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SOME PSEUDOPARABOLIC VARIATIONAL INEQUALITIES WITH HIGHER DERIVATIVES

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We consider a pseudoparabolic variational inequality with higher derivatives. We prove the existence and uniqueness of a solution of this inequality with a zero initial condition.

The process of filtration of liquid in a porous medium with cracks [1] and the process of heat transfer in a heterogeneous medium [2] are simulated by boundary-value problems for pseudoparabolic equations. Di Benedetto and Showalter [3] showed that the solution of the single-phase Stefan problem can be reduced to the solution of a pseudoparabolic variational inequality.

Scarpini [4] considered a linear pseudoparabolic inequality in the case of a single space variable with second derivatives with respect to a space coordinate. Problems with initial and boundary conditions for a pseudoparabolic equation with higher derivatives were investigated by Brill [5]. A general theory of pseudoparabolic equations was considered, in particular, in [6–10].

In the present paper, the well-posedness of a pseudoparabolic variational problem is investigated for the first time in the case of many space variables with derivatives of order $2n$, $n \geq 1$, with respect to space coordinates and with zero initial condition.

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, let V be a closed subspace in $\left(H^l(\Omega)\right)^N$ such that $\left(\overset{\circ}{H}{}^l(\Omega)\right)^N \subset V \subset \left(H^l(\Omega)\right)^N$ and V is continuously and compactly imbedded in $\left(L^2(\Omega)\right)^N$, and let K be a convex closed set in V that contains the zero element.

Let $\langle \cdot, \cdot \rangle$ denote the action of a functional from the space V^* on elements of the space V and let $(\cdot, \cdot)$ denote the scalar product in the Euclidean space $\mathbb{R}^N$.

In the domain $Q_T = \Omega \times (0, T)$, we consider the variational inequality

$$\int_0^\tau (\langle M(t)u_t, v - u_t \rangle + \langle L(t)u, v - u_t \rangle - \langle F, v - u_t \rangle)\,dt \;\geq\; 0 \tag{1}$$

with the initial condition

$$u(0) \;=\; 0, \qquad x \in \Omega. \tag{2}$$

Here, the operators $L(t)$, $M(t)$, and $F(t)$ are defined as follows:

For every $t \in (0, T)$, we have $L(t)\colon V \to V^*$ and, for arbitrary $u, v \in V$,

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$$\langle L(t)u, v\rangle = \int_\Omega \Bigg[\sum_{|\alpha|=|\beta|\le l} \Big(A_{\alpha,\beta}(x,t)D^\beta u(x,t), D^\alpha v(x,t)\Big) + \sum_{1\le|\alpha|\le l} \Big(C_\alpha(x,t)D^\alpha u(x,t), v(x,t)\Big)\Bigg]dx.$$

For every $t \in (0, T)$, we have $M(t): V \to V^*$ and, for arbitrary $u, v \in V$,

$$\langle M(t)u, v\rangle = \int_\Omega \sum_{|\alpha|=|\beta|\le l} \Big(B_{\alpha\beta}(x,t)D^\beta u(x,t), D^\alpha v(x,t)\Big)dx.$$

For every $v \in V$ and arbitrary $t \in (0, T)$, we have

$$\langle F(t), v\rangle = \int_\Omega \sum_{|\alpha|\le l} \Big(f_\alpha, D^\alpha v(x,t)\Big)dx.$$

Here, $A_{\alpha\beta}$, $B_{\alpha\beta}$, and C_α are square $N\times N$ matrices, $f_\alpha = (f_{\alpha,1},\ldots,f_{\alpha,N})$, and $l \ge 1$.
We introduce the space

$$W = \Big\{w: w \in L^2((0,T); V),\ w_t \in L^2\big((0,T); V^*\big)\Big\}$$

and denote $\Omega_\tau = Q_T \cap \{t = \tau\}$.

Definition 1. *A generalized solution of inequality (1) with initial condition (2) is understood as a function* u, $u \in L^2([0,T]; V)$, $u_t \in L^2((0,T); V)$, $u_t \in K$ *for almost all* $t \in (0,T)$, *that satisfies (1) and (2) for arbitrary* τ, $0 < \tau \le T$, *and for an arbitrary function* $v \in W$, $v \in K$ *for almost all* $t \in (0,T)$.

Assume that the coefficients of the operators L and M satisfy the following conditions:

(A) $A_{\alpha\beta} \in L^\infty(Q_T)$, $A_{\alpha\beta,t} \in L^\infty(Q_T)$, $A_{\alpha\beta}(x,t) = A_{\beta\alpha}(x,t)$, and $A_{\alpha\beta}(x,t) = A^T_{\alpha\beta}(x,t)$ for all α, β, $|\alpha| \le l$, $|\beta| \le l$, and $(x,t) \in Q_T$, and

$$\int_{\Omega_\tau} \sum_{|\alpha|=|\beta|\le l} \Big(A_{\alpha\beta}(x,t)D^\beta v, D^\alpha v\Big)dx \ge \int_{\Omega_\tau} \Bigg(a_l \sum_{|\alpha|=l} \big|D^\alpha v\big|^2 + a_0|v|^2\Bigg)dx$$

for almost all $\tau \in [0,T]$, $v \in V$, $a_l > 0$, $a_0 > 0$;

(B) $B_{\alpha\beta} \in L^\infty(Q_T)$, $B_{\alpha\beta,t} \in L^\infty(Q_T)$, $B_{\alpha\beta}(x,t) = B_{\beta\alpha}(x,t)$, and $B_{\alpha\beta}(x,t) = B_{\alpha\beta}(x,t)^T$ for all α, β, $|\alpha| \le l$, $|\beta| \le l$, and $(x,t) \in Q_T$, and

$$\int_{\Omega_\tau} \sum_{|\alpha|=|\beta|\le l} \Big(B_{\alpha\beta}(x,t)D^\beta v, D^\alpha v\Big)dx \ge \int_{\Omega_\tau} \Bigg(b_l \sum_{|\alpha|=l} \big|D^\alpha v\big|^2 + b_0|v|^2\Bigg)dx$$

for almost all $\tau \in [0,T]$, $v \in V$, $b_l > 0$, $b_0 > 0$;

(C) $C_\alpha \in C\big([0, T]; L^\infty(\Omega)\big)$.

According to the corollary presented in [11, p. 27], we can use the norm

$$\|u\|^2_{H^l(\Omega)} = \int\limits_\Omega \left(\sum_{|\alpha| = l} \left| D^\alpha u \right|^2 + |u|^2 \right) dx$$

in the space $\left(H^l(\Omega)\right)^N$. Let A be a square matrix, $A(x,t) = \left[a_{ij}(x,t) \right]_{i,j=1}^N$. By $\|A\|_{L^\infty(Q_T)}$ we denote the quantity $\operatorname{ess\,sup}_{Q_T} \left[\sum_{i,j=1}^N a_{ij}^2(x,t) \right]^{1/2}$.

Theorem 1. *Suppose that conditions (A)–(C) are satisfied. Then problem (1), (2) has at most one solution.*

Proof. Assume that there exist two solutions u^1 and u^2 of inequality (1). Then, for u^1 and u^2, we have

$$\begin{aligned} &\int\limits_0^\tau \left(\left\langle M(t)u_t^1, v - u_t^1 \right\rangle + \left\langle L(t)u^1, v - u_t^1 \right\rangle - \left\langle F, v - u_t^1 \right\rangle \right) dt \ \geq\ 0, \\ &\int\limits_0^\tau \left(\left\langle M(t)u_t^2, v - u_t^2 \right\rangle + \left\langle L(t)u^2, v - u_t^2 \right\rangle - \left\langle F, v - u_t^2 \right\rangle \right) dt \ \geq\ 0. \end{aligned} \tag{3}$$

In the inequalities obtained, we set $v = \left(u_t^1 + u_t^2\right)/2$ and add them together. As a result, we get

$$\int\limits_0^\tau \left[\left\langle M(t)\left(u_t^2 - u_t^1\right), u_t^2 - u_t^1 \right\rangle + \left\langle L(t)(u^2 - u^1), u_t^2 - u_t^1 \right\rangle \right] dt \ \leq\ 0. \tag{4}$$

Denoting $u(x,t) = u^2(x,t) - u^1(x,t)$ and performing certain transformations in inequality (4), we obtain

$$\mathfrak{T}_1 = \int\limits_0^\tau \langle Mu_t, u_t \rangle dt = \int\limits_{\Omega_\tau} \sum_{|\alpha| = |\beta| \leq l} \left(B_{\alpha\beta} D^\beta u_t, D^\alpha u_t \right) dx\,dt \ \geq\ \int\limits_{Q_\tau} \left[b_l \sum_{|\alpha| = l} \left| D^\alpha u_t \right|^2 + b_0 |u_t|^2 \right] dx\,dt,$$

$$\begin{aligned} \mathfrak{T}_2 &= \int\limits_0^\tau \langle L(t)u, u_t \rangle dt = \int\limits_{Q_\tau} \sum_{|\alpha| = |\beta| \leq l} \left(A_{\alpha\beta} D^\beta u, D^\alpha u_t \right) dx\,dt + \int\limits_{Q_\tau} \sum_{1 \leq |\alpha| \leq l} \left(C_\alpha D^\alpha u, u_t \right) dx\,dt \\ &\geq \frac{1}{2} \int\limits_{\Omega_\tau} \left[a_l \sum_{|\alpha| = l} \left| D^\alpha u \right|^2 + a_0 |u|^2 \right] dx - \frac{1}{2} \int\limits_{Q_\tau} \left[a_l^2 \sum_{|\alpha| = l} \left| D^\alpha u \right|^2 + a_0^2 |u|^2 \right] dx\,dt \\ &\quad - \frac{C}{2\delta_0} \int\limits_{Q_\tau} \left[\mu_1 \sum_{|\alpha| = l} \left| D^\alpha u \right|^2 + \mu_2 |u|^2 \right] dx\,dt - \frac{\delta_0}{2} \int\limits_{Q_\tau} |u_t|^2\, dx\,dt, \quad \delta_0 > 0, \end{aligned}$$

where $C = \max_{1 \le |\alpha| \le l} \| C_\alpha \|_{L^\infty(Q_T)}$ and μ_1 and μ_2 are constants dependent on Ω, l, and n, because the elements of the matrices $A_{\alpha\beta}$, $A_{\alpha\beta,t}$, $B_{\alpha\beta}$, and $B_{\alpha\beta,t}$ are bounded and, by virtue of Theorem 1.17 in [12, p. 177], the formula of integration by parts is valid in $\mathfrak{T}_2$. Moreover, by virtue of the corollary presented in [11, p. 27], we have

$$
\begin{aligned}
&\int_\Omega \sum_{|\alpha|=|\beta|\le l} \big(A_{\alpha\beta}(x,t)D^\alpha u, D^\beta u\big)dx \le \int_\Omega \Big(a_l^1 \sum_{|\alpha|=l} \big|D^\alpha u\big|^2 + a_0^1 |u|^2 \Big) dx,\\
&\int_\Omega \sum_{|\alpha|=|\beta|\le l} \big(A_{\alpha\beta,t}(x,t)D^\alpha u, D^\beta u\big)dx \le \int_\Omega \Big(a_l^2 \sum_{|\alpha|=l} \big|D^\alpha u\big|^2 + a_0^2 |u|^2 \Big) dx,\\
&\int_\Omega \sum_{|\alpha|=|\beta|\le l} \big(B_{\alpha\beta}(x,t)D^\alpha u, D^\beta u\big)dx \le \int_\Omega \Big(b_l^1 \sum_{|\alpha|=l} \big|D^\alpha u\big|^2 + b_0^1 |u|^2 \Big) dx,\\
&\int_\Omega \sum_{|\alpha|=|\beta|\le l} \big(B_{\alpha\beta,t}(x,t)D^\alpha u, D^\beta u\big)dx \le \int_\Omega \Big(b_l^2 \sum_{|\alpha|=l} \big|D^\alpha u\big|^2 + b_0^2 |u|^2 \Big) dx.
\end{aligned}
\tag{5}
$$

By using the estimates of the integrals $\mathfrak{T}_1$ and $\mathfrak{T}_2$, we get

$$
\int_{Q_\tau} \Big[b_l \sum_{|\alpha|=l} \big|D^\alpha u_t\big|^2 + \Big(b_0 - \frac{\delta_0}{2}\Big)|u_t|^2 \Big] dx\,dt + \int_{\Omega_\tau} \Big[a_l \sum_{|\alpha|=l} \big|D^\alpha u\big|^2 + a_0 |u|^2 \Big] dx
$$

$$
\le \int_{Q_\tau} \Big[\Big(a_l^2 + \frac{C\mu_1}{2\delta} \Big) \sum_{|\alpha|=l} \big|D^\alpha u\big|^2 + \Big(a_0^2 + \frac{C\mu_2}{2\delta_0} \Big) |u|^2 \Big] dx\,dt,
$$

where $\delta_0 \le b_0$. By using the Gronwall–Bellman lemma, we establish that $\|u(t)\|_V \le 0$ for almost all $t \in (0, T)$. Hence, $u_1 = u_2$ almost everywhere in Q_T.

Theorem 2. *Suppose that the conditions of Theorem 1 are satisfied and* $f_\alpha \in C\big([0,T];(L^2(\Omega))^N\big)$ *for all* $|\alpha| \le l$. *Then a solution of problem (1), (2) exists.*

Proof. We seek a solution of the variational inequality (1) with initial condition (2) by the penalty method and the method of hyperbolic regularization.

Consider the sequence of functions

$$
u^s(x,t) = \sum_{k=1}^{s} z_k^s(t)\varphi^k(x), \qquad s = 1, 2, \dots,
$$

where $\varphi^k(x)$ is a basis of the space V, and $z_1^s, \dots, z_s^s$ is a solution of the Cauchy problem

$$\varepsilon\left\langle u_{tt}, \varphi^k \right\rangle + \left\langle M u_t^s, \varphi^k \right\rangle + \left\langle L u^s, \varphi^k \right\rangle + \frac{1}{\varepsilon}\left\langle B(u_t^s), \varphi^k \right\rangle = \left\langle F, \varphi^k \right\rangle, \tag{6}$$

$$z_k^s(0) = 0, \qquad z_{k,t}^s(0) = 0, \quad k = 1, \dots, s. \tag{7}$$

Here, B is the penalty operator [13, p. 370], $Bu = J\left(u - P_{K_1} u\right)$, where J is the duality operator between the spaces V and V^*, P_K is the operator of projection to the set K, and ε is a positive parameter.

Multiplying equality (6) by $z_{k,t}^s(t)$ and integrating the result with respect to t over the interval $(0, \tau)$, $0 \le \tau \le T$, we obtain

$$\int_{Q_\tau} \Bigg[\varepsilon\left(u_{tt}^s, u_t^s\right) + \sum_{|\alpha|=|\beta|\le l} \left(B_{\alpha\beta} D^\beta u_t^s, D^\alpha u_t^s\right) + \sum_{|\alpha|=|\beta|\le l} \left(A_{\alpha\beta} D^\beta u^s, D^\alpha u_t^s\right)$$

$$+ \sum_{1\le|\alpha|\le l} \left(C_\alpha D^\alpha u^s, u_t^s\right) + \sum_{|\alpha|\le l} \left(f_\alpha, D^\alpha u_t^s\right) \Bigg] dx\, dt + \frac{1}{\varepsilon} \int_0^\tau \left\langle B\left(u_t^s\right), u_t^s \right\rangle dt = 0. \tag{8}$$

Taking (5) into account, we estimate each term of the last equality as follows:

$$\mathfrak{T}_3 = \varepsilon \int_{Q_\tau} \left(u_{tt}^s, u_t^s\right) dx\, dt = \frac{\varepsilon}{2} \int_{\Omega_\tau} \left|u_t^s\right|^2 dx,$$

$$\mathfrak{T}_4 = \int_{Q_\tau} \sum_{|\alpha|=|\beta|\le l} \left(A_{\alpha\beta} D^\beta u^s, D^\alpha u_t^s\right) dx\, dt = \frac{1}{2} \int_{\Omega_\tau} \sum_{|\alpha|=|\beta|\le l} \left(A_{\alpha\beta}(x,t) D^\beta u^s, D^\alpha u^s\right) dx$$

$$- \frac{1}{2} \int_{Q_\tau} \sum_{|\alpha|=|\beta|\le l} \left(A_{\alpha\beta,t}(x,t) D^\beta u^s, D^\alpha u^s\right) dx\, dt$$

$$\ge \frac{1}{2} \int_{\Omega_\tau} \Bigg[a_l \sum_{|\alpha|=l} \left|D^\alpha u^s\right|^2 + a_0 \left|u^s\right|^2 \Bigg] dx - \frac{1}{2} \int_{Q_\tau} \Bigg[a_l^2 \sum_{|\alpha|=l} \left|D^\alpha u^s\right|^2 + a_0^2 \left|u^s\right|^2 \Bigg] dx\, dt,$$

$$\mathfrak{T}_5 = \int_{Q_\tau} \sum_{|\alpha|=|\beta|\le l} \left(B_{\alpha\beta} D^\beta u_t^s, D^\alpha u_t^s\right) \ge \int_{Q_\tau} \Bigg[b_l \sum_{|\alpha|=l} \left|D^\alpha u_t^s\right|^2 + b_0 \left|u_t^s\right|^2 \Bigg] dx\, dt,$$

$$\mathfrak{T}_6 = \int_{Q_\tau} \sum_{1\le|\alpha|\le l} \left(C_\alpha D^\alpha u^s, u_t^s\right) dx\, dt \le \frac{\delta_0}{2} \int_{Q_\tau} \left|u_t^s\right|^2 dx\, dt + \frac{C}{2\delta_0} \int_{Q_\tau} \Bigg[\mu_1 \sum_{|\alpha|=l} \left|D^\alpha u^s\right|^2 + \mu_2 \left|u^s\right|^2 \Bigg] dx\, dt,$$

$$\mathfrak{T}_7 = \int_{Q_\tau} \sum_{|\alpha|\le l} \left(f_\alpha, D^\alpha u_t^s\right) dx\, dt \le \frac{1}{2\delta_1} \int_{Q_\tau} \sum_{|\alpha|\le l} |f_\alpha|^2 dx\, dt + \frac{\delta_1}{2} \int_{Q_\tau} \Bigg[\mu_3 \sum_{|\alpha|=l} \left|D^\alpha u_t^s\right|^2 + \mu_4 \left|u_t^s\right|^2 \Bigg] dx\, dt,$$

$$\mathfrak{T}_8 = \int_0^\tau \left\langle B\left(u_t^s\right), u_t^s \right\rangle dt, \qquad \delta_0 > 0, \quad \delta_1 > 0.$$

Using these transformations, we get

$$\varepsilon \int_{\Omega_\tau} \left|u_t^s\right|^2 dx + \int_{\Omega_\tau} \left(a_l \sum_{|\alpha|=l} \left|D^\alpha u^s\right|^2 + a_0 \left|u^s\right|^2 \right) dx + \int_{Q_\tau} (2b_l - \delta_1 \mu_3) \sum_{|\alpha|=l} \left|D^\alpha u_t^s\right|^2 dx\,dt$$

$$+ \int_{Q_\tau} (2b_0 - \delta_1 \mu_4 - \delta_0) \left|u_t^s\right|^2 dx\,dt + \frac{1}{\varepsilon} \int_0^\tau \left\langle B\left(u_t^s\right), u_t^s \right\rangle dt$$

$$\leq \int_{Q_\tau} \left[\left(a_l^2 + \frac{C\mu_1}{\delta_0} \right) \sum_{|\alpha|=l} \left|D^\alpha u^s\right|^2 + \left(a_0^2 + \frac{C\mu_2}{\delta_0} \right) \left|u^s\right|^2 \right] dx\,dt + \frac{1}{2\delta_1} \int_{Q_\tau} \sum_{|\alpha| \leq l} |f_\alpha|^2 dx\,dt. \tag{9}$$

Choosing δ_0 and δ_1 so that the corresponding terms are positive and using the Gronwall–Bellman lemma and inequality (9), we obtain the following estimates for $u^s(x,t)$:

$$\int_{\Omega_\tau} \left|u_t^s\right|^2 dx \leq \nu_1 \varepsilon, \quad \tau \in [0, T],$$

$$\int_{\Omega_\tau} \left[\sum_{|\alpha|=l} \left|D^\alpha u^s\right|^2 + \left|u^s\right|^2 \right] dx \leq \nu_1, \qquad \tau \in [0, T], \tag{10}$$

$$\int_{Q_T} \left[\sum_{|\alpha|=l} \left|D^\alpha u_t^s\right|^2 + \left|u_t^s\right|^2 \right] dx\,dt \leq \nu_1,$$

$$\int_0^T \left\langle B\left(u_t^s\right), u_t^s \right\rangle dt \leq \varepsilon \nu_1.$$

The condition $f_\alpha \in C\left([0,T]; \left(L^2(\Omega)\right)^N\right)$ and estimates (10) yield

$$\int_{\Omega_\delta} \left|u_t^s\right|^2 dx \leq \delta\mu, \qquad \int_{\Omega_\delta} \left[\sum_{|\alpha|=l} \left|D^\alpha u^s\right|^2 + \left|u^s\right|^2 \right] dx \leq \delta\mu.$$

Let us prove the uniform convergence of the sequence $\left\{u^s(x,t)\right\}$ in the space $C([0,T]; V)$ and the sequence $\left\{u_t^s(x,t)\right\}$ in the space $C\left([0,T]; \left(L^2(\Omega)\right)^N\right)$. For this purpose, we consider equality (6) for t and $t + \delta$ and multiply these equalities by $z_{k,t}^s(t+\delta) - z_{k,t}^s(t)$. We have

$$\varepsilon\Big\langle u_{tt}^s(t+\delta), \tilde{u}_t^s(t)\Big\rangle + \Big\langle Mu_t^s(t+\delta), \tilde{u}_t^s(t)\Big\rangle + \Big\langle Au^s(t+\delta), \tilde{u}_t^s(t)\Big\rangle + \frac{1}{\varepsilon}\Big\langle B(u_t^s(t+\delta)), \tilde{u}_t^s(t)\Big\rangle = \Big\langle F(t+\delta), \tilde{u}_t^s(t)\Big\rangle,$$

$$\varepsilon\Big\langle u_{tt}^s(t), \tilde{u}_t^s(t)\Big\rangle + \Big\langle Mu_t^s(t), \tilde{u}_t^s(t)\Big\rangle + \Big\langle Au^s(t), \tilde{u}_t^s(t)\Big\rangle + \frac{1}{\varepsilon}\Big\langle B(u_t^s(t)), \tilde{u}_t^s(t)\Big\rangle = \Big\langle F(t), \tilde{u}_t^s(t)\Big\rangle,$$

where $u^s(x,t) = u^s(x,t+\delta) - u^s(x,t)$.

Subtracting the last inequalities and integrating the result with respect to t, we obtain

$$\begin{aligned}
\int\limits_{Q_\tau}\Bigg[&\varepsilon\big(\tilde{u}_{tt}^s, \tilde{u}_t^s\big) + \sum_{|\alpha|=|\beta|\le l}\Big(B_{\alpha\beta}(x,t+\delta)D^\beta u_t^s(x,t+\delta) - B_{\alpha\beta}(x,t)D^\beta u_t^s(x,t),\, D^\alpha\tilde{u}_t^s\Big) \\
&+ \sum_{|\alpha|=|\beta|\le l}\Big(A_{\alpha\beta}(x,t+\delta)D^\beta u^s(x,t+\delta) - A_{\alpha\beta}(x,t)D^\beta u^s(x,t),\, D^\alpha\tilde{u}_t^s\Big) \\
&+ \sum_{1\le|\alpha|\le l}\Big(C_\alpha(x,t+\delta)D^\alpha u^s(x,t+\delta) - C_\alpha(x,t)D^\alpha u^s(x,t),\, \tilde{u}_t^s\Big) \\
&- \sum_{|\alpha|\le l}\Big(f_\alpha(x,t+\delta) - f_\alpha(x,t),\, D^\alpha\tilde{u}_t^s\Big)\Bigg]dx\,dt \\
&+ \frac{1}{\varepsilon}\int\limits_0^\tau\Big\langle B\big(u_t^s(x,t+\delta)\big) - B\big(u_t^s(x,t)\big), \tilde{u}_t^s\Big\rangle dt = 0.
\end{aligned} \tag{11}$$

We perform certain transformations using estimates (5):

$$\mathfrak{T}_9 = \int\limits_{Q_\tau}\varepsilon\big(\tilde{u}_{tt}^s, \tilde{u}_t^s\big)dx\,dt = \frac{\varepsilon}{2}\int\limits_{\Omega_\tau}\big|\tilde{u}_t^s\big|^2 dx - \frac{\varepsilon}{2}\int\limits_{\Omega_\delta}\big|u_t^s\big|^2 dx,$$

$$\begin{aligned}
\mathfrak{T}_{10} = &\int\limits_{Q_\tau}\sum_{|\alpha|=|\beta|\le l}\Big(B_{\alpha\beta}(x,t+\delta)D^\beta\tilde{u}_t^s,\, D^\alpha\tilde{u}_t^s\Big)dx\,dt \\
&+ \int\limits_{Q_\tau}\sum_{|\alpha|=|\beta|\le l}\Big(\big(B_{\alpha\beta}(x,t+\delta) - B_{\alpha\beta}(x,t)\big)D^\beta u_t^s(x,t),\, D^\alpha\tilde{u}_t^s(x,t)\Big)dx\,dt \\
\ge &\int\limits_{Q_\tau}\Bigg(b_l\sum_{|\alpha|=m}\big|D^\alpha\tilde{u}_t^s\big|^2 + b_0\big|\tilde{u}_t^s\big|^2\Bigg)dx\,dt \\
&- \max_{|\alpha|=|\beta|\le l}\big\|B_{\alpha\beta}(\cdot,t+\delta) - B_{\alpha\beta}(\cdot,t)\big\|_{L^\infty(\Omega)}\int\limits_{Q_\tau}\Bigg(\mu_6\sum_{|\alpha|=l}\big|D^\alpha u_t^s\big|^2 + \mu_7\big|u_t^s\big|^2\Bigg)dx\,dt \\
&- \max_{|\alpha|=|\beta|\le l}\big\|B_{\alpha\beta}(\cdot,t+\delta) - B_{\alpha\beta}(\cdot,t)\big\|_{L^\infty(\Omega)}\int\limits_{Q_\tau}\Bigg(\mu_8\sum_{|\alpha|=l}\big|D^\alpha\tilde{u}_t^s\big|^2 + \mu_9\big|\tilde{u}_t^s\big|^2\Bigg)dx\,dt,
\end{aligned}$$

$$\mathfrak{T}_{11} = \int\limits_{Q_\tau} \sum_{|\alpha|=|\beta|\le l} \Big(A_{\alpha\beta}(x, t+\delta)D^\beta \tilde{u}^s(x,t), D^\alpha \tilde{u}_t^s(x,t)\Big)dx\,dt$$

$$+ \int\limits_{Q_\tau} \sum_{|\alpha|=|\beta|\le l} \Big(\Big(A_{\alpha\beta}(x, t+\delta) - A_{\alpha\beta}(x,t)\Big)D^\beta u^s(x,t), D^\beta \tilde{u}_t^s(x,t)\Big)dx\,dt$$

$$\ge \frac{1}{2}\int\limits_{\Omega_\tau}\left(a_l \sum_{|\alpha|=l} \left|D^\alpha \tilde{u}^s\right|^2 + a_0\left|\tilde{u}^s\right|^2\right)dx - \frac{1}{2}\int\limits_{\Omega_\delta}\left(a_l^1 \sum_{|\alpha|=l}\left|D^\alpha u^s\right|^2 + a_0^1\left|u^s\right|^2\right)dx$$

$$- \frac{1}{2}\int\limits_{Q_\tau}\left(a_l^2 \sum_{|\alpha|=l}\left|D^\alpha \tilde{u}^s\right|^2 + a_0^2\left|\tilde{u}^s\right|^2\right)dx\,dt$$

$$- \max_{|\alpha|=|\beta|\le l}\left\| A_{\alpha\beta}(\cdot, t+\delta) - A_{\alpha\beta}(\cdot, t)\right\|_{L^\infty(\Omega)} \int\limits_{Q_\tau}\left(\mu_{10}\sum_{|\alpha|=l}\left|D^\alpha \tilde{u}_t^s\right|^2 + \mu_{11}\left|\tilde{u}_t^s\right|^2\right)dx\,dt$$

$$- \max_{|\alpha|=|\beta|\le l}\left\| A_{\alpha\beta}(\cdot, t+\delta) - A_{\alpha\beta}(\cdot, t)\right\|_{L^\infty(\Omega)} \int\limits_{Q_\tau}\left(\mu_{12}\sum_{|\alpha|=l}\left|D^\alpha u^s\right|^2 + \mu_{13}\left|u^s\right|^2\right)dx\,dt,$$

$$\mathfrak{T}_{12} = \int\limits_{Q_\tau} \sum_{1\le|\alpha|\le l} \Big((C_\alpha(x, t+\delta) - C_\alpha(x,t))D^\alpha u^s(x,t), \tilde{u}_t^s(x,t)\Big)dx\,dt$$

$$+ \int\limits_{Q_\tau} \sum_{1\le|\alpha|\le l} \Big(C_\alpha(x,t)D^\alpha \tilde{u}^l(t), \tilde{u}_t^s(x,t)\Big)dx\,dt$$

$$\ge -\frac{C}{2\delta_0}\int\limits_{Q_\tau}\left(\mu_{14}\sum_{|\alpha|=l}\left|D^\alpha \tilde{u}^s\right|^2 + \mu_{15}\left|\tilde{u}^s(x,t)\right|^2\right)dx\,dt - \frac{\delta_0}{2}\int\limits_{Q_\tau}\left|\tilde{u}_t^s\right|^2 dx\,dt$$

$$- \max_{1\le|\alpha|\le l}\left\| C_\alpha(\cdot, t+\delta) - C_\alpha(\cdot, t)\right\|_{L^\infty(\Omega)} \int\limits_{Q_\tau}\left(\mu_{16}\sum_{|\alpha|=l}\left|D^\alpha u^s\right|^2 + \mu_{17}\left|u^s(x,t)\right|^2\right)dx\,dt$$

$$- \max_{1\le|\alpha|\le l}\left\| C_\alpha(\cdot, t+\delta) - C_\alpha(\cdot, t)\right\|_{L^\infty(\Omega)} \int\limits_{Q_\tau}\left|\tilde{u}_t^s(x,t)\right|^2 dx\,dt,$$

$$\mathfrak{T}_{13} = \int\limits_{Q_\tau} \sum_{|\alpha|\le l} \Big(f_\alpha(x, t+\delta) - f_\alpha(x,t), D^\alpha \tilde{u}_t^s\Big)dx\,dt$$

$$\le \frac{1}{2\delta_1}\int\limits_{Q_\tau} \sum_{|\alpha|\le l} \left| f_\alpha(x, t+\delta) - f_\alpha(x,t)\right|^2 dx\,dt$$

$$+ \frac{\delta_1}{2}\int\limits_{Q_\tau}\left(\mu_{18}\sum_{|\alpha|=l}\left|D^\alpha \tilde{u}_t^s\right|^2 + \mu_{19}\left|\tilde{u}_t^s\right|^2\right)dx\,dt.$$

Since the functions $A_{\alpha\beta}(x,t)$, $A_{\alpha\beta,t}(x,t)$, $B_{\alpha\beta}(x,t)$, $B_{\alpha\beta,t}(x,t)$, $C_\alpha(x,t)$, and $f_\alpha(x,t)$ are continuous with respect to t, we conclude that, for any $\sigma > 0$, there exists δ such that

$$\max_{|\alpha|=|\beta|\le l} \left\| A_{\alpha\beta}(\cdot, t+\delta) - A_{\alpha\beta}(\cdot, t) \right\|_{L^\infty(\Omega)} \le \sigma,$$

$$\max_{|\alpha|=|\beta|\le l} \left\| B_{\alpha\beta}(\cdot, t+\delta) - B_{\alpha\beta}(\cdot, t) \right\|_{L^\infty(\Omega)} \le \sigma,$$

$$\max_{1\le|\alpha|\le l} \| C_\alpha(\cdot, t+\delta) - C_\alpha(\cdot, t) \|_{L^\infty(\Omega)} \le \sigma,$$

$$\max_{|\alpha|\le l} \| f_\alpha(\cdot, t+\delta) - f_\alpha(\cdot, t) \|_{L^\infty(\Omega)} \le \sigma.$$

Then

$$\varepsilon \int_{\Omega_\tau} \left| \tilde{u}_t^s \right|^2 dx + \int_{\Omega_\tau} \left[(a_l - 2\sigma\mu_{10}) \sum_{|\alpha|=l} \left| D^\alpha \tilde{u}^s \right|^2 + (a_0 - 2\sigma\mu_{11}) \left| \tilde{u}^s \right|^2 \right] dx$$

$$+ \int_{Q_\tau} \left[(2b_l - 2\sigma\mu_8 - \delta_1\mu_{18}) \sum_{|\alpha|=l} \left| D^\alpha \tilde{u}_t^s \right|^2 + (2b_0 - 2\sigma(\mu_9 + 1) - \delta_0 - \delta_1\mu_{19}) \left| \tilde{u}_t^s \right|^2 \right] dx\, dt$$

$$\le \varepsilon \int_{\Omega_\delta} \left| u_t^s \right|^2 dx + \int_{\Omega_\delta} \left(a_l^1 \sum_{|\alpha|=l} \left| D^\alpha u^s \right|^2 + a_0^1 \left| u^s \right|^2 \right) dx$$

$$+ 2\sigma \int_{Q_\tau} \left[(\mu_{12} + \mu_{16}) \sum_{|\alpha|=l} \left| D^\alpha u^s \right|^2 + (\mu_{13} + \mu_{17}) \left| u^s \right|^2 \right] dx\, dt$$

$$+ 2\sigma \int_{Q_\tau} \left(\mu_6 \sum_{|\alpha|=l} \left| D^\alpha u_t^s \right|^2 + \mu_7 \left| u_t^s \right|^2 \right) dx\, dt + \tilde{\nu}\sigma^2.$$

The last inequality yields

$$\varepsilon \int_{\Omega_\tau} \left| u_t^s(x, t+\delta) - u_t^s(x,t) \right|^2 dx \le \nu_2 \sigma,$$

$$\int_{\Omega_\tau} \left(\sum_{|\alpha|=l} \left| D^\alpha \tilde{u}^s(x,t) \right|^2 + \left| \tilde{u}^s \right|^2 \right) dx \le \nu_2 \sigma, \qquad \tau \in [0, T]. \tag{12}$$

It follows from estimates (10) and (12) that there exists a subsequence $\left\{ u^p(x,t) \right\}$ of the sequence $\left\{ u^s(x,t) \right\}$ such that

$$u^p \to u^\varepsilon \quad \text{weakly in} \quad L^2((0,T);V),$$

$$u_t^p \to u_t^\varepsilon \quad \text{weakly in} \quad L^2((0,T);V),$$

$$u^p \to u^\varepsilon \quad \text{uniformly in} \quad C([0,T];V), \tag{13}$$

$$u_t^p \to u_t^\varepsilon \quad \text{uniformly in} \quad C\Big([0,T];\big(L^2(\Omega)\big)^N\Big),$$

$$B\big(u_t^p\big) \to \omega \quad \text{weakly in} \quad L^2((0,T);V^*)$$

as $p \to \infty$.

Assume that k in Eqs. (6) varies from 1 to s_0. According to (13), we have

$$\big\langle L(t)u^s, \varphi^k \big\rangle \to \big\langle L(t)u^\varepsilon, \varphi^k \big\rangle \quad \text{weakly in} \quad L^2(0,T),$$

$$\big\langle M(t)u_t^s, \varphi^k \big\rangle \to \big\langle M(t)u_t^\varepsilon, \varphi^k \big\rangle \quad \text{weakly in} \quad L^2(0,T),$$

$$\big\langle B\big(u_t^s\big), \varphi^k \big\rangle \to \big\langle \omega^\varepsilon, \varphi^k \big\rangle \quad \text{weakly in} \quad L^2(0,T),$$

$$\big\langle u_t^s, \varphi^k \big\rangle \to \big\langle u_t^\varepsilon, \varphi^k \big\rangle \quad \text{weakly in} \quad L^2(0,T)$$

as $s \to \infty$. Therefore,

$$\big\langle u_{tt}^s, \varphi^k \big\rangle = \frac{d}{dt}\big\langle u_t^s, \varphi^k \big\rangle \to \big\langle u_{tt}^\varepsilon, \varphi^k \big\rangle \quad \text{in} \quad \mathcal{D}'(0,T),$$

and, taking into account that $\big\{\varphi^k(x)\big\}$ is a basis in V, we get

$$\varepsilon\big\langle u_{tt}^\varepsilon, v \big\rangle = -\big\langle M(t)u_t^\varepsilon + L(t)u^\varepsilon - F(t), v \big\rangle - \frac{1}{\varepsilon}\big\langle \omega^\varepsilon, v \big\rangle$$

for arbitrary $v \in V$. This equality and the definition of the operators M, L, F, and B yield the inclusion $u_{tt}^\varepsilon \in L^2((0,T);V^*)$. Since the spaces $V \subset L^2(\Omega) \subset V^*$ are compactly imbedded and $u_t \in L^2((0,T);V)$, we conclude that, according to Theorem 1.17 in [12, p. 177], the integral $\int_0^\tau \big\langle u_{tt}^\varepsilon, v \big\rangle dt$ is meaningful for $v \in W$, and

$$\int_0^\tau \big\langle u_{tt}^\varepsilon, v \big\rangle dt = \int_{\Omega_\tau} \big(u_t^\varepsilon, v\big) dx - \int_0^\tau \big\langle u_t^\varepsilon, v_t \big\rangle dt, \qquad \tau \in [0,T].$$

Hence, u^ε satisfies the equality

$$\varepsilon\int_0^\tau \left\langle u_{tt}^\varepsilon, v\right\rangle dt + \int_{Q_\tau}\left[\sum_{|\alpha|=|\beta|\le l}\left(B_{\alpha\beta}D^\beta u_t^s, D^\alpha v\right) + \sum_{|\alpha|=|\beta|\le l}\left(A_{\alpha\beta}D^\beta u^\varepsilon, D^\alpha v\right)\right.$$

$$\left. + \sum_{1\le|\alpha|\le l}\left(C_\alpha D^\alpha u^\varepsilon, v\right) - \sum_{|\alpha|\le l}\left(f_\alpha, D^\alpha v\right)\right]dx\,dt + \frac{1}{\varepsilon}\int_0^\tau \left\langle \omega^\varepsilon, v\right\rangle dt = 0 \qquad (14)$$

for an any function $v \in W$ and any $\tau \in (0, T]$. Since the operator B is monotone [13, p. 384], one can easily prove by analogy with [13, p. 171] that $\omega^\varepsilon = B(u_t^\varepsilon)$.

We set $v = u_t^\varepsilon$ in equality (14). As a result, we get

$$\varepsilon\int_0^\tau \left\langle u_{tt}^\varepsilon, u_t^\varepsilon\right\rangle dt + \int_{Q_\tau}\left[\sum_{|\alpha|=|\beta|\le l}\left(B_{\alpha\beta}D^\beta u_t^\varepsilon, D^\alpha u_t^\varepsilon\right) + \sum_{|\alpha|=|\beta|\le l}\left(A_{\alpha\beta}D^\beta u^\varepsilon, D^\alpha u_t^\varepsilon\right)\right.$$

$$\left. + \sum_{1\le|\alpha|\le l}\left(C_\alpha D^\alpha u^\varepsilon, u_t^\varepsilon\right) - \sum_{|\alpha|\le l}\left(f_\alpha, D^\alpha u_t^\varepsilon\right)\right]dx\,dt + \frac{1}{\varepsilon}\int_0^\tau \left\langle B(u_t^\varepsilon), u_t^\varepsilon\right\rangle dt = 0. \qquad (15)$$

Note that equality (14) for the functions u^ε coincides with equality (8) for the functions u^s. Therefore, by analogy with the case of the sequence $\{u^s\}$, for $\{u^\varepsilon\}$ we get

$$\int_{\Omega_\tau}\left[\left|u^\varepsilon\right|^2 + \sum_{|\alpha|=l}\left|D^\alpha u^\varepsilon\right|^2\right]dx \le \nu_3, \qquad \tau\in[0,T],$$

$$\int_{\Omega_\tau}\left[\left|u^\varepsilon(t+\delta) - u^\varepsilon(t)\right|^2 + \sum_{|\alpha|=l}\left|D^\alpha u^\varepsilon(t+\delta) - D^\alpha u^\varepsilon(t)\right|^2\right]dx \le \nu_4\sigma, \qquad \tau\in[0,T],$$

$$\int_{Q_T}\left[\sum_{|\alpha|=l}\left|D^\alpha u_t^\varepsilon\right|^2 + \left|u_t^\varepsilon\right|^2\right]dx\,dt \le \nu_3, \qquad (16)$$

$$\int_0^T \left\langle B(u_t^\varepsilon), u_t^\varepsilon\right\rangle dt = \varepsilon\nu_3, \qquad \int_{\Omega_\tau}\left|u_t^\varepsilon\right|^2 dx \le \varepsilon\nu_3, \qquad \tau\in[0,T].$$

It follows from estimates (16) that there exists a subsequence $\left\{u^{\varepsilon_m}(x,t)\right\}$ of the set $\left\{u^\varepsilon(x,t)\right\}$ such that

$$u^{\varepsilon_m} \to u \quad \text{weakly in} \quad L^2((0,T);V),$$

$$u_t^{\varepsilon_m} \to u_t \quad \text{weakly in} \quad L^2((0,T);V),$$

$$u^{\varepsilon_m} \to u \quad \text{strongly in} \quad L^2((0,T);V),$$

$$u^{\varepsilon_m} \to u \quad \text{uniformly in} \quad C([0, T]; V),$$

$$\varepsilon_m u^{\varepsilon_m} \to 0 \quad \text{weakly in} \quad L^\infty\Big((0, T); \big(L^2(\Omega)\big)^N\Big) \tag{17}$$

as $\varepsilon_m \to 0$. In equality (14), we set $v = w - u_t^{\varepsilon_m}$, where $w \in W$ and $w \in K$ for almost all $t \in (0, T)$, and transform individual terms as follows:

$$\mathfrak{T}_{14} = -\varepsilon_m \int_0^\tau \left\langle u_t^{\varepsilon_m}, w_t - u_{tt}^{\varepsilon_m} \right\rangle dt + \varepsilon_m \int_{\Omega_\tau} \left\langle u_t^{\varepsilon_m}, w - u_t^{\varepsilon_m} \right\rangle dx$$

$$= -\varepsilon_m \int_0^\tau \left\langle u_t^{\varepsilon_m}, w_t \right\rangle dt - \frac{\varepsilon_m}{2} \int_{\Omega_\tau} \left| u_t^{\varepsilon_m} \right|^2 dx + \varepsilon_m \int_{\Omega_\tau} \left(u_t^{\varepsilon_m}, w \right) dx,$$

$$\mathfrak{T}_{15} = \int_{Q_\tau} \sum_{|\alpha| = |\beta| \le l} \left(B_{\alpha\beta} D^\beta u_t^{\varepsilon_m}, D^\alpha w - D^\alpha u_t^{\varepsilon_m} \right) dx\, dt$$

$$= \int_{Q_\tau} \sum_{|\alpha| = |\beta| \le l} \left(B_{\alpha\beta} D^\beta u_t^{\varepsilon_m}, D^\alpha w \right) dx\, dt - \int_{Q_\tau} \sum_{|\alpha| = |\beta| \le l} \left(B_{\alpha\beta} D^\beta u_t^{\varepsilon_m}, D^\alpha u_t^{\varepsilon_m} \right) dx\, dt,$$

$$\mathfrak{T}_{16} = \int_{Q_\tau} \sum_{|\alpha| = |\beta| \le l} \left(A_{\alpha\beta} D^\beta u^{\varepsilon_m}, D^\alpha w - D^\alpha u_t^{\varepsilon_m} \right) dx\, dt$$

$$= -\frac{1}{2} \int_{\Omega_\tau} \sum_{|\alpha| = |\beta| \le l} \left(A_{\alpha\beta} D^\beta u^{\varepsilon_m}, D^\alpha u^{\varepsilon_m} \right) dx + \frac{1}{2} \int_{Q_\tau} \sum_{|\alpha| = |\beta| \le l} \left(A_{\alpha\beta, t} D^\beta u^{\varepsilon_m}, D^\alpha u^{\varepsilon_m} \right) dx\, dt$$

$$+ \int_{Q_\tau} \sum_{|\alpha| = |\beta| \le l} \left(A_{\alpha\beta} D^\beta u^{\varepsilon_m}, D^\alpha w \right) dx\, dt,$$

$$\mathfrak{T}_{17} = \int_{Q_\tau} \sum_{1 \le |\alpha| \le l} \left(C_\alpha D^\alpha u^{\varepsilon_m}, w - u_t^{\varepsilon_m} \right) dx\, dt$$

$$= \int_{Q_\tau} \sum_{1 \le |\alpha| \le l} \left(C_\alpha D^\alpha u^{\varepsilon_m}, w \right) dx\, dt - \int_{Q_\tau} \sum_{1 \le |\alpha| \le l} \left(C_\alpha D^\alpha u^{\varepsilon_m}, u_t^{\varepsilon_m} \right) dx\, dt.$$

Since $B(w) = 0$ and $\langle B(u_t) - B(w), u_t - w \rangle \ge 0$, taking into account the expressions for $\mathfrak{T}_{14}, \ldots, \mathfrak{T}_{17}$ we get

$$-\varepsilon_m \int_0^\tau \left\langle u_t^{\varepsilon_m}, w_t \right\rangle dt + \varepsilon_m \int_{\Omega_\tau} \left(u_t^{\varepsilon_m}, w \right) dx + \int_{Q_\tau} \Bigg[\sum_{|\alpha| = |\beta| \le l} \left(B_{\alpha\beta} D^\beta u_t^{\varepsilon_m}, D^\alpha w \right)$$

$$+ \sum_{|\alpha| = |\beta| \le l} \left(A_{\alpha\beta} D^\beta u^{\varepsilon_m}, D^\alpha w \right) + \sum_{1 \le |\alpha| \le l} \left(C_\alpha D^\alpha u^{\varepsilon_m}, w \right) \Bigg] dx\, dt$$

$$- \int_{Q_\tau} \Bigg[\sum_{|\alpha|=|\beta|\le l} \left(B_{\alpha\beta} D^\beta u_t^{\varepsilon_m}, D^\alpha u_t^{\varepsilon_m} \right)$$

$$- \sum_{|\alpha|=|\beta|\le l} \left(A_{\alpha\beta,\,t} D^\beta u^{\varepsilon_m}, D^\alpha u^{\varepsilon_m} \right) + \sum_{1\le|\alpha|\le l} \left(C_\alpha D^\alpha u^{\varepsilon_m}, u_t^{\varepsilon_m} \right) \Bigg] dx\,dt$$

$$- \int_{Q_\tau} \sum_{|\alpha|\le l} \left(f_\alpha, D^\beta w - D^\alpha u_t^{\varepsilon_m} \right) dx\,dt - \frac{1}{2} \int_{\Omega_\tau} \sum_{|\alpha|=|\beta|\le l} \left(A_{\alpha\beta} D^\alpha u^{\varepsilon_m}, D^\alpha u^{\varepsilon_m} \right) dx \ \ge\ 0. \tag{18}$$

By virtue of condition (B), we have

$$-\liminf_{\varepsilon_m\to 0} \int_{Q_\tau} \sum_{|\alpha|=|\beta|\le l} \left(B_{\alpha\beta} D^\beta u_t^{\varepsilon_m}, D^\alpha u_t^{\varepsilon_m} \right) dx\,dt \ \le\ -\int_{Q_\tau} \sum_{|\alpha|=|\beta|\le l} \left(B_{\alpha\beta} D^\beta u_t, D^\alpha u_t \right) dx\,dt. \tag{19}$$

Hence, by using (17) and (19), we determine the lower bound in inequality (18) as $\varepsilon_m \to 0$. We have

$$\int_{Q_\tau} \Bigg[\sum_{|\alpha|=|\beta|\le l} \left(B_{\alpha\beta} D^\beta u_t, D^\alpha w \right) + \sum_{|\alpha|=|\beta|\le l} \left(A_{\alpha\beta} D^\beta u, D^\alpha w \right) + \sum_{1\le|\alpha|\le l} \left(C_\alpha D^\beta u, w \right) \Bigg] dx\,dt$$

$$- \int_{Q_\tau} \Bigg[\sum_{|\alpha|\le|\beta|\le l} \left(B_{\alpha\beta} D^\beta u_t, D^\alpha u_t \right) - \frac{1}{2} \sum_{|\alpha|\le|\beta|\le l} \left(A_{\alpha\beta,\,t} D^\beta u, D^\alpha u \right) + \sum_{1\le|\alpha|\le l} \left(C_\alpha D^\alpha u, u_t \right) \Bigg] dx\,dt$$

$$- \int_{Q_\tau} \sum_{|\alpha|\le l} \left(f_\alpha, D^\alpha w - D^\alpha u_t \right) dx\,dt - \frac{1}{2} \int_{\Omega_\tau} \sum_{|\alpha|=|\beta|\le l} \left(A_{\alpha\beta} D^\beta u, D^\alpha u \right) dx \ \ge\ 0,$$

or

$$\int_{Q_\tau} \Bigg[\sum_{|\alpha|=|\beta|\le l} \left(B_{\alpha\beta}(x,t) D^\beta u_t, D^\alpha (w - u_t) \right) + \sum_{|\alpha|=|\beta|\le l} \left(A_{\alpha\beta} D^\beta u, D^\alpha (w - u_t) \right)$$

$$+ \sum_{1\le|\alpha|\le l} \left(C_\alpha(x,t) D^\alpha u, w - u_t \right) + \sum_{|\alpha|\le l} \left(f_\alpha, D^\alpha (w - u_t) \right) \Bigg] dx\,dt \ \ge\ 0$$

for any $w \in W$ and $w \in K$ for almost all $t \in (0, T)$. Moreover, by analogy with [13, p. 386], one can prove that $u_t \in K$ for almost all $t \in (0, T)$. Hence, $u(x, t)$ is a solution of problem (1), (2), and Theorem 2 is proved.

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