown in [20–22] that a solution of the Neumann problem for the Laplace equation in $H^1(\Omega)$ has the form of a harmonic single layer potential with density from $H^{-1/2}(\partial\Omega)$. All these results enable us to show that each solution of the Robin problem in $H^1(\Omega)$ is representable by a harmonic single layer potential with density $\psi \in H^{-1/2}(\partial\Omega)$, and the corresponding integral operator is continuously invertible.

The potential theory for the hydrodynamics was first developed to study classical solutions of the Dirichlet and Neumann problems for the Stokes system [23–27]. Later, solutions of the Dirichlet problem, the Neumann problem, and the transmission problem for the Stokes system have been looked for in the form of hydrodynamical boundary layers also for p-integrable boundary conditions and for solutions from Sobolev and Besov spaces [28–34]. We have used this theory to study a solution $(\boldsymbol{v}, p) \in [H^1(\Omega)]^3 \times L^2(\Omega)$ of the Navier–Dirichlet problem for the Stokes system. We have proved that the Navier–Dirichlet problem for the Stokes system is uniquely solvable and the corresponding solution can be represented using a modified hydrodynamic single layer potential with density $\Psi \in [H^{-1/2}(\partial\Omega)]^3$, and the corresponding integral operator is continuously invertible, too.

2. Single layer potentials

For $0 \neq \boldsymbol{x} \in \mathbb{R}^3$ consider the fundamental solution $h_\triangle$ of the Laplace equation $-\Delta u = 0$, defined by

$$h_\triangle(\boldsymbol{x}) = \frac{1}{4\pi|\boldsymbol{x}|}\,.$$

Assume that $G \subset \mathbb{R}^3$ is a bounded open set with Lipschitz boundary. Then for $\psi \in H^{-1/2}(\partial G)$, we can define the harmonic single layer potential with density ψ as the convolution $\mathcal{S}_G\psi = h_\triangle * \psi$. In particular, if $\psi \in L^2(\partial G)$, then

$$(\mathcal{S}_G\psi)(\boldsymbol{x}) = \int_{\partial G} h_\triangle(\boldsymbol{x}-\boldsymbol{y})\psi(\boldsymbol{y})\,d\sigma_{\boldsymbol{y}} \qquad \text{for} \quad \mathbf{x} \in G. \tag{7}$$

If $\psi \in H^{-1/2}(\partial G)$, then $u := \mathcal{S}_G\psi$ is a solution of the Dirichlet problem for the Laplace equation

$$\begin{aligned} -\Delta u &= 0 \qquad \text{in} \quad G\,, \\ u &= \operatorname{tr}(\mathcal{S}_G\psi) \quad \text{on} \quad \partial G\,, \end{aligned}$$

where $\operatorname{tr}(\mathcal{S}_G\psi) \in H^{1/2}(\partial G)$ denotes the usual trace of $\mathcal{S}_G\psi \in W^{1,2}(G)$ (see, e.g., [9, Lemma 6.6]).

For $\psi \in L^2(\partial G)$ and $\boldsymbol{x} \in \partial G$, we set

$$K_G^\triangle\psi(\boldsymbol{x}) = \lim_{r\downarrow 0} \int\limits_{\partial G\setminus B(\boldsymbol{x};r)} \frac{\boldsymbol{n}^G(\boldsymbol{x})\cdot(\boldsymbol{x}-\boldsymbol{y})}{4\pi|\boldsymbol{x}-\boldsymbol{y}|^3}\psi(\boldsymbol{y})\,\mathrm{d}\sigma_{\boldsymbol{y}} \tag{8}$$

with $\boldsymbol{n}^G(\boldsymbol{x})$ as the exterior unit normal vector with respect to G and $B(\boldsymbol{x};r)$ as the ball with radius $r > 0$ and center at $x \in \mathbb{R}^3$. This limit is defined for almost all $\boldsymbol{x} \in \partial G$, and $K_G^\triangle$ is a bounded linear operator on $L^2(\partial G)$, which can be extended to a bounded linear operator on $H^{-1/2}(\partial G)$ (see, e.g., [8, Theorem5.6.2]). For a harmonic function $u \in W^{1,2}(G)$ and $g \in H^{-1/2}(\partial G)$, we have that $\nabla u\cdot\boldsymbol{n} = g$ if and only if

$$\int_G \nabla u\cdot\nabla\varphi\,\mathrm{d}\boldsymbol{x} = \langle g, \operatorname{tr}(\varphi)\rangle_{H^{-1/2},H^{1/2}} \qquad \forall\varphi \in W^{1,2}(G).$$

See [19] for details. Thus, we can conclude that for $\psi \in H^{-1/2}(\partial G)$ it holds

$$\nabla(\mathcal{S}_G\psi)\cdot\boldsymbol{n} = \frac{\psi}{2} - K_G^\triangle\psi \quad \text{on} \quad \partial G. \tag{9}$$

See [9, Lemma 6.8].

Next, we consider the (4×3) fundamental tensor E of the Stokes system, given by

$$E_{j,k}(\boldsymbol{x}) = \frac{1}{8\pi}\left\{\delta_{jk}\frac{1}{|\boldsymbol{x}|} + \frac{x_j x_k}{|\boldsymbol{x}|^3}\right\}, \quad E_{4,k}(\boldsymbol{x}) = \frac{x_k}{4\pi|\boldsymbol{x}|^3} \quad \text{for } 0 \neq \mathbf{x} \in \mathbb{R}^3,\ j,k = 1,2,3. \tag{10}$$

Then for $\Psi = [\Psi_1, \Psi_2, \Psi_3] \in [H^{-1/2}(\partial G)]^3$, we can define the hydrodynamical single layer potential with density Ψ as the convolution $E_G\Psi = E * \Psi$. In particular, for $\Psi \in [L^2(\partial G)]^3$, we obtain

$$(E_G\Psi)(\boldsymbol{x}) = \int_{\partial G} E(\boldsymbol{x}-\boldsymbol{y})\Psi(\boldsymbol{y})\, d\sigma_{\boldsymbol{y}}. \tag{11}$$

By $E_G^{\bullet}\Psi = E' * \Psi$, we denote the velocity part of this potential, that is, the three components of the velocity field. Here, the 3×3 matrix $E'(\boldsymbol{z})$ is obtained from $E(\boldsymbol{z})$ by eliminating the last row, which corresponds to the pressure part.

If $\Psi \in [H^{-1/2}(\partial G)]^3$, then for $\boldsymbol{v} = E_G^{\bullet}\Psi$ and $p = [E_G\Psi]_4$, we obtain that $\boldsymbol{v} \in [W^{1,2}(G)]^3$, $p \in L^2(G)$ is a solution of the Stokes system

$$\begin{aligned} \Delta \boldsymbol{v} &= \nabla p, && \text{in } G\,, \\ \operatorname{div} \boldsymbol{v} &= 0 && \text{in } G\,, \\ \boldsymbol{v} &= \operatorname{tr}\left(E_G^{\bullet}\Psi\right) && \text{on } \partial G\,. \end{aligned}$$

See [9, §6.8] or [33, Theorem 4.4] for details.

For $\boldsymbol{x}, \boldsymbol{y} \in \partial G$, $\boldsymbol{y} \neq \boldsymbol{x}$ and $j,k = 1,2,3$ we consider the kernel matrix

$$K_{jk}^S(\boldsymbol{x},\boldsymbol{y}) = \frac{3}{4\pi}\frac{(x_j-y_j)(x_k-y_k)(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{n}^G(\boldsymbol{x})}{|\boldsymbol{x}-\boldsymbol{y}|^5}\,,$$

and for $\Psi \in [L^2(\partial G)]^3$ and $\boldsymbol{x} \in \partial G$, we set

$$K_G^S\Psi(\boldsymbol{x}) = \lim_{r\downarrow 0}\int_{\partial G\setminus B(\boldsymbol{x};r)} K^S(\boldsymbol{x},\boldsymbol{y})\Psi(\boldsymbol{y})\,\mathrm{d}\sigma_{\boldsymbol{y}}.$$

The limit in the last equality is well defined for almost all $\boldsymbol{x} \in \partial G$, and K_G^S is a bounded linear operator on $[L^2(\partial G)]^3$ [29, 33, 35], which can be extended to a bounded linear operator on $[H^{-1/2}(\partial G)]^3$ [36].

For $\boldsymbol{u} \in [W^{1,2}(G)]^3$, $p \in L^2(G)$, and $\boldsymbol{g} \in [H^{-1/2}(\partial G)]^3$, we have that $T(\boldsymbol{u},p)\,\boldsymbol{n} = \boldsymbol{g}$ if and only if

$$2\int_G \mathbf{D}\,\boldsymbol{u} : \mathbf{D}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{y} - \int_G p\ \operatorname{div}\boldsymbol{v}\,\mathrm{d}\boldsymbol{y} = \langle \boldsymbol{g}, \boldsymbol{v}\rangle_{H^{-1/2},H^{1/2}} \qquad \forall \boldsymbol{v} \in [H^1(G)]^3,$$

see [36] for details, where here and in the following we use $A : B = \sum_{i,j=1}^3 A_{ij}B_{ij}$ for 3×3 matrices A, B. Thus, using [36, Proposition 4.2], for $\Psi \in [H^{-1/2}(\partial G)]^3$, we obtain that

$$T(E_G\Psi)\boldsymbol{n} = \frac{\Psi}{2} - K_G^S\Psi \quad \text{on} \quad \partial G. \tag{12}$$

3. The Robin problem for the Laplace equation

We need to study two auxiliary problems and express their solutions in the form of appropriate potentials. The first problem is the Robin problem for the Laplace equation.

Let $G \subset \mathbb{R}^3$ be a bounded open set with Lipschitz boundary ∂G. For a given $g \in H^{-1/2}(\partial G)$ and a given positive constant a, we study the following Robin problem: Find a function $u \in H^1(G)$ with

$$\begin{aligned} -\Delta u &= 0 && \text{in } G, \\ \frac{\partial u}{\partial \boldsymbol{n}} + au &= g && \text{on } \partial G, \end{aligned} \tag{13}$$

that is, with

$$\int_G \nabla u\cdot\nabla\varphi\,\mathrm{d}\boldsymbol{y} + \int_{\partial G} a\,u\,\varphi\,\mathrm{d}\sigma_{\boldsymbol{y}} = \langle g, \operatorname{tr}(\varphi)\rangle_{H^{-1/2},H^{1/2}} \qquad \forall\,\varphi \in H^1(G).$$

Concerning the solvability of this problem, we find the following propositions:

Proposition 1
For $g \in H^{-1/2}(\partial G)$, there exists a unique solution $u \in H^1(G)$ of the Robin problem (13).

See [19] for the proof.

Proposition 2
Let $u \in H^1(G)$ and $-\Delta u = 0$ in G. Then, there exists a unique $f \in H^{-1/2}(\partial G)$ such that $u = \mathcal{S}_G f$.

Proof
If $f \in H^{-1/2}(\partial G)$, then $\mathcal{S}_G f \in H^1(G)$ with the trace $\mathrm{tr}(\mathcal{S}_G f) \in H^{1/2}(\partial G)$. The operator $\mathcal{S}_G : H^{-1/2}(\partial G) \to H^{1/2}(\partial G)$ is a Fredholm operator with index 0 [37, Theorem 4.1], and the kernel of $\mathcal{S}_G$ is trivial [38, Chapter VI]. This implies that $\mathcal{S}_G(H^{-1/2}(\partial G)) = H^{1/2}(\partial G)$. Therefore, for any $u|_{\partial G} \in H^{1/2}(\partial G)$, there exists a unique $f \in H^{-1/2}(\partial G)$ such that $u = \mathrm{tr}(\mathcal{S}_G f)$ on ∂G. Because the Dirichlet problem for the Laplace equation is uniquely solvable in $H^1(G)$ [19], we deduce that $u = \mathcal{S}_G f$ in G. □

Proposition 3
The operator $\frac{1}{2} I - K_G^{\Delta} + a\mathcal{S}_G$ is a continuously invertible bounded linear operator on $H^{-1/2}(\partial G)$, where I is the identity operator.

Proof
For $f, g \in H^{-1/2}(\partial G)$, we have that $\mathcal{S}_G f$ is a solution of the Robin problem (13) if and only if $[1/2\, I - K_G^{\Delta} + a\mathcal{S}_G] f = g$. On the other hand, by Proposition 1, for $g \in H^{-1/2}(\partial G)$, there exists a unique solution $u \in H^1(G)$ of the problem (13). Then, because of Proposition 2, there exists a unique $f \in H^{-1/2}(\partial G)$ such that $u = \mathcal{S}_G f$. Thus, because the operator $(1/2)\, I - K_G^{\Delta} + a\mathcal{S}_G$ on $H^{-1/2}(\partial G)$ is onto and one-to-one, it is continuously invertible [39, Theorem 3.8]. □

4. A mixed Navier–Dirichlet problem for the Stokes system

The second auxiliary problem we consider is a mixed Navier–Dirichlet problem for the Stokes system. Suppose that $G \subset \mathbb{R}^3$ is a bounded domain with Lipschitz boundary. Let $\Gamma \subset \partial G$ be a closed part of the boundary. For given $\boldsymbol{f} \in [H^{1/2}(\partial G)]^3$, $\boldsymbol{g} \in [H^{-1/2}(\partial G)]^3$, and a positive constant a, we look for a weak solutions $(\boldsymbol{v}, p) \in [H^1(G)]^3 \times L^2(G)$ of the problem

$$\begin{aligned} \Delta \boldsymbol{v} &= \nabla p, \qquad \operatorname{div} \boldsymbol{v} = 0 \quad \text{in } G, \\ \boldsymbol{v} &= \boldsymbol{f} \quad \text{on } \partial G \setminus \Gamma, \\ \boldsymbol{v_\tau} &= \boldsymbol{f_\tau} \quad \text{on } \Gamma, \\ [T(\boldsymbol{v}, p)\boldsymbol{n} + a\boldsymbol{v}] \cdot \boldsymbol{n} &= \boldsymbol{g} \cdot \boldsymbol{n} \quad \text{on } \Gamma, \end{aligned} \tag{14}$$

that is, the boundary conditions $\boldsymbol{v} = \boldsymbol{f}$ on $\partial G \setminus \Gamma$, $\boldsymbol{v_\tau} = \boldsymbol{f_\tau}$ on Γ are fulfilled in the sense of traces, and it holds

$$2 \int_G \mathbf{D}\boldsymbol{v} : \mathbf{D}\Phi \, \mathrm{d}\boldsymbol{y} - \int_G p \, \operatorname{div} \Phi \, \mathrm{d}\boldsymbol{y} + \int_{\partial G} a\, \boldsymbol{v} \cdot \Phi \, \mathrm{d}\sigma_{\boldsymbol{y}} = \langle \boldsymbol{g}, \Phi \rangle_{H^{-1/2}, H^{1,2}}$$

for all $\Phi \in V_\Gamma(G)$, where

$$V_\Gamma(G) = \left\{ \Phi \in [H^1(G)]^3 : \ \Phi = 0 \text{ on } \partial G \setminus \Gamma, \ \Phi_{\boldsymbol{\tau}} = 0 \text{ on } \Gamma \right\}.$$

If Γ is a set of the surface measure zero (for example, a set consisting from finitely many points), then the mixed problem (14) reduces to the Dirichlet problem. To avoid this case, we assume that there exists a function $\Theta \in [H^1(G)]^3$ with $\Theta = 0$ on $\partial G \setminus \Gamma$ and $\Theta_{\boldsymbol{\tau}} = 0$ on Γ satisfying

$$\int_{\partial G} \Theta \cdot \boldsymbol{n} \, \mathrm{d}\sigma_{\boldsymbol{y}} = 1. \tag{15}$$

(Notice that this condition is fulfilled if Γ contains a smooth surface.) If this condition is not satisfied, then $\boldsymbol{v} = (0,0,0)$ and $p = 1$ would be a nontrivial solution of the problem (14) with homogeneous boundary condition $\boldsymbol{f} = \boldsymbol{g} = (0,0,0)$.

In the case ∂G is connected, we shall look for a solution of (14) in the form of a hydrodynamical single layer potential $(\boldsymbol{v}, p)^T = E_G \Psi$ with an appropriate $\Psi \in [H^{-1/2}(\partial G)]^3$. If ∂G is not connected, then solutions of the problem (14) cannot be represented by a pure hydrodynamical single layer potential. In order to obtain a representation formula for solutions of (14) in this case, we can use some modifications as follows. We denote by $C_1, \ldots, C_k$ all bounded connected components of $\mathbb{R}^3 \setminus \overline{G}$ and consider for $j = 1, \ldots, k$ and fixed $\boldsymbol{z}^j \in C_j$ the functions

$$\boldsymbol{w}_j^{\bullet}(\boldsymbol{x}) = \frac{\boldsymbol{x} - \boldsymbol{z}^j}{|\boldsymbol{x} - \boldsymbol{z}^j|^3}, \qquad \boldsymbol{w}_j(\boldsymbol{x}) = \begin{pmatrix} \boldsymbol{w}_j^{\bullet}(\boldsymbol{x}) \\ 0 \end{pmatrix}. \tag{16}$$

An easy calculation yields that $\Delta \boldsymbol{w}_j^\bullet = 0$ with $\operatorname{div} \boldsymbol{w}_j^\bullet = 0$ in $\mathbb{R}^3 \setminus \{z^j\}$. Now, for $\Psi \in [H^{-1/2}(\partial G)]^3$, we define

$$\tilde{E}_G \Psi = E_G \Psi + \sum_{j=1}^{k} \boldsymbol{w}_j \langle \Psi, \boldsymbol{w}_j^\bullet \rangle_{H^{-1/2},H^{1/2}}, \tag{17}$$

and if ∂G is connected, we set $\tilde{E}_G \Psi = E_G \Psi$. Because of the definition of E_G and $\boldsymbol{w}_j$, in both cases, it is ensured that $\tilde{E}_G \Psi$ is a solution of the Stokes system in G.

Denote by $V_\Gamma(\partial G)$ the space of traces of $V_\Gamma(G)$, that is,

$$V_\Gamma(\partial G) = \left\{ \boldsymbol{v} \in [H^{1/2}(\partial G)]^3; \boldsymbol{v} = 0 \text{ on } \partial G \setminus \Gamma, \boldsymbol{v}_{\boldsymbol{\tau}} = 0 \text{ on } \Gamma \right\},$$

and by $V'_\Gamma(\partial G)$ the dual space of $V_\Gamma(\partial G)$. According to the Hahn–Banach theorem, the space $V'_\Gamma(\partial G)$ can be interpreted as the space of restrictions $\{\boldsymbol{g_n}|_\Gamma; \boldsymbol{g} \in [H^{-1/2}(\partial G)]^3\}$. (We use the usual notation $\boldsymbol{g_n} = (\boldsymbol{g} \cdot \boldsymbol{n})\boldsymbol{n}$ for the normal part of $\boldsymbol{g}$.) Clearly, $V'_\Gamma(\partial G) \subset V'_\Gamma(G)$ (the dual space of $V_\Gamma(G)$). In fact, $V'_\Gamma(\partial G)$ is the space of all $\boldsymbol{f} \in V'_\Gamma(G)$ supported on ∂G.

Denote the space of restrictions

$$W_\Gamma(\partial G) = \left\{ [\boldsymbol{v}|_{(\partial G \setminus \Gamma)}, \boldsymbol{v}_{\boldsymbol{\tau}}|_\Gamma]; \boldsymbol{v} \in [H^{1/2}(\partial G)]^3 \right\}$$

equipped with the norm

$$\|\boldsymbol{v}\|_{W_\Gamma(\partial G)} = \inf \left\{ \|\boldsymbol{u}\|_{H^{1/2}(\partial G)}; \boldsymbol{u} \in [H^{1/2}(\partial G)]^3, \boldsymbol{u} = \boldsymbol{v} \text{ on } \partial G \setminus \Gamma, \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{v}_{\boldsymbol{\tau}} \text{ on } \Gamma \right\}.$$

Because $W_\Gamma(\partial G)$ is the factor space $[H^{1/2}(\partial G)]^3 / V_\Gamma(\partial G)$, it is a Banach space.

The operator

$$\mathcal{T}_1 \Psi = \left[\tilde{E}_G^\bullet \Psi|_{\partial G \setminus \Gamma}, \left(\tilde{E}_G^\bullet \Psi \right)_{\boldsymbol{\tau}} |_\Gamma \right] \tag{18}$$

is a bounded linear operator from $[H^{-1/2}(\partial G)]^3$ to $W_\Gamma(\partial G)$. We now define a bounded operator $\mathcal{T}_2^a : [H^{-1/2}(\partial G)]^3 \to V'_\Gamma(G)$ as

$$\langle \mathcal{T}_2^a \Psi, \Phi \rangle = 2 \int_G \mathbf{D}\Phi \cdot \mathbf{D}\tilde{E}_G^\bullet \Psi \, \mathrm{d}\boldsymbol{y} - \int_G [E_G \Psi]_4 \operatorname{div} \Phi \, \mathrm{d}\boldsymbol{y} + \int_{\partial G} a\Phi \cdot \tilde{E}_G^\bullet \Psi \, \mathrm{d}\sigma_{\boldsymbol{y}}, \quad \Phi \in V_\Gamma(G). \tag{19}$$

Because $\tilde{E}\Psi$ is a solution of the Stokes system, we have $\langle \mathcal{T}_2^a \Psi, \Phi \rangle = 0$ for $\Phi \in [C^\infty(G)]^3$ with compact support in G. So, $\mathcal{T}_2^a \Psi$ is supported on ∂G. Hence, $\mathcal{T}_2^a : [H^{-1/2}(\partial G)]^3 \to V'_\Gamma(\partial G)$ is a bounded linear operator.

For $\Psi \in [H^{-1/2}(\partial G)]^3$, we obtain that $\tilde{E}_G \Psi$ is a solution of (14) if $\mathcal{T}_1 \Psi = \left[\boldsymbol{f}|_{\partial G \setminus \Gamma}, \boldsymbol{f_\tau}|_\Gamma \right]$ and $\mathcal{T}_2^a \Psi = \boldsymbol{g_n}|_\Gamma$.

Proposition 4

We have $\tilde{E}_G^\bullet([H^{-1/2}(\partial G)]^3) = \{\boldsymbol{f} \in [H^{1/2}(\partial G)]^3 : \int_{\partial G} \boldsymbol{f} \cdot \boldsymbol{n}^G \, \mathrm{d}\sigma_{\boldsymbol{y}} = 0\}$. If $\boldsymbol{v} \in [H^1(G)]^3$, $p \in L^2(G)$, and $\Delta \boldsymbol{v} = \nabla p$, $\operatorname{div} \boldsymbol{v} = 0$ in G, then there exists a unique $\Psi \in [H^{-1/2}(\partial G)]^3$ such that $[\boldsymbol{v}, p] = \tilde{E}_G \Psi$ and

$$\|\Psi\|_{[H^{-1/2}(\partial G)]^3} \le C \left[\|\boldsymbol{v}\|_{[H^{1/2}(\partial G)]^3} + \left| \int_G p \, \mathrm{d}\boldsymbol{y} \right| \right],$$

where a constant C depends only on G.

Proof

We define the space

$$X \equiv \left\{ \boldsymbol{f} \in [H^{1/2}(\partial G)]^3 : \int_{\partial G} \boldsymbol{f} \cdot \boldsymbol{n}^G \, \mathrm{d}\sigma_{\boldsymbol{y}} = 0 \right\}.$$

The operator $E_G^\bullet : [H^{-1/2}(\partial G)]^3 \to [H^{1/2}(\partial G)]^3$ is a Fredholm operator with index 0 [40]. Because $\tilde{E}_G^\bullet - E_G^\bullet$ is a finite dimensional operator, we obtain that $\tilde{E}_G^\bullet : [H^{-1/2}(\partial G)]^3 \to [H^{1/2}(\partial G)]^3$ is also a Fredholm operator with index 0 [41, § 16, Theorem 16]. For $\Psi \in [H^{-1/2}(\partial G)]^3$, we have that $\tilde{E}_G \Psi$ is a solution of the Stokes system in G and $\tilde{E}_G^\bullet \Psi \in X$ [42, Chapter IV]. Thus, the codimension of the range of $\tilde{E}_G^\bullet$ is at least 1.

We denote by $C_1, \ldots, C_{k+1}$ all components of $\mathbb{R}^3 \setminus \overline{G}$, where C_{k+1} denotes the unbounded component and consider $\boldsymbol{n}^j = \boldsymbol{n}$ on ∂C_j, whereas $\boldsymbol{n}^j = 0$ elsewhere. Then, $E_G \boldsymbol{n}^j = 0$ for $j = 1, \ldots, k$ and $E_G \boldsymbol{n}^{k+1} = [0, 0, 0, -1]$ in G (see, e.g., [26, §3.2]). Now, we define the space

$$Y = \{\Psi \in [H^{-1/2}(\partial G)]^3 : \int_G [E_G \Psi]_4 \, \mathrm{d}\boldsymbol{y} = 0\}.$$

Because $[E_G \boldsymbol{n}^{k+1}]_4 = -1$, the space $[H^{-1/2}(\partial G)]^3$ is the direct sum of Y and $\{c\boldsymbol{n}^{k+1}; c \in \mathbb{R}^1\}$.

Denote

$$Z = \left\{\Psi \in \left[H^{-1/2}(\partial G)\right]^3; \langle \Psi, \boldsymbol{w}_j^\bullet \rangle = 0 \;\forall j = 1, \dots, k\right\},$$

that is, $Z = \{\Psi \in [H^{-1/2}(\partial G)]^3; \tilde{E}_G^\bullet \Psi = E_G^\bullet \Psi\}$. Let $j, l \in \{1, \dots, k\}, j \neq l$. Because $\operatorname{div} \boldsymbol{w}_l^\bullet = 0$ in $\mathbb{R}^3 \setminus C_l$, Green's formula gives

$$\int_{\partial G} \boldsymbol{w}_l^\bullet \cdot \boldsymbol{n}^j \, d\sigma_{\boldsymbol{y}} = -\int_{\partial C_j} \boldsymbol{w}_l^\bullet \cdot \boldsymbol{n} \, d\sigma_{\boldsymbol{y}} = -\int_{C_j} \operatorname{div} \boldsymbol{w}_l^\bullet \, d\boldsymbol{y} = 0.$$

For $r > 0$ such that $B(\boldsymbol{z}^l; r) \equiv \{\boldsymbol{y}; |\boldsymbol{y} - \boldsymbol{z}^l| < r\} \subset C_l$, applying easy calculation, we obtain

$$\begin{aligned}\int_{\partial G} \boldsymbol{w}_l^\bullet \cdot \boldsymbol{n}^l \, d\sigma_{\boldsymbol{y}} &= -\int_{\partial(C_l \setminus B(\boldsymbol{z}^l; r))} \boldsymbol{w}_l^\bullet \cdot \boldsymbol{n} \, d\sigma_{\boldsymbol{y}} - \int_{\partial B(\boldsymbol{z}^l; r)} \boldsymbol{w}_l^\bullet \cdot \boldsymbol{n} \, d\sigma_{\boldsymbol{y}} \\ &= -\int_{\partial B(\boldsymbol{z}^l; r)} \boldsymbol{w}_l^\bullet \cdot \boldsymbol{n} \, d\sigma_{\boldsymbol{y}} \neq 0.\end{aligned}$$

Thus, $[H^{-1/2}(\partial G)]^3$ is the direct sum of Z, and the linear hull of $\{\boldsymbol{n}^1, \dots, \boldsymbol{n}^k\}$. So, $[H^{-1/2}(\partial G)]^3$ is the direct sum of $Y \cap Z$ and the linear hull of $\{\boldsymbol{n}^1, \dots, \boldsymbol{n}^{k+1}\}$.

Suppose now that $\tilde{E}_G^\bullet \Psi = 0$ on ∂G. Then, we obtain that $\tilde{E}_G^\bullet \Psi = 0$ in G [42, Chapter IV]. Because $\operatorname{div} E^\bullet \Psi = 0$ in $\mathbb{R}^3 \setminus \partial G$, we conclude

$$\int_{\partial G} \boldsymbol{n}^j \cdot E^\bullet \Psi \, d\sigma_{\boldsymbol{y}} = 0, \quad \text{for } j = 1, \dots, k+1.$$

See [42, Chapter IV]. If $l = 1, \dots, k$, then

$$0 = \int_{\partial G} \boldsymbol{n}^l \cdot \tilde{E}_G^\bullet \Psi \, d\sigma_{\boldsymbol{y}} = \sum_{j=1}^k \langle \Psi, \boldsymbol{w}_j^\bullet \rangle \int_{\partial G} \boldsymbol{w}_j^\bullet \cdot \boldsymbol{n}^l \, d\sigma_{\boldsymbol{y}} = \langle \Psi, \boldsymbol{w}_l^\bullet \rangle \int_{\partial G} \boldsymbol{w}_l^\bullet \cdot \boldsymbol{n}^l \, d\sigma_{\boldsymbol{y}}.$$

Because

$$\int_{\partial G} \boldsymbol{w}_l^\bullet \cdot \boldsymbol{n}^l \, d\sigma_{\boldsymbol{y}} \neq 0,$$

this forces that $\langle \Psi, \boldsymbol{w}_l^\bullet \rangle = 0$. Thus, $\Psi \in Z$ and $\tilde{E}_G^{\bullet\Psi} = E_G^\bullet \Psi$, and therefore, $\tilde{E}_G \Psi = E_G \Psi$. Because $E_G^\bullet$ is injective on $Y \cap Z$ by [40] and the codimension of Y is equal to 1, we deduce that the dimension of the kernel of $\tilde{E}_G^\bullet$ is at most 1. Because $\tilde{E}_G^\bullet$ is a Fredholm operator with index 0, the dimension of the kernel of $\tilde{E}_G^\bullet$ and the codimension of the range of $\tilde{E}_G^\bullet$ are equal to 1. Because $\tilde{E}_G^\bullet([H^{-1/2}(\partial G)]^3) \subset X$, we infer that $\tilde{E}_G^\bullet([H^{-1/2}(\partial G)]^3) = X$. Because the dimension of the kernel of $\tilde{E}_G^\bullet$ is equal to 1, there exists $\Phi \in Z \setminus Y$ such that $\tilde{E}_G^\bullet \Phi = 0$, that is, there exists Φ such that $\tilde{E}_G^\bullet \Phi = [0, 0, 0]$ and

$$\int_G [E_G \Phi]_4 \, d\boldsymbol{y} \neq 0.$$

Because $\tilde{E}_G \Phi$ is a solution of the Stokes system in G, we deduce that $[E_G \Phi]_4$ is constant in G. So, we can choose Φ such that $\tilde{E}_G \Phi = [0, 0, 0, 1]$ in G. Therefore,

$$\Psi \mapsto \left[\tilde{E}_G^\bullet \Psi, \int_G [E_G \Psi]_4 \, d\boldsymbol{y}\right]$$

is an injective mapping $[H^{-1/2}(\partial G)]^3$ onto $X \times R$. This mapping is continuously invertible by [39], Theorem 3.8. So, there exists a positive constant C such that

$$\|\Psi\|_{[H^{-1/2}(\partial G)]^3} \leq C \left[\|\tilde{E}_G^\bullet \Psi\|_{[H^{1/2}(\partial G)]^3} + \left|\int_G [E_G \Psi]_4 \, d\boldsymbol{y}\right|\right].$$

Let us now assume that $\boldsymbol{v} \in [H^1(G)]^3, p \in L^2(\Omega)$ is a solution of the Stokes system in G. Then, we obtain that the trace of $\boldsymbol{v}$ is in X (see [42], Chapter IV) and there exists $\Psi \in [H^{-1/2}(\partial G)]^3$ such that $\tilde{E}_G^\bullet \Psi = \boldsymbol{v}$ on ∂G. Because $(\boldsymbol{v}, p) - \tilde{E}_G \Psi$ is a solution of the Dirichlet problem for the Stokes system with the zero boundary condition, we have $\boldsymbol{v} = \tilde{E}_G^\bullet \Psi$ in G and $p - [E_G \Psi]_4$ is constant in G. Therefore, there exists a constant c such that $(\boldsymbol{v}, p) = \tilde{E}_G(\Psi + c\Phi)$. □

Proposition 5
Suppose that there exists $\Theta \in [H^1(G)]^3$ such that $\Theta = 0$ on $\partial G \setminus \Gamma$, $\Theta_{\boldsymbol{\tau}} = 0$ on Γ, and the assumption (15) is satisfied.

- Then, $\mathcal{T} : \Psi \mapsto [\mathcal{T}_1\Psi, \mathcal{T}_2^a\Psi]$ is a continuously invertible bounded linear operator from $\left[H^{-1/2}(\partial G)\right]^3$ onto $W_\Gamma(\partial G) \times V'_\Gamma(\partial G)$.
- If $\boldsymbol{f} \in \left[H^{1/2}(\partial G)\right]^3$, $\boldsymbol{g} \in \left[H^{-1/2}(\partial G)\right]^3$, then there exists a unique solution $\boldsymbol{v} \in \left[H^1(G)\right]^3$, $p \in L^2(G)$ of the problem (14). Moreover, $(\boldsymbol{v}, p) = \tilde{E}_G\Psi$, where Ψ is a unique solution of the integral equations $\mathcal{T}_1\Psi = \left[\boldsymbol{f}|_{\partial G\setminus\Gamma}, \boldsymbol{f}_{\boldsymbol{\tau}}|_\Gamma\right]$ and $\mathcal{T}_2^a\Psi = \boldsymbol{g}_{\boldsymbol{n}}|_\Gamma$.

Proof

□

Suppose first that $(\boldsymbol{v}, p)$ is a solution of the problem (14) with $\boldsymbol{f} = \boldsymbol{g} = (0,0,0)$. Then,

$$0 = \langle \boldsymbol{g}, \boldsymbol{v}\rangle_{H^{-1/2},H^{1/2}} = 2\int_G |\mathbf{D}\,\boldsymbol{v}|^2\,\mathrm{d}\boldsymbol{y} + \int_{\partial G} a|\boldsymbol{v}|^2\,\mathrm{d}\sigma_{\boldsymbol{y}}.$$

Denote the inner product

$$(\boldsymbol{w}, \boldsymbol{u}) = 2\int_G \mathbf{D}\,\boldsymbol{w}\cdot\mathbf{D}\,\boldsymbol{u}\,\mathrm{d}\boldsymbol{y} + \int_{\partial G} a\,\boldsymbol{w}\cdot\boldsymbol{u}\,\mathrm{d}\sigma_{\boldsymbol{y}}. \tag{20}$$

Then, $\|\boldsymbol{w}\| = \sqrt{(\boldsymbol{w}, \boldsymbol{w})}$ is an equivalent norm in $[H^1(G)]^3$ (see, for example, [43, Theorem 5.2]). Thus, $\boldsymbol{v} = 0$ in G. Hence, $\nabla p = \Delta\boldsymbol{v} = 0$ in G and $p = c$ with some constant c [44, Lemma 6.4]. Therefore, $T(\boldsymbol{v}, p)\boldsymbol{n} + a\boldsymbol{v} = -c\boldsymbol{n}$, and using boundary condition in (14), we obtain

$$0 = \langle (T(\boldsymbol{v}, p)\boldsymbol{n} + a\boldsymbol{v})\cdot\boldsymbol{n}, \Theta\rangle = -c$$

and $c = 0$.

We consider now $\boldsymbol{g} \in \left[H^{-1/2}(\partial G)\right]^3$ and $\boldsymbol{f} \in \left[H^{1/2}(\partial G)\right]^3$ and define

$$\alpha = \int_{\partial G} \boldsymbol{f}\cdot\boldsymbol{n}^G\,\mathrm{d}\sigma_{\boldsymbol{y}}\,.$$

Then for $\tilde{\boldsymbol{f}} = \boldsymbol{f} - \alpha\Theta$, there exists a solution $\tilde{\boldsymbol{v}} \in \left[H^{1,2}(G)\right]^3$, $\tilde{p} \in L^2(G)$ of the Stokes system in G such that $\tilde{\boldsymbol{v}} = \tilde{\boldsymbol{f}}$ on ∂G [42, Chapter IV]. Considering $\boldsymbol{v} = \tilde{\boldsymbol{v}} + \boldsymbol{u}$ and $p = \tilde{p} + q$, we can conclude that $(\boldsymbol{v}, p)$ is a solution of the mixed problem (14) if and only if $(\boldsymbol{u}, q) \in \left[H^1(G)\right]^3 \times L^2(G)$ is a solution of the mixed problem

$$\begin{aligned} \Delta\boldsymbol{u} &= \nabla q, \quad \operatorname{div}\boldsymbol{u} = 0 \quad &&\text{in } G,\\ \boldsymbol{u} &= 0 &&\text{on } \partial G\setminus\Gamma,\\ \boldsymbol{u}_{\boldsymbol{\tau}} &= 0 &&\text{on } \Gamma,\\ [T(\boldsymbol{u}, q)\boldsymbol{n} + a\boldsymbol{u}]\cdot\boldsymbol{n} &= \tilde{\boldsymbol{g}}\cdot\boldsymbol{n} &&\text{on } \Gamma, \end{aligned} \tag{21}$$

where $\tilde{\boldsymbol{g}} = \boldsymbol{g} - [T(\tilde{\boldsymbol{v}}, \tilde{p})\boldsymbol{n} + a\tilde{\boldsymbol{v}}]$.

Denote

$$X_\Gamma = \left\{\boldsymbol{v} \in V_\Gamma(\partial G); \int_{\partial G} \boldsymbol{v}\cdot\boldsymbol{n}^G\,\mathrm{d}\sigma_{\boldsymbol{y}} = 0\right\}.$$

Clearly, $V_\Gamma(\partial G)$ and X_Γ are closed subspaces of $\left[H^{1/2}(\partial G)\right]^3$, and $V_\Gamma(\partial G)$ is the direct sum of X_Γ and $\{c\Theta; c \in \mathbb{R}\}$. We denote also the spaces

$$Y_\Gamma = \left\{\Psi \in \left[H^{-1/2}(\partial G)\right]^3; \tilde{E}_G^{\bullet}\Psi \in X_\Gamma\right\}, \qquad Y_\Gamma^0 = \left\{\Psi \in Y_\Gamma; \int_G \left[\tilde{E}_G\Psi\right]_4\,\mathrm{d}\boldsymbol{y} = 0\right\}.$$

For $\boldsymbol{f} \in X_\Gamma$, there exists a unique solution $\boldsymbol{v} \in \left[H^1(G)\right]^3$ and $p \in L^2(G)$ of the Stokes system in G such that $\boldsymbol{v} = \boldsymbol{f}$ on ∂G and

$$\int_G p\,\mathrm{d}\boldsymbol{y} = 0.$$

See, for example, [42, Chapter IV]. Proposition 4 implies that $\tilde{E}_G^{\bullet}$ is a bounded continuously invertible operator from Y_Γ^0 onto X_Γ. Thus, $\{\tilde{E}_G^{\bullet}\Psi; \Psi \in Y_\Gamma\} = X_\Gamma$.

If $\Psi \in Y_\Gamma$, then $\tilde{E}_G\Psi$ is a solution of the mixed problem (21) if and only if $\mathcal{T}_2^a\Psi = \tilde{\boldsymbol{g}}_{\boldsymbol{n}}|_\Gamma$. Because $V'_\Gamma(\partial G)$ is the dual space of $V_\Gamma(\partial G)$, we have $\mathcal{T}_2^a\Psi = \tilde{\boldsymbol{g}}_{\boldsymbol{n}}|_\Gamma$ if and only if $\langle \mathcal{T}_2^a\Psi, \boldsymbol{w}\rangle = \langle \tilde{\boldsymbol{g}}, \boldsymbol{w}\rangle$ for all $\boldsymbol{w} \in V_\Gamma(\partial G)$ (i.e., for $\boldsymbol{w} = \Theta$ and $\boldsymbol{w} = \tilde{E}_G^\bullet\Phi$ with $\Phi \in Y_\Gamma$).

Denote

$$Z_\Gamma = \left\{\tilde{E}_G^\bullet\Psi|_G; \Psi \in Y_\Gamma\right\}.$$

Then, Z_Γ is a closed subspace of $\left[H^1(G)\right]^3$. Because the inner product (,) given by (20) defines an equivalent norm in $[H^1(G)]^3$, the Riesz representation theorem implies that there exists a unique $\boldsymbol{w} \in Z_\Gamma$ such that $(\boldsymbol{w}, \tilde{\boldsymbol{w}}) = \langle \tilde{\boldsymbol{g}}, \tilde{\boldsymbol{w}}\rangle$ for all $\tilde{\boldsymbol{w}} \in Z_\Gamma$. Fix $\Psi \in Y_\Gamma$ such that $\boldsymbol{w} = \tilde{E}_G^\bullet\Psi$. Then, $\langle \mathcal{T}_2^a\Psi, \tilde{\boldsymbol{w}}\rangle = \langle \tilde{\boldsymbol{g}}, \tilde{\boldsymbol{w}}\rangle$ for all $\tilde{\boldsymbol{w}} = \tilde{E}_G^\bullet\Phi$ with $\Phi \in Y_\Gamma$. Denote by ω the unbounded component of $\mathbb{R}^3 \setminus \overline{G}$. Then, $E_G\boldsymbol{n}^\omega = [0,0,0,1]$ in G (see, for example, [26, §3.2]) and $\tilde{E}_G\boldsymbol{n}^\omega = [0,0,0,1]$ in G. If $c \in \mathbb{R}$, then $\tilde{E}_G^\bullet(\Psi + c\boldsymbol{n}^\omega) = \boldsymbol{w}$, and therefore, $\langle \mathcal{T}_2^a(\Psi + c\boldsymbol{n}^\omega), \tilde{\boldsymbol{w}}\rangle = \langle \tilde{\boldsymbol{g}}, \tilde{\boldsymbol{w}}\rangle$ for all $\tilde{\boldsymbol{w}} = \tilde{E}_G^\bullet\Phi$ with $\Phi \in Y_\Gamma$. Now, we choose $c \in \mathbb{R}$ such that $\langle \mathcal{T}_2^a(\Psi + c\boldsymbol{n}^\omega), \Theta\rangle = \langle \tilde{\boldsymbol{g}}, \Theta\rangle$. We have proved that there exists a solution of the problem (14).

If $\boldsymbol{f} \in \left[H^{1/2}(\partial G)\right]^3$ and $\boldsymbol{g} \in \left[H^{-1/2}(\partial G)\right]^3$, then there exists a unique solution $\boldsymbol{v} \in \left[H^1(G)\right]^3$, $p \in L^2(\Omega)$ of the problem (14). According to Proposition 4, there exists a unique $\Psi \in \left[H^{-1/2}(\partial G)\right]^3$ such that $(\boldsymbol{v}, p) = \tilde{E}_G\Psi$. Remark that $\tilde{E}_G\Psi$ is a solution of the problem (14) if and only if $\mathcal{T}\Psi = \left[\boldsymbol{f}|_{\partial G\setminus\Gamma}, \boldsymbol{f}_\tau|_\Gamma, \boldsymbol{g}_{\boldsymbol{n}}|_\Gamma\right]$. Thus, the operator $\mathcal{T}$ is a continuous injective operator from $\left[H^{-1/2}(\partial G)\right]^3$ onto $W_\Gamma(\partial G) \times V'_\Gamma(\partial G)$. Therefore, according to [39, Theorem 3.8], the operator $\mathcal{T}$ is continuously invertible.

5. Stokes–Darcy problem

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain and suppose that Ω_S is a subdomain of Ω with Lipschitz boundary such that $\Omega_D = \Omega \setminus \overline{\Omega_S}$ has Lipschitz boundary. Let Γ be a nonempty closed subset of $\partial\Omega_S \cap \partial\Omega_D$. Let k and η be positive constants. For given $\boldsymbol{g} \in \left[H^{-1/2}(\partial\Omega^S)\right]^3$, $\boldsymbol{f} \in \left[H^{1/2}(\partial\Omega^S)\right]^3$, and $h \in H^{-1/2}(\partial\Omega_D)$, we shall look for a solution $(\boldsymbol{v}^S, p^S) \in \left[H^1(\Omega_S)\right]^3 \times L^2(\Omega_S)$ and $(\boldsymbol{v}^D, p^D) \in \left[L^2(\Omega_D)\right]^3 \times H^1(\Omega_D)$ of the coupled Stokes–Darcy problem (1)–(6). Here, $\boldsymbol{n} = \boldsymbol{n}^S$ on $\partial\Omega_S$, $\boldsymbol{n} = -\boldsymbol{n}^D$ on $\partial\Omega_D$. We suppose that there exists $\Theta \in \left[H^1(\Omega_S)\right]^3$ such that $\Theta = 0$ on $\partial\Omega \setminus \Gamma$ with $\Theta_{\boldsymbol{\tau}} = 0$ on Γ and satisfies

$$\int_\Gamma \Theta \cdot \boldsymbol{n}\, \mathrm{d}\sigma_{\boldsymbol{y}} = 1\,.$$

Notice that this condition is fulfilled if Γ contains a nontrivial smooth surface.

Denote by $\dot{H}^{1/2}(\partial\Omega_D\setminus\Gamma)$ the closure of infinitely differentiable functions with compact support in $\partial\Omega_D\setminus\Gamma$ and by $H^{-1/2}(\partial\Omega_D\setminus\Gamma)$ its dual space. Suppose that $h^D \in H^{-1/2}(\partial\Omega_D \setminus \Gamma)$ and $h^\Gamma \in H^{-1/2}(\partial\Omega_D)$ supported in Γ. Denote by $P_{\partial\Omega_D\setminus\Gamma}$ the orthogonal projection of $H^{1/2}(\partial\Omega_D)$ onto $H^{1/2}(\partial\Omega_D \setminus \Gamma)$. Define $h = h^D \circ P_{\partial\Omega_D\setminus\Gamma} + h^\Gamma$. Then, $h \in H^{-1/2}(\partial\Omega_D)$. Because h^Γ is supported on Γ, we have $h|_{\partial\Omega_D\setminus\Gamma} = h^D$. If $\varphi \in H^{1/2}(\partial\Omega_D, \varphi = 0$ in $\partial\Omega_D \setminus \Gamma$, then $\langle h, \varphi\rangle = \langle h^\Gamma, \varphi\rangle$. Thus, $h|_\Gamma = h^\Gamma$. Let $h \in H^{-1/2}(\partial\Omega_D)$. We prove that there exist $h^D \in H^{-1/2}(\partial\Omega_D \setminus \Gamma)$ and $h^\Gamma \in H^{-1/2}(\partial\Omega_D)$ such that $h = h^D \circ P_{\partial\Omega_D\setminus\Gamma} + h^\Gamma$. Define $\langle h^D, \varphi\rangle = \langle h, \varphi\rangle$ for $\varphi \in H^{1/2}(\partial\Omega_D \setminus \Gamma)$. Then, $h^D \in H^{-1/2}(\partial\Omega_D \setminus \Gamma)$. If we define $h^\Gamma = h - h^D \circ P_{\partial\Omega_D\setminus\Gamma}$, then $h^D \in H^{-1/2}(\partial\Omega_D)$ is supported on Γ.

Suppose now that $(\boldsymbol{v}^S, p^S) \in \left[H^1(\Omega_S)\right]^3 \times L^2(\Omega_S)$, $(\boldsymbol{v}^D, p^D) \in \left[L^2(\Omega)\right]^3 \times H^1(\Omega_D)$ is a solution of the problem (1)–(6). We notice that $\Delta p^D = \mathrm{div}\nabla p^D = -\mathrm{div}\, \boldsymbol{v}^D = 0$ in Ω_D. According to Proposition 2, there exists $\psi \in H^{-1/2}(\partial\Omega_D)$ such that $p^D = \mathcal{S}\psi$, where $\mathcal{S}\psi = \mathcal{S}_G\psi$ and $G = \Omega_D$.

If $\partial\Omega_S$ is connected, we denote $\tilde{E}\Psi = E_G\Psi$ with $G = \Omega_S$. In the case $\partial\Omega_S$ is not connected, we denote by $C_1, \dots, C_k$ all bounded components of $\mathbb{R}^3 \setminus \overline{\Omega_S}$ and consider fixed points $z^j \in C_j$, for $j = 1, \dots, k$. Then, as in (16) and (17), for $\Psi \in \left[H^{-1/2}(\partial\Omega_S)\right]^3$, we can define $\tilde{E}\Psi := \tilde{E}_G\Psi$ with $G = \Omega_S$. According to Proposition 4, there exists a unique $\Psi \in \left[H^{-1/2}(\partial\Omega_S)\right]^3$ such that $(\boldsymbol{v}^S, p^S) = \tilde{E}\Psi$. Thus, for integral representation of the solutions of (1)–(6), we shall look for a solution in that form.

Now, we denote by K^Δ the operator K_G^Δ defined by (8) for $G = \Omega_D$. Let $W_\Gamma(\partial\Omega_S)$, $V_\Gamma(\partial\Omega_S)$ and $V'_\Gamma(\partial\Omega_S)$ be spaces from the Section 4. We consider $\mathcal{T}_1$ the bounded linear operator from $\left[H^{-1/2}(\partial\Omega_S)\right]^3$ to $W_\Gamma(\partial\Omega_S)$ given by (18) for $G = \Omega_S$. For a constant $a \in \mathbb{R}$, we denote by $\mathcal{T}_2^a$ the bounded operator from $\left[H^{-1/2}(\partial\Omega_S)\right]^3$ to $V'_\Gamma(\partial\Omega_S)$ defined by (19) with $G = \Omega_S$.

For $\psi \in H^{-1/2}(\partial\Omega_D)$ and $\Psi \in [H^{-1/2}(\partial\Omega_S)]^3$, we define

$$\mathcal{T}_3(\psi, \Psi) = \left[\psi/2 - K^\Delta\psi - \chi_\Gamma \boldsymbol{n}\cdot\tilde{E}^\bullet\Psi, \mathcal{T}_1\Psi, \eta\mathcal{T}_2^0\Psi + k^{-1}\mathcal{S}\psi + \psi/2 - K^\Delta\psi\right],$$

where χ_Γ is the characteristic function of Γ.

Proposition 6

If $\psi \in H^{-1/2}(\partial\Omega_D)$, $\Psi \in \left[H^{-1/2}(\partial\Omega_S)\right]^3$, $h^D = h|_{\partial\Omega_D\setminus\Gamma}$, $h^\Gamma = h|_\Gamma$, then $(\boldsymbol{v}^S, p^S) = \tilde{E}\Psi$, and $p^D = \mathcal{S}\psi$, $\boldsymbol{v}^D = -\nabla p^D$ is a solution of the problem (1)–(6) if and only if $\mathcal{T}_3(\psi, \Psi) = \left[h, \boldsymbol{f}|_{\partial\Omega_S\setminus\Gamma}, \boldsymbol{f}_\tau|_\Gamma, \boldsymbol{g}_{\boldsymbol{n}}|_\Gamma\right]$. The operator $\mathcal{T}_3 : H^{-1/2}(\partial\Omega_D) \times \left[H^{-1/2}(\partial\Omega_S)\right]^3 \to H^{-1/2}(\partial\Omega_D) \times W_\Gamma(\partial\Omega_S) \times V'_\Gamma(\partial\Omega_S)$ is a Fredholm operator with index 0.

Proof

For $\psi \in H^{-1/2}(\partial\Omega_D)$ and $\Psi \in \left[H^{-1/2}(\partial\Omega_S)\right]^3$, easy calculation ensures that $(\boldsymbol{v}^S, p^S) = \tilde{E}\Psi$, and $p^D = \mathcal{S}\psi$, $\boldsymbol{v}^D = -\nabla p^D$ is a solution of the problem (1)–(6) if and only if $\mathcal{T}_3(\psi, \Psi) = \left[h, \boldsymbol{f}|_{\partial\Omega_S\setminus\Gamma}, \boldsymbol{f}_\tau|_\Gamma, \boldsymbol{g}_{\boldsymbol{n}}|_\Gamma\right]$.

For $\psi \in H^{-1/2}(\partial\Omega_D)$ and $\Psi \in \left[H^{-1/2}(\partial\Omega_S)\right]^3$, we define the operator

$$\mathcal{T}_4(\psi,\Psi) = \left[\psi/2 - K^\Delta\psi + \mathcal{S}\psi, \mathcal{T}_1\Psi, \eta\mathcal{T}_2^1\Psi + k^{-1}\mathcal{S}\psi + \psi/2 - K^\Delta\psi\right]$$

and shall show that $\mathcal{T}_4$ is a continuously invertible bounded linear operator from $H^{-1/2}(\partial\Omega_D) \times \left[H^{-1/2}(\partial\Omega_S)\right]^3$ to $H^{-1/2}(\partial\Omega_D) \times W_\Gamma(\partial\Omega_S) \times V'_\Gamma(\partial\Omega_S)$.

For $h \in H^{-1/2}(\partial\Omega_D)$, $\boldsymbol{f} \in \left[H^{1/2}(\partial\Omega_S)\right]^3$, and $\boldsymbol{g} \in [H^{-1/2}(\partial\Omega_S)]^3$, because of Proposition 3, there exists a unique $\psi \in H^{-1/2}(\partial\Omega_D)$ such that $K^\Delta\psi - \frac{1}{2}\psi - \mathcal{S}\psi = h$. Then, Proposition 5 ensures that there exists a unique $\Psi \in \left[H^{-1/2}(\partial\Omega_S)\right]^3$ such that $\mathcal{T}_1\Psi = \left[\boldsymbol{f}|_{\partial\Omega_S\setminus\Gamma}, \boldsymbol{f_\tau}|_\Gamma\right]$ and $\eta\mathcal{T}_2^1\Psi = \boldsymbol{g_n} - k^{-1}\mathcal{S}\psi - \frac{1}{2}\psi + K^\Delta\psi$. Because $\mathcal{T}_4$ is an injective bounded linear operator from $H^{-1/2}(\partial\Omega_D) \times \left[H^{-1/2}(\partial\Omega_S)\right]^3$ onto $H^{-1/2}(\partial\Omega_D) \times W_\Gamma(\partial\Omega_S) \times V'_\Gamma(\partial\Omega_S)$, applying Theorem 3.8 in [39], we obtain that $\mathcal{T}_4$ is continuously invertible.

For $\psi \in H^{-1/2}(\partial\Omega_D)$ and $\Psi \in \left[H^{-1/2}(\partial\Omega_S)\right]^3$, we have that

$$[\mathcal{T}_3 - \mathcal{T}_4](\psi,\Psi) = \left[-\mathcal{S}\psi - \chi_\Gamma\boldsymbol{n}\cdot\tilde{E}^\bullet\Psi, 0, -\eta\tilde{E}^\bullet\Psi\right].$$

$\mathcal{S}$ is a bounded linear operator from $H^{-1/2}(\partial\Omega_D)$ to $H^{1/2}(\partial\Omega_D)$ (see, for example, [37, Theorem 4.1]), and therefore, a compact operator on $H^{-1/2}(\partial\Omega_D)$. Similarly, $\tilde{E}^\bullet$ is a bounded linear operator from $\left[H^{-1/2}(\partial\Omega_S)\right]^3$ to $\left[H^{1/2}(\partial\Omega_S)\right]^3$ [36, Proposition 4.10] and a compact operator on $\left[H^{-1/2}(\partial\Omega_S)\right]^3$. Thus, $\chi_\Gamma\boldsymbol{n}\cdot\tilde{E}^\bullet$ is a compact operator from $\left[H^{-1/2}(\partial\Omega_S)\right]^3$ to $H^{-1/2}(\partial\Omega_D)$. Altogether, $[\mathcal{T}_3 - \mathcal{T}_4]$ is a compact linear operator from $H^{-1/2}(\partial\Omega_D) \times \left[H^{-1/2}(\partial\Omega_S)\right]^3$ to $H^{-1/2}(\partial\Omega_D) \times W_\Gamma(\partial\Omega_S) \times V'_\Gamma(\partial\Omega_S)$. Because $\mathcal{T}_4$ is invertible, $\mathcal{T}_3$ is a Fredholm operator with index 0 [41, § 16, Theorem 16]. □

Proposition 7

Let $\left(\boldsymbol{v}^S, p^S\right) \in \left[H^1(\Omega_S)\right]^3 \times L^2(\Omega_S)$ and $\left(\boldsymbol{v}^D, p^D\right) \in \left[L^2(\Omega_D)\right]^3 \times H^1(\Omega_D)$ be a solution of the problem (1)–(6) with $\boldsymbol{f} \equiv 0$, $h^D \equiv 0$, $h^\Gamma \equiv 0$, and $\boldsymbol{g} \equiv 0$. Then, there exists a constant c such that $p^S = c$, $\boldsymbol{v}^S \equiv 0$, $\boldsymbol{v}^D \equiv 0$, and $p^D = k\eta c$. On the other hand, if $p^S = c$, $\boldsymbol{v}^S \equiv 0$, $\boldsymbol{v}^D \equiv 0$, $p^D = k\eta c$ for some constant c, then $\left(\boldsymbol{v}^S, p^S, \boldsymbol{v}^D, p^D\right)$ is a solution of the problem (1)–(6) with $\boldsymbol{f} \equiv 0$, $h^D \equiv 0$, $h^\Gamma \equiv 0$ and $\boldsymbol{g} \equiv 0$.

Proof

Because $\boldsymbol{v}^S\cdot\boldsymbol{n} = \boldsymbol{v}^D\cdot\boldsymbol{n} = -\partial p^D/\partial\boldsymbol{n}^S = \partial p^D/\partial\boldsymbol{n}^D$, we have, using Green's formula,

$$\begin{aligned} 0 = &\int_\Gamma \left(\boldsymbol{v}^S\cdot\boldsymbol{n}\right)\left\{\eta\left[T(\boldsymbol{v}^S,p^S)\boldsymbol{n}^S\right]\cdot\boldsymbol{n} + p^D/k + \boldsymbol{v}^D\cdot\boldsymbol{n}\right\}\,\mathrm{d}\sigma_{\boldsymbol{y}} \\ &+ \int_\Gamma \boldsymbol{v}^S_{\boldsymbol{\tau}}\left[\eta T(\boldsymbol{v}^S,p^S)\boldsymbol{n}^S\right]_{\boldsymbol{\tau}}\,\mathrm{d}\sigma_{\boldsymbol{y}} + \int_{\partial\Omega_S\setminus\Gamma} \eta\boldsymbol{v}^S\cdot T\left(\boldsymbol{v}^S,p^S\right)\boldsymbol{n}^S\,\mathrm{d}\sigma_{\boldsymbol{y}} \\ &+ \int_{\partial\Omega_D\setminus\Gamma}\left(\boldsymbol{v}^D\cdot\boldsymbol{n}\right)\frac{p^D}{k}\,\mathrm{d}\sigma_{\boldsymbol{y}} = \int_{\partial\Omega_S}\eta\boldsymbol{v}^S\cdot T\left(\boldsymbol{v}^S,p^S\right)\boldsymbol{n}^S\,\mathrm{d}\sigma_{\boldsymbol{y}} + \int_{\partial\Omega_D}\frac{p^D}{k}\frac{\partial p^D}{\partial\boldsymbol{n}^D}\,\mathrm{d}\sigma_{\boldsymbol{y}} \\ &+ \int_\Gamma |\boldsymbol{v}^S\cdot\boldsymbol{n}|^2\,\mathrm{d}\sigma_{\boldsymbol{y}} = \int_{\Omega_S} 2\eta|\mathbf{D}\,\boldsymbol{v}^S|^2\,\mathrm{d}\boldsymbol{y} + \int_{\Omega_D}\frac{|\nabla p^D|^2}{k}\,\mathrm{d}\boldsymbol{y} + \int_\Gamma |\boldsymbol{v}^S\cdot\boldsymbol{n}|^2\,\mathrm{d}\sigma_{\boldsymbol{y}}. \end{aligned} \tag{22}$$

Therefore, $\boldsymbol{v}^S\cdot\boldsymbol{n} = 0$ on Γ, $\mathbf{D}\,\boldsymbol{v}^S = 0$ in Ω_S, and $\nabla p^D = 0$ in Ω_D. According to (3) and (5), we have $\boldsymbol{v}^S = 0$ on $\partial\Omega_S$. Because $\mathbf{D}\,\boldsymbol{v}^S \equiv 0$, we obtain that the functions $\boldsymbol{v}^S_j$, for $j = 1,2,3$ are affine [45, Lemma 6] and therefore harmonic. The maximum principle for harmonic functions gives that $\boldsymbol{v}^{\boldsymbol{S}}_{\boldsymbol{j}} \equiv 0$, for $j = 1,2,3$. Because $\nabla p^S = \Delta\boldsymbol{v}^S = 0$, there exists a constant c such that $p^S = c$. Because $\nabla p^D = 0$ in Ω_D, the function p^S is constant on each component of Ω_D. Therefore, $\boldsymbol{v}^D = -\nabla p^D = 0$. Using the boundary conditions $0 = \eta[T(\boldsymbol{v}^S,p^S)\boldsymbol{n}^S]\cdot\boldsymbol{n} + p^D/k + \boldsymbol{v}^D\cdot\boldsymbol{n} = -\eta c + p^D/k$ on Γ, we can conclude that $p^D = k\eta c$. □

Theorem 1

For $\boldsymbol{g} \in \left[H^{-1/2}(\partial\Omega^S)\right]^3$, $\boldsymbol{f} \in \left[H^{1/2}(\partial\Omega^S)\right]^3$, $h \in H^{-1/2}(\partial\Omega_D)$, $h^\Gamma = h|_\Gamma$, and $h^D = h|_{\partial\Omega^D\setminus\Gamma}$, there exists a solution of the problem (1)–(6) if and only if

$$\langle h, 1\rangle = \int_{\partial\Omega_S\setminus\Gamma}\boldsymbol{n}^S\cdot\boldsymbol{f}\,\mathrm{d}\sigma_{\boldsymbol{y}}. \tag{23}$$

Proof

Let $(\boldsymbol{v}^S, p^S) \in \left[H^1(\Omega_S)\right]^3 \times L^2(\Omega_S)$ and $\boldsymbol{v}^D \in \left[L^2(\Omega_D)\right]^3$, $p^D \in H^1(\Omega_D)$ be a solution of the problem (1)–(6). Because $\Delta p^D = 0$ for $\varphi \equiv 1$, we obtain that

$$\langle\partial p^D/\partial\boldsymbol{n}^D, 1\rangle = \int_{\Omega_D}\nabla p^D\cdot\nabla\varphi\,\mathrm{d}\boldsymbol{y} = 0.$$

Considering $\operatorname{div}\boldsymbol{v}^S = 0$, Green's theorem gives

$$\int_{\partial\Omega_S}\boldsymbol{n}^S\cdot\boldsymbol{v}^S\,\mathrm{d}\sigma_{\boldsymbol{y}} = 0,$$

compare [42, Chapter IV]. Because $\boldsymbol{n} = \boldsymbol{n}^S$ on $\partial\Omega_S$, $\boldsymbol{n} = -\boldsymbol{n}^D$ on $\partial\Omega_D$, and $\partial p^D/\partial \boldsymbol{n}^D = -\boldsymbol{n}^D \cdot \boldsymbol{v}^D = \boldsymbol{n} \cdot \boldsymbol{v}^D$, we have

$$0 = \langle \partial p^D/\partial \boldsymbol{n}^D, 1\rangle = \langle h, 1\rangle + \int_\Gamma \boldsymbol{n}^S \cdot \boldsymbol{v}^S \, d\sigma_{\boldsymbol{y}} - \int_{\partial\Omega_S} \boldsymbol{v}^S \cdot \boldsymbol{n}^S \, d\sigma_{\boldsymbol{y}}$$
$$= \langle h, 1\rangle - \int_{\partial\Omega_S \setminus \Gamma} \boldsymbol{f} \cdot \boldsymbol{n}^S \, d\sigma_{\boldsymbol{y}}.$$

Now, for $\psi \in H^{-1/2}(\partial\Omega_D)$ and $\Psi \in \left[H^{-1/2}(\partial\Omega_S)\right]^3$, we consider $\left(\boldsymbol{v}^S, p^S\right) = \tilde{E}\Psi$ and $p^D = \mathcal{S}\psi$, $\boldsymbol{v}^D = -\nabla p^D$. Then by Proposition 6, $(\boldsymbol{v}^S, p^S)$ and $\left(\boldsymbol{v}^D, p^D\right)$ are the solutions of the problem (1)–(6) if and only if $\mathcal{T}_3\left(\psi, \Psi\right) = \left[h, \boldsymbol{f}|_{\partial\Omega_S\setminus\Gamma}, \boldsymbol{f}_\tau|_\Gamma, \boldsymbol{g}_{\boldsymbol{n}}|_\Gamma\right]$. Suppose now that $\mathcal{T}_3(\psi, \Psi) = 0$. According to Proposition 7, there exists a constant c such that $\tilde{E}\Psi = [0, 0, 0, c]$ and $\mathcal{S}\psi = k\eta c$. This, together with Propositions 2 and 4, yields that the dimension of the kernel of $\mathcal{T}_3$ is at most 1. The condition (23) forces that the codimension of the range of $\mathcal{T}_3$ is at least 1. Because $\mathcal{T}_3$ is a Fredholm operator with index 0, we infer that codim $\mathcal{T}_3\left(H^{-1/2}(\partial\Omega_D) \times [H^{-1/2}(\partial\Omega_S)]^3\right) =$ dim Ker $\mathcal{T}_3 = 1$. Hence, the Stokes–Darcy problem is solvable if and only if the compatibility condition (23) holds true. □

Acknowledgements

The research of D. M. was supported by RVO no. 67985840.

References

1. Chavarría-Krauser A, Ptashnyk M. Homogenization approach to water transport in plant tissues with periodic microstructures. *Mathematical Modelling of Natural Phenomena* 2013; **8**:80–111. DOI: 10.1051/mmnp/20138406.
2. Steudle E. Water uptake by plant roots: an integration of views. *Plant Soil* 2000; **226**:45–56.
3. Layton WJ, Schieweck F, Yotov I. Coupling fluid flow with porous media flow. *SIAM Journal on Numerical Analysis* 2003; **40**:2195–2218. DOI: 10.1137/S0036142901392766.
4. Lipnikov K, Vassilev D, Yotov I. Discontinuous Galerkin and mimetic finite difference methods for coupled Stokes–Darcy flows on polygonal and polyhedral grids. *Numerical Mathematics* 2014; **126**:321–360. DOI: 10.1007/s00211-013-0563-3.
5. Quarteroni A, Discacciati M. Navier–Stokes/Darcy coupling: modeling, analysis, and numerical approximation. *Revista Matemática Complutense* 2009; **22**:315–426.
6. Arbogast T, Lehr H. Homogenization of a Darcy–Stokes system modeling vuggy porous media. *Computational Geosciences* 2006; **10**:291–302.
7. Jäger W, Mikelic A. On the interface boundary condition of Beavers, Joseph and Saffman. *SIAM Journal on Applied Mathematics* 2000; **60**:1111–1127. DOI: 10.1137/S003613999833678X.
8. Hsiao GC, Wendland WL. *Boundary Integral Equations*. Springer: Heidelberg, 2008.
9. Steinbach O. *Numerical Approximation Methods for Elliptic Boundary Value Problems. Finite and Boundary Elements*. Springer: New York, 2008.
10. Kenig CE. *Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems*. American Mathematical Society: Providence, Rhode Island, 1994,1–146.
11. *Integral Operators in Potential Theory*. Springer-Verlag: Berlin, 1980.
12. Maz'ya VG. *Boundary Integral Equations. Analysis IV. Encyclopaedia of Mathematical Sciences*. Springer-Verlag: New York, 1991,127–222.
13. Netuka I. An operator connected with the third boundary value problem in potential theory. *Czechoslovak Mathematical Journal* 1972; **22**:462–489.
14. Netuka I. The third boundary value problem in potential theory. *Czechoslovak Mathematical Journal* 1972; **22**:554–580.
15. Netuka I. Generalized Robin problem in potential theory. *Czechoslovak Mathematical Journal* 1972; **22**:312–324.
16. Lanzani L, Mendez O. The Poisson's problem for the Laplacian with Robin boundary condition in non-smooth domains. *Revista Matemática Iberoamericana* 2006; **22**:181–204. DOI: 10.4171/RMI/453.
17. Lanzani L, Shen Z. On the Robin boundary condition for Laplace's equation in Lipschitz domains. *Communications in PDEs* 2004; **29**:91–109. DOI: 10.1081/PDE-120028845.
18. Medková D. The third problem for the Laplace equation with a boundary condition from L_p. *Integral Equations and Operator Theory* 2006; **54**: 235–258. DOI: 10.1007/s00020-003-1354-5.
19. Nečas J. *Direct methods in the theory of elliptic equations*. Springer: Berlin, 2012.
20. Medková D. The Neumann problem for the Laplace equation on general domains. *Czechoslovak Mathematical Journal* 2007; **57**:1107–1139.
21. Medková D. The integral equation method and the Neumann problem for the Poisson equation on NTA domains. *Integral Equations and Operator Theory* 2009; **63**:227–247. DOI: 10.1007/s00020-008-1651-0.
22. Steinbach O, Wendland WL, On C. Neumann's method for second-order elliptic systems in domains with non-smooth boundaries. *Journal of Mathematical Analysis and Applications* 2001; **262**:733–748.
23. Kohr M, Pop I. *Viscous Incompressible Flow for Low Reynolds Numbers*. WIT Press: Southampton, 2004.
24. Maremonti P, Russo R, Starita G. On the Stokes equations: the boundary value problem. In *Advances in Fluid Dynamics*, Maremonti P (ed). Dipartimento di Matematica Seconda Università di Napoli: Caserta, 1999; 69–140.
25. Odquist FKG. Über die Randwertaufgaben in der Hydrodynamik zäher Flüssigkeiten. *Mathematische Zeitschrift* 1930; **32**:329–375.
26. Pozrikidis C. *Boundary integral and singularity methods for linearized viscous flow*. Cambridge University Press: New York, 1992.
27. Varnhorn W. *The Stokes Equations*. Akademie Verlag: Berlin, 1994.
28. Dindoš M, Mitrea M. The stationary Navier–Stokes system in nonsmooth manifolds: the Poisson problem in Lipschitz and C^1 domains. *Archive for Rational Mechanics and Analysis* 2004; **174**:1–37. DOI: 10.1007/978-1-4419-1343-2_5.
29. Fabes EB, Kenig CE, Verchota GC. The Dirichlet problem for the Stokes system on Lipschitz domains. *Duke Mathematical Journal* 1988; **57**:769–793. DOI: 10.1215/S0012-7094-88-05734-1.

30. Kilty J. The L^p Dirichlet problem for the Stokes system on Lipschitz domain. *Indiana University Mathematics Journal* 2009; **58**:1219–1234. DOI: 10.1512/iumj.2009.58.3568.
31. Kohr M, Lanza de Cristoforis M, Wendland WL. Nonlinear Neumann-transmission problems for Stokes and Brinkman equations on Euclidean Lipschitz domains. *Potential Anal* 2013; **38**:1123–1171. DOI: 10.1007/s11118-012-9310-0.
32. Kohr M, Pintea C, Wendland WL. Stokes–Brinkman transmission problems on Lipschitz and C^1 domains in Riemannian manifolds. *Communications on Pure and Applied Analysis* 2010; **9**:493–537. DOI: 10.3934/cpaa.2010.9.493.
33. Maz'ya V, Mitrea M, Shaposhnikova T. The inhomogenous Dirichlet problem for the Stokes system in Lipschitz domains with unit normal close to *VMO*∗. *Functional Analysis and its Applications* 2009; **43**:217–235. DOI: 10.1007/s10688-009-0029-7.
34. Mitrea M, Wright M. *Boundary value problems for the Stokes system in arbitrary Lipschitz domains*. Astérisque: Paris, 2012.
35. Dahlberg BEJ, Kenig C, Verchota GC. Boundary value problems for the systems of elastics in Lipschitz domains. *Duke Mathematical Journal* 1988; **57**(3):795–818. DOI: 10.1215/S0012-7094-8805735-3.
36. Medková D. Convergence of the Neumann series in BEM for the Neumann problem of the Stokes system. *Acta Applicandae Mathematicae* 2011; **116**:281–304. DOI: 10.1007/s10440-011-9643-5.
37. Mitrea D. The method of layer potentials for non-smooth domains with arbitrary topology. *Integral Equations and Operator Theory* 1997; **29**:320–338.
38. Landkof NL. *Foundations of Modern Potential Theory*. Springer–Verlag: Berlin–Heidelberg–New York, 1972.
39. Schechter M. *Principles of Functional Analysis*. American Mathematical Society: Providence, Rhode Island, 2002.
40. Reidinger B, Steinbach O. A symmetric boundary element method for the Stokes problem in multiple connected domains. *Mathematical Methods in the Applied Sciences* 2003; **26**:77–93. DOI: 10.1002/mma.347.
41. Müller V. *Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras. Operator*. Birkhäuser: Basel, 2007.
42. Galdi GP. *An introduction to the Mathematical Theory of the Navier-Stokes Equations I, Linearised Steady Problems*. Springer Verlag: Berlin - Heidelberg - New York, 1998.
43. Alessandrini G, Morassi A, Rosset E. The linear constraint in Poincaré and Korn type inequalities. *Forum Math* 2008; **20**:557–569. DOI: 10.1515/FORUM.2008.028.
44. Tartar L. *An Introduction to Sobolev Spaces and Interpolation Spaces*. Springer-Verlag: Berlin Heidelberg, 2007.
45. Medková D. Integral representation of a solution of the Neumann problem for the Stokes system. *Numerical Algorithms* 2010; **54**:459–484. DOI: 10.1007/s11075-009-9346-4.