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Homogenization of some degenerate pseudoparabolic variational inequalities

Mariya Ptashnyk [a,b,*]

[a] *Department of Mathematics, School of Mathematical and Computer Sciences, Heriot-Watt University, Edinburgh EH14 4AS, UK*
[b] *Division of Mathematics, University of Dundee, Dundee, DD1 4HN, UK*

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ABSTRACT

Multiscale analysis of a degenerate pseudoparabolic variational inequality, modelling the two-phase flow with dynamical capillary pressure in a perforated domain, is the main topic of this work. Regularisation and penalty operator methods are applied to show the existence of a solution of the nonlinear degenerate pseudoparabolic variational inequality defined in a domain with microscopic perforations, as well as to derive a priori estimates for solutions of the microscopic problem. The main challenge is the derivation of a priori estimates for solutions of the variational inequality, uniformly with respect to the regularisation parameter and to the small parameter defining the scale of the microstructure. The method of two-scale convergence is used to derive the corresponding macroscopic obstacle problem.

1. Introduction

In this paper we consider multiscale analysis of a nonlinear degenerate pseudoparabolic variational inequality modelling unsaturated flow with dynamic capillary pressure in a perforated porous medium. Models for two-phase flow with dynamical capillary pressure, originally proposed by [16,38], consider Darcy's law for the flux of the moisture content u given by

$$J = -A\, k(u)(\nabla p + \mathbf{e}_n),$$

and assume that the pressure p in the wetting phase is a function of the moisture content u and its time derivative $\partial_t u$, i.e. in a simplified form,

$$p = -\tilde{P}_c(u) + \tau \partial_t u,$$

* Correspondence to: Department of Mathematics, School of Mathematical and Computer Sciences, Heriot-Watt University, Edinburgh EH14 4AS, UK.
E-mail address: m.ptashnyk@hw.ac.uk.

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where the permeability function $k(u)$ depends on the moisture content, the vector $\mathbf{e}_n = (0, \dots, 0, 1)$ determines the direction of flow due to gravity, and A and τ are positive constants. Then for the moisture content u we obtain a pseudoparabolic equation of the form

$$\partial_t u = \nabla \cdot \big(A\, k(u)[P_c(u)\nabla u + \tau \nabla \partial_t u + \mathbf{e}_n]\big), \tag{1}$$

where $P_c(u) = -\tilde{P}_c'(u)$.

If considering a two-phase flow problem in a perforated porous medium with Signorini's type conditions on the surfaces of perforations

$$\begin{aligned} &u \geq 0, \quad A\, k(u)(P_c(u)\nabla u + \tau \nabla \partial_t u + \mathbf{e}_n) \cdot \nu \geq -f(t, x, u), \\ &u\big[A\, k(u)(P_c(u)\nabla u + \tau \nabla \partial_t u + \mathbf{e}_n) \cdot \nu + f(t, x, u)\big] = 0, \end{aligned} \tag{2}$$

then a weak formulation of equation (1) together with conditions (2) results in a pseudoparabolic variational inequality of the form

$$\langle \partial_t u, v - u\rangle_{G^\varepsilon} + \langle A\, k(u)[P_c(u)\nabla u + \tau \partial_t \nabla u + \mathbf{e}_n], \nabla(v - u)\rangle_{G^\varepsilon} + \langle f(t, x, u), v - u\rangle_{\Gamma^\varepsilon} \geq 0, \tag{3}$$

where $G^\varepsilon \subset \mathbb{R}^n$, with $n = 2, 3$, denotes the perforated domain and Γ^ε defines the boundaries of perforations.

As an example of a porous medium with microscopic perforations we can consider a part of the soil perforated by a root network, where conditions (2) model water (solute) uptake by plant roots.

In our analysis of the obstacle problem (1) and (2), or equivalently variational inequality (3), defined in a heterogeneous perforated domain G^ε, where ε denotes a characteristic size of perforations, we shall consider a function $A(x)$ describing the heterogeneity of the medium, instead of a constant A, and a more general convection term, describing flow transport by a given velocity field.

Along with models for two-phase flow with dynamic capillary pressure [12,16,38], pseudoparabolic equations are also used to model fluid filtration in fissured porous media [3], heat transfer in a heterogeneous medium [43], or to regularise ill-posed transport problems [4,36]. Pseudoparabolic variational inequalities are considered to describe obstacle [45] and free boundary problems [13]. The well-posedness for non-degenerate pseudoparabolic equations and variational inequalities was studied by many authors [6,8,13,22,31,40,41,45, 48]. Global existence results for degenerate pseudoparabolic equations are obtained in [8,31]. The multiscale analysis for non-degenerate pseudoparabolic equations was considered in [39] and the method of two-scale convergence was applied to derive the corresponding macroscopic equations. To the best of our knowledge, there are no results on homogenization of pseudoparabolic variational inequalities. Several results are known on multiscale analysis of elliptic [9,19,20,37,44,50] and parabolic [18,29,42,46] variational inequalities. In [9] the periodic unfolding method was used to derive macroscopic variational inequality for the microscopic Signorini–Tresca problem. The method of two-scale convergence was applied to derive macroscopic problems for microscopic linear elasticity equations with boundary conditions of Signorini types [19], elliptic variational inequalities for obstacle problems [44], and evolutionary variational inequalities [29]. Weak convergence and construction of a corrector were considered in [20,46,50] to derive macroscopic problems for microscopic elliptic and parabolic variational inequalities under certain conditions on the relation between the period and the size of the microstructure. In [42] the multiscale analysis of a parabolic variation inequality corresponding to the Stefan problem was performed using the H-convergence method [33]. Homogenization of variational inequalities in domains with thick junctions, for which standard extension results do not hold, was studied in [28,27,26] using the method of monotone operators and construction of appropriate auxiliary functions.

To prove existence of a solution of the microscopic problem, considered here, the regularisation of degenerate coefficients in the pseudoparabolic variational inequality together with a proper choice of test functions, similar to those proposed in [8,31] for pseudoparabolic equations, is considered. In the case of variational inequalities additional care is required due to the fact that admissible test functions have to belong to a convex subset of the corresponding function space. The penalty operator method is applied to show existence of a solution of the pseudoparabolic variational inequality with regularised coefficients. To pass to the limit in the nonlinear penalty operator we prove strong convergence of approximations of solutions of the corresponding nonlinear pseudoparabolic equation. The main step in the analysis and derivation of the macroscopic variational inequality, for the microscopic problem considered here, is to derive a priori estimates uniformly with respect to small parameter ε. The main idea in the derivation of a priori estimates for the time derivative of the gradient of a solution of variational inequality, similar to [31], is to use the specific structure of the degenerate coefficients which allows to prove that some negative power of a solution of the variational inequality is a L^p-function with $1 < p < 2$. The uniqueness result is obtained in the case when the coefficient $k(u)$ in front of the pseudoparabolic term is non-degenerate and under additional regularity assumptions on solutions of the pseudoparabolic variational inequality.

The paper is organised as follows. In Section 2 we formulate the microscopic obstacle problem defined in a perforated domain G^ε. In Section 3 we prove existence and uniqueness results for the regularised problem, derive a priori estimates, and show existence of a solution of the original degenerate pseudoparabolic variational inequality defined in the perforated domain G^ε. In Section 4 we prove convergence results as $\varepsilon \to 0$ and derive macroscopic problem defined in a homogeneous domain G with the constraint $u(t,x) \geq 0$ in $(0,T) \times G$. In Appendix we summarise the main compactness results for the two-scale convergence used in the derivation of the macroscopic pseudoparabolic variational inequality.

2. Formulation of mathematical problem

A general obstacle problem can be formulated as a variational inequality

$$\begin{aligned} &u \in \mathcal{K}(t), \\ &\langle \partial_t b(u), v - u\rangle + \langle \mathcal{A}(x, \nabla u, \partial_t \nabla u), \nabla(v-u)\rangle \geq \langle R(t,x,u), v-u\rangle \end{aligned} \tag{4}$$

for $v \in L^2(0,T;\mathcal{K}(t))$, where $\mathcal{K}(t)$ is a closed convex set in $H^1(G)$. We shall consider variational inequality (4) defined in a perforated domain G^ε with a periodic distribution of perforations.

To define the domain G^ε, where ε denotes the characteristic size of perforations, we consider a bounded domain $G \subset \mathbb{R}^n$, for $n = 2, 3$, where G is quasi-convex or $\partial G \subset C^{1,\alpha}$ for some $0 < \alpha < 1$, a 'unit cell' $Y \subset \mathbb{R}^n$, a subset Y^0, with $\overline{Y^0} \subset Y$ and Lipschitz boundary $\Gamma = \partial Y^0$, and denote $Y^* = Y \setminus \overline{Y^0}$. Then

$$G_0^\varepsilon = \bigcup_{\xi \in \Xi^\varepsilon} \varepsilon(Y^0 + \xi), \quad \widetilde{G}^\varepsilon = \operatorname{Int} \bigcup_{\xi \in \Xi^\varepsilon} \varepsilon(\overline{Y} + \xi),$$

where $\Xi^\varepsilon = \{\xi \in \mathbb{Z}^n \ : \ \varepsilon(\overline{Y^0} + \xi) \subset G\}$, and $G^\varepsilon = G \setminus \overline{G_0^\varepsilon}$. The boundaries of perforations are defined by

$$\Gamma^\varepsilon = \bigcup_{\xi \in \Xi^\varepsilon} \varepsilon(\Gamma + \xi).$$

For the nonlinear function $\mathcal{A}$ in the variational inequality in (4) we consider

$$\mathcal{A}(x, \nabla u^\varepsilon, \partial_t \nabla u^\varepsilon) = A^\varepsilon(x) k(u^\varepsilon)(P_c(u^\varepsilon)\nabla u^\varepsilon + \partial_t \nabla u^\varepsilon) - F^\varepsilon(t,x,u^\varepsilon),$$

and assume that $R(t,x,u^\varepsilon) = 0$, where the functions b, A^ε, k, P_c, and F^ε are specified below. On the microscopic boundaries Γ^ε we specify the following Signorini type conditions

$$
\begin{aligned}
u^\varepsilon &\geq 0, \\
\big(A^\varepsilon(x)k(u^\varepsilon)[P_c(u^\varepsilon)\nabla u^\varepsilon + \partial_t \nabla u^\varepsilon] - F^\varepsilon(t,x,u^\varepsilon)\big)\cdot \nu + \varepsilon f^\varepsilon(t,x,u^\varepsilon) &\geq 0, \\
u^\varepsilon \left[\big(A^\varepsilon(x)k(u^\varepsilon)[P_c(u^\varepsilon)\nabla u^\varepsilon + \partial_t \nabla u^\varepsilon] - F^\varepsilon(t,x,u^\varepsilon)\big)\cdot \nu + \varepsilon f^\varepsilon(t,x,u^\varepsilon)\right] &= 0,
\end{aligned}
$$

where function f^ε is specified below. Then the closed convex set $\mathcal{K}^\varepsilon$ is defined as

$$
\mathcal{K}^\varepsilon = \{v \in H^1(G^\varepsilon) \ : \ v = \kappa_D \ \text{ on } \partial G, \ v \geq 0 \text{ on } \Gamma^\varepsilon\}, \tag{5}
$$

with some constant $0 < \kappa_D \leq 1$, and the corresponding variational inequality reads

$$
\begin{aligned}
&\langle \partial_t b(u^\varepsilon), v - u^\varepsilon\rangle_{G^\varepsilon_T} + \langle A^\varepsilon(x)k(u^\varepsilon)[P_c(u^\varepsilon)\nabla u^\varepsilon + \partial_t\nabla u^\varepsilon], \nabla(v-u^\varepsilon)\rangle_{G^\varepsilon_T} \\
&\quad - \langle F^\varepsilon(t,x,u^\varepsilon), \nabla(v-u^\varepsilon)\rangle_{G^\varepsilon_T} + \langle \varepsilon f^\varepsilon(t,x,u^\varepsilon), v-u^\varepsilon\rangle_{\Gamma^\varepsilon_T} \geq 0,
\end{aligned} \tag{6}
$$

for $v - \kappa_D \in L^2(0,T;V)$ and $v(t) \in \mathcal{K}^\varepsilon$ for $t \in (0,T)$, where

$$
V = \{v \in H^1(G^\varepsilon) : \ v = 0 \ \text{ on } \partial G\}.
$$

Here we use notation $G_T = (0,T)\times G$, $G^\varepsilon_T = (0,T)\times G^\varepsilon$, $\Gamma_T = (0,T)\times\Gamma$, $\Gamma^\varepsilon_T = (0,T)\times\Gamma^\varepsilon$, $Y_T = (0,T)\times Y$, $Y^*_T = (0,T)\times Y^*$, and

$$
\begin{aligned}
\langle \phi, \psi\rangle_{G^\varepsilon_T} &= \int_0^T\!\!\int_{G^\varepsilon} \phi\,\psi\, dxdt, \qquad \text{for } \phi \in L^p(0,T;L^q(G^\varepsilon)), \ \psi \in L^{p'}(0,T;L^{q'}(G^\varepsilon)), \\
\langle \phi, \psi\rangle_{\Gamma^\varepsilon_T} &= \int_0^T\!\!\int_{\Gamma^\varepsilon} \phi\,\psi\, d\gamma dt, \qquad \text{for } \phi \in L^p(0,T;L^q(\Gamma^\varepsilon)), \ \psi \in L^{p'}(0,T;L^{q'}(\Gamma^\varepsilon)),
\end{aligned}
$$

where $1 < p, p', q, q' < \infty$ with $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$.

Remark. Notice that $\langle\cdot,\cdot\rangle_{G^\varepsilon_T}$ and $\langle\cdot,\cdot\rangle_{\Gamma^\varepsilon_T}$ are used as short notation for an integral of a product of two functions. In most cases we will consider a product of two L^2-functions, however we shall use the same notation for the integral of a product of L^p- and $L^{p'}$-functions, which is well defined.

We shall consider the following assumptions on functions A^ε, b, k, P_c, F^ε, and f^ε.

Assumption 2.1.

1) $k : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, nondecreasing, with $k(z) > 0$ for $z > 0$ and $k(0) = 0$, e.g.

$$
k(z) = \frac{\vartheta_k z^\beta}{1 + \gamma_k z^\beta} \qquad \text{for some } \vartheta_k, \, \gamma_k > 0 \text{ and } \beta \geq 1,
$$

$$
P_c(z) = \frac{\vartheta_p z^{-\lambda}}{1 + \gamma_p(z) z^\lambda} \quad \text{for } \ \vartheta_p, \lambda > 0, \ \text{nonnegative } \gamma_p \in C^\infty_0(\mathbb{R}), \ \text{and } |k(z)P_c(z)| \leq C < \infty \ \text{ for } z \geq 0.
$$

2) $A \in L^\infty(Y)$ is extended Y-periodically to $\mathbb{R}^n$, and $A(y) \geq a_0 > 0$ for $y \in Y$, with $A^\varepsilon(x) = A(x/\varepsilon)$ for $x \in \mathbb{R}^n$.
3) $b : \mathbb{R} \to \mathbb{R}$ is continuous, nondecreasing, and twice continuously differentiable for $z > 0$, with $b(z) > 0$ for $z > 0$, $b(0) = 0$, and $|b'(z)| \leq \gamma_b(1 + z^2)$ for $z \geq 1$ and $\gamma_b > 0$, e.g. $b(z) = \vartheta_b z^\alpha$, with $0 < \alpha \leq 3$ and $\vartheta_b > 0$.

4) $F^\varepsilon : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is Lipschitz continuous, $F^\varepsilon(t,x,z) = Q^\varepsilon(t,x)H(z) + k(z)g$, where $|H'(z)(b'(z))^{-\frac{1}{2}}| \le C < \infty$ for $z \ge 0$, $g \in \mathbb{R}^n$ is a constant vector, $\nabla_x \cdot Q^\varepsilon(t,x) = 0$ for $(t,x) \in G^\varepsilon_T$, $Q^\varepsilon(t,x)\cdot\nu = 0$ on Γ^ε_T, $Q^\varepsilon \in L^\infty(G^\varepsilon_T)$, and $Q^\varepsilon(t,x) \to Q(t,x,y)$ strongly two-scale, $Q \in L^2(G_T, H_{\rm div}(Y^*)) \cap L^\infty(G_T \times Y^*)$, where $H_{\rm div}(Y^*) = \{v \in L^2(Y^*)^n,\ \nabla_y \cdot v = 0 \text{ in } Y^*,$ and v is Y-periodic$\}$.
5) $f^\varepsilon(t,x,\xi) = f_0(t,x/\varepsilon)f_1(\xi)$, where $f_0 \in C^1([0,T]; C^1_{\rm per}(\Gamma))$, with $f_0(t,y) \ge 0$ for $(t,y) \in \Gamma_T$, and $f_1 \in C^1_0(\mathbb{R})$, with $\xi f_1(\xi) \ge 0$, $f_1(0) = 0$, and

$$\Big| f_1(\xi) \int_\xi^{\kappa_D} \frac{d\eta}{k(\eta)} \Big| \le C \quad \text{for} \ \ 0 \le \xi \le \kappa_D.$$

6) Initial condition $u_0 \in \mathcal{K}$ and

$$\int_{\kappa_D}^{u_0} b'(\xi) \int_{\kappa_D}^{\xi} \frac{dz}{k(z)}\, d\xi \ \in L^1(G),$$

$$\text{where} \quad \mathcal{K} = \{w \in H^1(G):\ w = \kappa_D \ \text{ on } \partial G \ \text{ and } \ w \ge 0 \text{ in } G\}. \tag{7}$$

Remark. Notice that assumptions 1) and 4) in Assumption 2.1 are similar to the corresponding assumptions in [8,31], however for the vector field Q^ε additional assumptions are required due to the perforated microstructure of domain G^ε. Function F^ε describes the directed flow due to a given velocity field Q^ε and gravity g. As an example of a function Q^ε satisfying assumption 4) we can consider a solution of the Stokes problem

$$\begin{aligned} -\mu\,\varepsilon^2 \Delta Q^\varepsilon + \nabla p^\varepsilon = 0 \quad &\text{in } \ G^\varepsilon, \qquad \operatorname{div} Q^\varepsilon = 0 \quad \text{in } \ G^\varepsilon, \\ Q^\varepsilon = 0 \quad &\text{on } \ \Gamma^\varepsilon, \qquad\quad Q^\varepsilon = v \quad \text{on } \ \partial G, \end{aligned} \tag{8}$$

for $t \in (0,T)$ and a given velocity $v \in L^\infty(0,T;H^2(G))^n$ with $\operatorname{div} v(t,x) = 0$ for $x \in G$ and $t \in (0,T)$. The regularity theory for Stokes equations, see e.g. [7,14,32,51], implies that for each fixed ε there exists a solution $(Q^\varepsilon, p^\varepsilon) \in L^\infty(0,T;W^{1,p}(G^\varepsilon)) \times L^\infty(0,T;L^p(G^\varepsilon)/\mathbb{R})$, with $2 \le p < n + \delta_1$ and some $\delta_1 > 0$, of system (8). Then using the Sobolev embedding theorem we obtain $Q^\varepsilon \in L^\infty(0,T;L^\infty(G^\varepsilon))$. The multiscale analysis results for the Stokes system, see e.g. [17], imply existence of a velocity field $Q \in L^\infty(0,T;L^2(G;H^1_{\rm per}(Y^*)))$, pressure $p \in L^\infty(0,T;L^2(G)/\mathbb{R})$, and $\pi \in L^\infty(0,T;L^2(G\times Y^*)/\mathbb{R})$, such that $Q^\varepsilon \rightharpoonup Q$ two-scale, $p^\varepsilon \rightharpoonup p$ weakly-$*$ in $L^\infty(0,T;L^2(G))$, and Q is a solution of

$$\begin{aligned} -\mu\,\Delta_y Q + \nabla_y \pi + \nabla p = 0 \qquad\quad &\text{in } \ Y^*, \\ \operatorname{div}_y Q = 0 \quad \text{in } \ Y^*, \quad Q = 0 \qquad &\text{on } \ \Gamma, \end{aligned} \tag{9}$$

and $\operatorname{div} \int_{Y^*} Q(t,x,y)dy = 0$ for $(t,x) \in G_T$, with

$$\operatorname{div}(K\nabla p) = 0 \text{ in } G, \qquad K\nabla p \cdot \nu = v \cdot \nu \quad \text{on } \partial G,$$

for $t \in [0,T]$ and constant permeability tensor K determined by the corresponding 'unit cell' problems. Using the regularity theory for elliptic equations with Neumann boundary conditions, together with the assumptions on G and v, we obtain $\nabla p \in L^\infty(0,T;L^\infty(G))^n$, see e.g. [5,15,21]. Then applying the regularity results for the Stokes system, see e.g. [7,14,32], to problem (9) yields $Q \in L^\infty(G_T;L^\infty(Y^*))$. Notice that in (9) variables t and x play the role of parameters in the Stokes operator with respect to the microscopic variable y.

To show strong two-scale convergence of Q^ε we consider $Q^\varepsilon - R^\varepsilon_{Y^*}(v)$, where $R^\varepsilon_{Y^*}(v)(x) = R_{Y^*}(v^\varepsilon_\xi)(x/\varepsilon)$, with $v^\varepsilon_\xi(y) = v(\varepsilon y)$ for $y \in \varepsilon(Y+\xi)$ and $\xi \in \Xi^\varepsilon$, and $R_{Y^*} : W^{1,p}(Y)^n \to W^{1,p}_\Gamma(Y^*)^n$, for $1 < p < \infty$, is a restriction operator, see e.g. [30,49], as a test function in (8) and obtain

$$\mu\|\nabla_y Q\|^2_{L^2(G\times Y^*)} \le \mu \liminf_{\varepsilon\to 0} \|\varepsilon\nabla Q^\varepsilon\|^2_{L^2(G^\varepsilon)} \le \mu \limsup_{\varepsilon\to 0} \|\varepsilon\nabla Q^\varepsilon\|^2_{L^2(G^\varepsilon)} = \lim_{\varepsilon\to 0} \varepsilon^2\mu\langle \nabla Q^\varepsilon, \nabla R^\varepsilon_{Y^*}(v)\rangle_{G^\varepsilon} \tag{10}$$

for $t \in [0,T]$. Here $W^{1,p}_\Gamma(Y^*)^n = \{w \in W^{1,p}(Y^*)^n \; : \; w = 0 \text{ on } \Gamma\}$. Notice that $R^\varepsilon_{Y^*}(v) = v$ in $G \setminus \widetilde{G}^\varepsilon$ and the construction of the restriction operator ensures

$$\begin{aligned}&\|\mathcal{T}^{\varepsilon_n} R^{\varepsilon_n}_{Y^*}(v) - \mathcal{T}^{\varepsilon_m} R^{\varepsilon_m}_{Y^*}(v)\|_{L^2(G\times Y^*)} + \|\nabla_y(\mathcal{T}^{\varepsilon_n} R^{\varepsilon_n}_{Y^*}(v) - \mathcal{T}^{\varepsilon_m} R^{\varepsilon_m}_{Y^*}(v))\|_{L^2(G\times Y^*)} \\ &\le C\big[\|\mathcal{T}^{\varepsilon_n} v - \mathcal{T}^{\varepsilon_m} v\|_{L^2(G\times Y)} + \|\nabla_y \mathcal{T}^{\varepsilon_n} v - \nabla_y \mathcal{T}^{\varepsilon_m} v\|_{L^2(G\times Y)}\big] \to 0\end{aligned}$$

as $n, m \to \infty$ and for $t \in [0,T]$, where $\mathcal{T}^\varepsilon$ is the periodic unfolding operator, see e.g. [11]. Hence $R^\varepsilon_{Y^*}(v) \to \hat{R}_{Y^*}(v)$ and $\varepsilon\nabla R^\varepsilon_{Y^*}(v) \to \nabla_y \hat{R}_{Y^*}(v)$ strongly two-scale as $\varepsilon \to 0$, with $\hat{R}_{Y^*}(v) \in L^\infty(G_T; H^1_{\text{per}}(Y^*))$. Then using the two-scale convergence of Q^ε we obtain

$$\lim_{\varepsilon\to 0} \varepsilon^2\mu\langle \nabla Q^\varepsilon, \nabla R^\varepsilon_{Y^*}(v)\rangle_{G^\varepsilon} = \mu\langle \nabla_y Q, \nabla_y \hat{R}_{Y^*}(v)\rangle_{L^2(G\times Y^*)} \tag{11}$$

for $t \in [0,T]$. Taking $Q - \hat{R}_{Y^*}(v)$ as a test function in (9) and using the fact that $\hat{R}_{Y^*}(v)(t,x,\cdot)$ is Y-periodic, $\hat{R}_{Y^*}(v) = 0$ on Γ, $\text{div}_y \hat{R}_{Y^*}(v) = 0$, and $\text{div} \int_{Y^*} \hat{R}_{Y^*}(v)(t,x,y)dy = 0$, yield

$$\mu \,\langle \nabla_y Q, \nabla_y Q - \nabla_y \hat{R}_{Y^*}(v)\rangle_{G\times Y^*} = 0$$

for $t \in [0,T]$. Combining the last equality with inequality (10) and convergence in (11) implies

$$\lim_{\varepsilon\to 0} \|\varepsilon\nabla Q^\varepsilon\|_{L^2(G^\varepsilon)} = \|\nabla_y Q\|_{L^2(G\times Y^*)}$$

for $t \in [0,T]$, and we have the strong two-scale convergence of $\varepsilon\nabla Q^\varepsilon$ and strong convergence of unfolded sequence $\nabla_y \mathcal{T}^\varepsilon Q^\varepsilon$ in $L^2(G_T \times Y^*)$. Using zero Dirichlet boundary conditions on Γ^ε and applying the Poincaré inequality we obtain

$$\|\mathcal{T}^{\varepsilon_m} Q^{\varepsilon_m} - \mathcal{T}^{\varepsilon_n} Q^{\varepsilon_n}\|_{L^2(G_T\times Y^*)} \le C\|\nabla_y(\mathcal{T}^{\varepsilon_m} Q^{\varepsilon_m} - \mathcal{T}^{\varepsilon_n} Q^{\varepsilon_n})\|_{L^2(G_T\times Y^*)} \to 0$$

as $n, m \to \infty$. Thus we have strong convergence of $\mathcal{T}^\varepsilon Q^\varepsilon$ in $L^2(G_T \times Y^*)$ and strong two-scale convergence of Q^ε to Q.

As next we give the definition of a solution of the microscopic inequality (6).

Definition 2.2. A solution of inequality (6) is a function u^ε such that $u^\varepsilon - \kappa_D \in L^2(0,T;V)$, $\partial_t b(u^\varepsilon) \in L^2(0,T;L^r(G^\varepsilon))$, with $6/5 \le r < 4/3$, $\sqrt{k(u^\varepsilon)}\nabla\partial_t u^\varepsilon \in L^2(G^\varepsilon_T)$, and $u^\varepsilon(t) \in \mathcal{K}^\varepsilon$ for $t \in [0,T]$, and u^ε satisfies variational inequality (6) for $v \in L^2(0,T;\mathcal{K}^\varepsilon)$ and initial condition $u^\varepsilon(t) \to u_0$ in $L^2(G^\varepsilon)$ as $t \to 0$.

3. A priori estimates and existence result

Similar to [31], in order to prove the existence result for variational inequality (6), we first consider regularisation of functions b, k, and P_c, given by $b_\delta(v) = b(v^+ + \delta)$, with $b_\delta(v) = b(v)$ if $b(v) = \vartheta_b v$ for some constant $\vartheta_b > 0$, $k_\delta(v) = k(v^+ + \delta)$, and $P_{c,\delta}(v) = P_c(v^+ + \delta)$, where $\delta > 0$ and $v^+ = \max\{v, 0\}$.

Then the corresponding regularised problem reads

$$\begin{aligned}
&\langle \partial_t b_\delta(u^\varepsilon_\delta), v - u^\varepsilon_\delta\rangle_{G^\varepsilon_T} + \langle A^\varepsilon(x) k_\delta(u^\varepsilon_\delta)[P_{c,\delta}(u^\varepsilon_\delta)\nabla u^\varepsilon_\delta + \partial_t \nabla u^\varepsilon_\delta], \nabla(v - u^\varepsilon_\delta)\rangle_{G^\varepsilon_T} \\
&\quad - \langle F^\varepsilon(t, x, u^\varepsilon_\delta), \nabla(v - u^\varepsilon_\delta)\rangle_{G^\varepsilon_T} + \langle \varepsilon f^\varepsilon(t, x, u^\varepsilon_\delta), v - u^\varepsilon_\delta\rangle_{\Gamma^\varepsilon_T} \geq 0, \\
&\text{and } u^\varepsilon_\delta(t) \in \mathcal{K}^\varepsilon, \quad \text{for } v \in L^2(0, T; \mathcal{K}^\varepsilon),
\end{aligned} \tag{12}$$

and $u^\varepsilon_\delta(0) = u_0$ in L^2-sense.

To show the existence of a solution of problem (12) we apply the penalty operator method [23,24] and consider

$$\begin{aligned}
&\partial_t b_\delta(u^\varepsilon_{\delta,\mu}) - \nabla \cdot \big(A^\varepsilon(x) k_\delta(u^\varepsilon_{\delta,\mu})[P_{c,\delta}(u^\varepsilon_{\delta,\mu})\nabla u^\varepsilon_{\delta,\mu} + \partial_t \nabla u^\varepsilon_{\delta,\mu}]\big) + \nabla \cdot F^\varepsilon(t, x, u^\varepsilon_{\delta,\mu}) \\
&\qquad + \frac{1}{\mu}\mathcal{B}(u^\varepsilon_{\delta,\mu} - \kappa_D) = 0 \quad \text{in } G^\varepsilon_T, \\
&\big(A^\varepsilon(x) k_\delta(u^\varepsilon_{\delta,\mu})[P_{c,\delta}(u^\varepsilon_{\delta,\mu})\nabla u^\varepsilon_{\delta,\mu} + \partial_t \nabla u^\varepsilon_{\delta,\mu}] - F^\varepsilon(t, x, u^\varepsilon_{\delta,\mu})\big) \cdot \nu = -\varepsilon f^\varepsilon(t, x, u^\varepsilon_{\delta,\mu}) \quad \text{on } \Gamma^\varepsilon_T,
\end{aligned} \tag{13}$$

where $\mu > 0$ and a penalty operator $\mathcal{B} : L^2(0, T; V) \to L^2(0, T; V')$ is monotone, bounded, hemicontinuous, and $\mathcal{B}(v - \kappa_D) = 0$ for $v(t) \in \mathcal{K}^\varepsilon$.

Lemma 3.1. *Under Assumption 2.1 there exists a solution $u^\varepsilon_\delta \in L^2(0, T; \mathcal{K}^\varepsilon)$ of (12) completed with initial condition $u^\varepsilon_\delta(0) = u_0$ in G^ε, with $\partial_t u^\varepsilon_\delta \in L^2(0, T; H^1(G^\varepsilon))$ and $\partial_t b_\delta(u^\varepsilon_\delta) \in L^2(G^\varepsilon_T)$. Under additional regularity assumption $\partial_t u^\varepsilon_\delta \in L^2(0, T; W^{1,p}(G^\varepsilon))$ and $u_0 \in W^{1,p}(G^\varepsilon)$ for $p > n$, or if $k(\xi) = \mathrm{const}$, P_c is Lipschitz continuous for $\xi > 0$, and $\nabla u^\varepsilon_\delta \in L^2(0, T; L^p(G^\varepsilon))$, variational inequality (12) has a unique solution.*

Proof. First we shall apply the Rothe and Galerkin methods to show existence of a weak solution of (13). Then by letting $\mu \to 0$ we will obtain the existence result for variational inequality (12). The discretisation in time of equations in (13) yields the following elliptic problem for $u^{\varepsilon,j}_{\delta,\mu}(x) := u^\varepsilon_{\delta,\mu}(t_j, x)$, for $x \in G^\varepsilon$,

$$\begin{aligned}
&b'_\delta(u^{\varepsilon,j}_{\delta,\mu})\frac{1}{h}(u^{\varepsilon,j}_{\delta,\mu} - u^{\varepsilon,j-1}_{\delta,\mu}) - \nabla \cdot \big(A^\varepsilon(x) k_\delta(u^{\varepsilon,j}_{\delta,\mu})[P_{c,\delta}(u^{\varepsilon,j}_{\delta,\mu})\nabla u^{\varepsilon,j}_{\delta,\mu} + \frac{1}{h}\nabla(u^{\varepsilon,j}_{\delta,\mu} - u^{\varepsilon,j-1}_{\delta,\mu})]\big) \\
&\qquad + \nabla \cdot F^\varepsilon(t_j, x, u^{\varepsilon,j-1}_{\delta,\mu}) + \frac{1}{\mu}\mathcal{B}(u^{\varepsilon,j}_{\delta,\mu} - \kappa_D) = 0 \quad \text{in } G^\varepsilon, \\
&\Big(A^\varepsilon(x) k_\delta(u^{\varepsilon,j}_{\delta,\mu})\big[P_{c,\delta}(u^{\varepsilon,j}_{\delta,\mu})\nabla u^{\varepsilon,j}_{\delta,\mu} + \frac{1}{h}\nabla(u^{\varepsilon,j}_{\delta,\mu} - u^{\varepsilon,j-1}_{\delta,\mu})\big] - F^\varepsilon(t_j, x, u^{\varepsilon,j-1}_{\delta,\mu})\Big) \cdot \nu \\
&\qquad = -\varepsilon f^\varepsilon(t_j, x, u^{\varepsilon,j-1}_{\delta,\mu}) \quad \text{on } \Gamma^\varepsilon, \\
&\qquad u^{\varepsilon,j}_{\delta,\mu} = \kappa_D \quad \text{on } \partial G,
\end{aligned} \tag{14}$$

where $h = T/N$ and $t_j = jh$, for $j = 1, \dots, N$ and $N \in \mathbb{N}$, and $u^{\varepsilon,0}_{\delta,\mu}(x) = u_0(x)$ for $x \in G^\varepsilon$. Since in this proof we assume that δ and ε are fixed, for the clarity of presentation we shall omit indices δ and ε in the calculations below. Now applying the Galerkin method to (14), we consider the orthogonal system of basis functions $\{\psi_i\}_{i\in\mathbb{N}}$ of the space V and are looking for functions

$$u^j_{\mu,m}(x) = \kappa_D + \sum_{i=1}^m \alpha^j_{mi}\psi_i(x)$$

in the subspace $V_m = \mathrm{span}\{\psi_1, \dots, \psi_m\}$ such that

$$\langle b'_\delta(u^j_{\mu,m})\frac{1}{h}(u^j_{\mu,m}-u^{j-1}_{\mu,m}),\zeta\rangle_{G^\varepsilon}+\langle A^\varepsilon(x)k_\delta(u^j_{\mu,m})[P_{c,\delta}(u^j_{\mu,m})\nabla u^j_{\mu,m}+\frac{1}{h}\nabla(u^j_{\mu,m}-u^{j-1}_{\mu,m})],\nabla\zeta\rangle_{G^\varepsilon}$$
$$-\langle F^\varepsilon(t_j,x,u^{j-1}_{\mu,m}),\nabla\zeta\rangle_{G^\varepsilon}+\frac{1}{\mu}\langle\mathcal{B}(u^j_{\mu,m}-\kappa_D),\zeta\rangle_{V',V}=-\langle\varepsilon f^\varepsilon(t_j,x,u^{j-1}_{\mu,m}),\zeta\rangle_{\Gamma^\varepsilon} \tag{15}$$

for all functions $\zeta\in V_m$. Here $u^0_{\mu,m}$, with $u^0_{\mu,m}-\kappa_D\in V_m$ and $u^0_{\mu,m}\in\mathcal{K}^\varepsilon$, is a finite-dimensional approximation of u_0. Thus we have a system of algebraic equations for unknown coefficients $\alpha=(\alpha^j_{m1},\dots,\alpha^j_{mm})$ and

$$J(\alpha)\alpha=\big\langle b'_\delta(v+\kappa_D)\frac{1}{h}(v+\kappa_D-u^{j-1}_{\mu,m}),v+\kappa_D\big\rangle_{G^\varepsilon}-\big\langle b'_\delta(v+\kappa_D)\frac{1}{h}(v+\kappa_D-u^{j-1}_{\mu,m}),\kappa_D\big\rangle_{G^\varepsilon}$$
$$+\big\langle A^\varepsilon(x)k_\delta(v+\kappa_D)\big[P_{c,\delta}(v+\kappa_D)\nabla v+\frac{1}{h}\nabla(v-u^{j-1}_{\mu,m})\big],\nabla v\rangle_{G^\varepsilon}-\langle F^\varepsilon(t_j,x,u^{j-1}_{\mu,m}),\nabla v\big\rangle_{G^\varepsilon}$$
$$+\frac{1}{\mu}\big\langle\mathcal{B}(v),v\big\rangle_{V',V}+\langle\varepsilon f^\varepsilon(t_j,x,u^{j-1}_{\mu,m}),v\rangle_{\Gamma^\varepsilon},$$

where $v=\sum_{i=1}^m\alpha^j_{mi}\psi_i(x)$. Assumptions on the nonlinear functions and monotonicity of $\mathcal{B}$ ensure

$$J(\alpha)\alpha\ge\frac{C_1}{4h}\|(v+\kappa_D)\chi_{\{v+\kappa_D>0\}}\|^2_{L^2(G^\varepsilon)}+\frac{C_2}{h}\delta^\beta\|\nabla v\|^2_{L^2(G^\varepsilon)}-\frac{C_3}{h}\|u^{j-1}_{\mu,m}\chi_{\{v+\kappa_D>0\}}\|^2_{L^2(G^\varepsilon)}$$
$$-\frac{C_4}{h}\|\nabla u^{j-1}_{\mu,m}\|^2_{L^2(G^\varepsilon)}-C_5\big[\|F^\varepsilon(t_j,x,u^{j-1}_{\mu,m})\|^2_{L^2(G^\varepsilon)}+\varepsilon\|f^\varepsilon(t_j,x,u^{j-1}_{\mu,m})\|^2_{L^2(\Gamma^\varepsilon)}\big]-\frac{C_6}{h}\kappa_D^2 \tag{16}$$
$$\ge C_7\big[\|(v+\kappa_D)\chi_{\{v+\kappa_D>0\}}\|^2_{L^2(G^\varepsilon)}+\|\nabla v\|^2_{L^2(G^\varepsilon)}\big]-C_8.$$

Thus for sufficiently large $|\alpha|$ we obtain that $J(\alpha)\alpha\ge 0$ and there exists a zero of $J(\alpha)$ and hence there is a $u^j_{\mu,m}\in\kappa_D+V_m$ satisfying (15), see e.g. [47]. Notice that if $b_\delta(w)=\vartheta_b w$, we have $\|v+\kappa_D\|^2_{L^2(G^\varepsilon)}$, instead of $\|(v+\kappa_D)\chi_{\{v+\kappa_D>0\}}\|^2_{L^2(G^\varepsilon)}$, in the last estimate.

Considering $u^j_{\mu,m}-u^{j-1}_{\mu,m}$ as a test function in (15) and summing over $j=1,\dots,l$, with $1<l\le N$, yield

$$\sum_{j=1}^{l}\langle A^\varepsilon(x)k_\delta(u^j_{\mu,m})[P_{c,\delta}(u^j_{\mu,m})\nabla u^j_{\mu,m}+\frac{1}{h}\nabla(u^j_{\mu,m}-u^{j-1}_{\mu,m})],\nabla(u^j_{\mu,m}-u^{j-1}_{\mu,m})\rangle_{G^\varepsilon}$$
$$+\sum_{j=1}^{l}\frac{1}{h}\langle b'_\delta(u^j_{\mu,m})(u^j_{\mu,m}-u^{j-1}_{\mu,m}),u^j_{\mu,m}-u^{j-1}_{\mu,m}\rangle_{G^\varepsilon}-\sum_{j=1}^{l}\langle F^\varepsilon(t_j,x,u^{j-1}_{\mu,m}),\nabla(u^j_{\mu,m}-u^{j-1}_{\mu,m})\rangle_{G^\varepsilon} \tag{17}$$
$$+\sum_{j=1}^{l}\frac{1}{\mu}\langle\mathcal{B}(u^j_{\mu,m}-\kappa_D),u^j_{\mu,m}-u^{j-1}_{\mu,m}\rangle_{V',V}=-\sum_{j=1}^{l}\varepsilon\langle f^\varepsilon(t_j,x,u^{j-1}_{\mu,m}),u^j_{\mu,m}-u^{j-1}_{\mu,m}\rangle_{\Gamma^\varepsilon}.$$

For penalty operator $\mathcal{B}$ given by $\mathcal{B}=J(I-P_{\mathcal{K}^\varepsilon})$, with $P_{\mathcal{K}^\varepsilon}:V\to\mathcal{K}^\varepsilon-\kappa_D$ being the projection operator on $\mathcal{K}^\varepsilon-\kappa_D$ and $J:V\to V'$ a dual mapping, which can be chosen as

$$\langle J(u),v\rangle_{V',V}=\int_{G^\varepsilon}\big(u\,v+\nabla u\nabla v\big)dx, \tag{18}$$

considering that $u^0_{\mu,m}\in\mathcal{K}^\varepsilon$ and using the property of the projection operator

$$\langle J(u-P_{\mathcal{K}^\varepsilon}u),P_{\mathcal{K}^\varepsilon}u-v\rangle_{V',V}\ge 0\quad\text{for }\ v\in\mathcal{K}^\varepsilon-\kappa_D, \tag{19}$$

we obtain the following estimate

$$
\begin{aligned}
&\sum_{j=1}^{l} \big\langle \mathcal{B}(u^j_{\mu,m} - \kappa_D), u^j_{\mu,m} - u^{j-1}_{\mu,m}\big\rangle_{V',V} \\
&= \sum_{j=1}^{l} \Big[\Big\langle J(\widetilde{u}^j_{\mu,m} - P_{\mathcal{K}^\varepsilon}\widetilde{u}^j_{\mu,m}), (\widetilde{u}^j_{\mu,m} - P_{\mathcal{K}^\varepsilon}\widetilde{u}^j_{\mu,m}) - (\widetilde{u}^{j-1}_{\mu,m} - P_{\mathcal{K}^\varepsilon}\widetilde{u}^{j-1}_{\mu,m})\Big\rangle_{V',V} \\
&\quad + \big\langle J(\widetilde{u}^j_{\mu,m} - P_{\mathcal{K}^\varepsilon}\widetilde{u}^j_{\mu,m}), P_{\mathcal{K}^\varepsilon}\widetilde{u}^j_{\mu,m} - P_{\mathcal{K}^\varepsilon}\widetilde{u}^{j-1}_{\mu,m}\big\rangle_{V',V}\Big] \\
&\geq \frac{1}{2}\int\limits_{G^\varepsilon} \Big[|(\widetilde{u}^l_{\mu,m} - P_{\mathcal{K}^\varepsilon}\widetilde{u}^l_{\mu,m})|^2 + |\nabla(\widetilde{u}^l_{\mu,m} - P_{\mathcal{K}^\varepsilon}\widetilde{u}^l_{\mu,m})|^2\Big] dx \geq 0,
\end{aligned}
$$

where $\widetilde{u}^j_{\mu,m} = u^j_{\mu,m} - \kappa_D$. Then using in (17) the monotonicity of b, Lipschitz continuity of F^ε and f^ε, regularity of initial data, and the uniform boundedness from below of k_δ, ensures

$$
\begin{aligned}
&\sum_{j=1}^{l} h\Big\|\frac{\nabla(u^j_{\mu,m} - u^{j-1}_{\mu,m})}{h}\Big\|^2_{L^2(G^\varepsilon)} \leq C_\sigma \sum_{j=1}^{l} h\big(\|\nabla u^j_{\mu,m}\|^2_{L^2(G^\varepsilon)} + \|u^{j-1}_{\mu,m}\|^2_{L^2(G^\varepsilon)} + \varepsilon\|u^{j-1}_{\mu,m}\|^2_{L^2(\Gamma^\varepsilon)}\big) \\
&\quad + \sigma_1 \sum_{j=1}^{l} h\Big\|\frac{\nabla(u^j_{\mu,m} - u^{j-1}_{\mu,m})}{h}\Big\|^2_{L^2(G^\varepsilon)} + \sigma_2\varepsilon \sum_{j=1}^{l} h\Big\|\frac{u^j_{\mu,m} - u^{j-1}_{\mu,m}}{h}\Big\|^2_{L^2(\Gamma^\varepsilon)} \\
&\leq C_1 \sum_{j=1}^{l} h \sum_{i=1}^{j} h\Big\|\frac{\nabla(u^i_{\mu,m} - u^{i-1}_{\mu,m})}{h}\Big\|^2_{L^2(G^\varepsilon)} + \sigma \sum_{j=1}^{l} h\Big\|\frac{\nabla(u^j_{\mu,m} - u^{j-1}_{\mu,m})}{h}\Big\|^2_{L^2(G^\varepsilon)} + C_2.
\end{aligned}
$$

In the last estimate we also used the trace and Poincaré inequalities. Choosing $\sigma > 0$ sufficiently small and applying the discrete Gronwall inequality we obtain

$$
\sum_{j=1}^{l} h\Big\|\frac{\nabla(u^j_{\mu,m} - u^{j-1}_{\mu,m})}{h}\Big\|^2_{L^2(G^\varepsilon)} \leq C, \tag{20}
$$

with $1 < l \leq N$ and a constant C independent of h, m, and μ. Estimate (20) together with the Poincaré inequality implies

$$
\sum_{j=1}^{l} h\Big\|\frac{u^j_{\mu,m} - u^{j-1}_{\mu,m}}{h}\Big\|^2_{L^2(G^\varepsilon)} \leq C_1 \sum_{j=1}^{l} h\Big\|\frac{\nabla(u^j_{\mu,m} - u^{j-1}_{\mu,m})}{h}\Big\|^2_{L^2(G^\varepsilon)} \leq C. \tag{21}
$$

Considering now $u^j_{\mu,m} - \kappa_D$ as a test function in (15) yields

$$
\begin{aligned}
&\sum_{j=1}^{l} \langle A^\varepsilon(x)k_\delta(u^j_{\mu,m})[P_{c,\delta}(u^j_{\mu,m})\nabla u^j_{\mu,m} + \frac{1}{h}\nabla(u^j_{\mu,m} - u^{j-1}_{\mu,m})], \nabla u^j_{\mu,m}\rangle_{G^\varepsilon} \\
&\quad + \sum_{j=1}^{l} \frac{1}{h}\langle b'_\delta(u^j_{\mu,m})(u^j_{\mu,m} - u^{j-1}_{\mu,m}), u^j_{\mu,m} - \kappa_D\rangle_{G^\varepsilon} - \sum_{j=1}^{l} \langle F^\varepsilon(t,x,u^{j-1}_{\mu,m}), \nabla u^j_{\mu,m}\rangle_{G^\varepsilon} \\
&\quad + \frac{1}{\mu}\sum_{j=1}^{l} \langle \mathcal{B}(u^j_{\mu,m} - \kappa_D), u^j_{\mu,m} - \kappa_D\rangle_{V',V} = -\sum_{j=1}^{l} \langle \varepsilon f^\varepsilon(t,x,u^{j-1}_{\mu,m}), u^j_{\mu,m} - \kappa_D\rangle_{\Gamma^\varepsilon}.
\end{aligned} \tag{22}
$$

Then assumptions on A, k, P_c, b, F^ε and f^ε, together with the trace and Poincaré inequalities, monotonicity of $\mathcal{B}$, and estimates (20) and (21), ensure

$$\sum_{j=1}^{l} h\big[\|\nabla u^j_{\mu,m}\|^2_{L^2(G^\varepsilon)} + \|u^j_{\mu,m}\|^2_{L^2(G^\varepsilon)}\big] + \frac{1}{\mu}\sum_{j=1}^{l} h\langle \mathcal{B}(u^j_{\mu,m}-\kappa_D), u^j_{\mu,m}-\kappa_D\rangle_{V',V} \le C, \tag{23}$$

with a constant C independent of μ, m, and h. The second term in (22) is estimated, using the assumptions on b and the continuous embedding $H^1(G^\varepsilon) \subset L^6(G^\varepsilon)$ for $n \le 3$, in the following way

$$\begin{aligned}
&\sum_{j=1}^{l} h\big|\langle b'_\delta(u^j_{\mu,m})\frac{1}{h}(u^j_{\mu,m}-u^{j-1}_{\mu,m}), u^j_{\mu,m}-\kappa_D\rangle_{G^\varepsilon}\big| \\
&\le C_1\sum_{j=1}^{l} h\Big\|\frac{u^j_{\mu,m}-u^{j-1}_{\mu,m}}{h}\Big\|^2_{L^2(G^\varepsilon)} + C_2\sum_{j=1}^{l} h\big(\|u^j_{\mu,m}\|^6_{L^6(G^\varepsilon)}+1\big) \\
&\le C_3\sum_{j=1}^{l} h\Big\|\frac{u^j_{\mu,m}-u^{j-1}_{\mu,m}}{h}\Big\|^2_{L^2(G^\varepsilon)} + C_4\Big[\sum_{j=1}^{l} h\Big\|\frac{\nabla(u^j_{\mu,m}-u^{j-1}_{\mu,m})}{h}\Big\|^2_{L^2(G^\varepsilon)}\Big]^3 + C_5.
\end{aligned}$$

To show that a subsequence of approximate solutions $\{u^j_{\mu,m}\}$ converges to a solution of problem (13) we define piecewise linear and piecewise constant interpolations with respect to the time variable

$$\begin{aligned}
&u^N_{\mu,m}(t,x) := u^{j-1}_{\mu,m}(x) + (t-t_{j-1})\frac{u^j_{\mu,m}(x)-u^{j-1}_{\mu,m}(x)}{h} && \text{for } t\in(t_{j-1},t_j],\\
&\bar{u}^N_{\mu,m}(t,x) := u^j_{\mu,m}(x) && \text{for } t\in(t_{j-1},t_j].
\end{aligned}$$

Then a priori estimates in (20), (21), and (23) and the boundedness of the penalty operator $\mathcal{B}$ ensure

$$\begin{aligned}
&\|\bar{u}^N_{\mu,m}\|_{L^2(G^\varepsilon_T)} + \|\nabla\bar{u}^N_{\mu,m}\|_{L^2(G^\varepsilon_T)} + \|\partial_t u^N_{\mu,m}\|_{L^2(G^\varepsilon_T)} + \|\partial_t\nabla u^N_{\mu,m}\|_{L^2(G^\varepsilon_T)} \le C,\\
&\frac{1}{\mu}\int_0^T \langle\mathcal{B}(\bar{u}^N_{\mu,m}-\kappa_D), \bar{u}^N_{\mu,m}-\kappa_D\rangle_{V',V}dt + \int_0^T \|\mathcal{B}(\bar{u}^N_{\mu,m}-\kappa_D)\|^2_{V'}dt \le C,
\end{aligned} \tag{24}$$

with a constant C independent of N, m, and μ. Integrating problem (15) over $(0,T)$ yields

$$\begin{aligned}
&\langle b'_\delta(\bar{u}^N_{\mu,m})\partial_t u^N_{\mu,m}, \zeta\rangle_{G^\varepsilon_T} + \langle A^\varepsilon(x)k_\delta(\bar{u}^N_{\mu,m})[P_{c,\delta}(\bar{u}^N_{\mu,m})\nabla\bar{u}^N_{\mu,m} + \partial_t\nabla u^N_{\mu,m}], \nabla\zeta\rangle_{G^\varepsilon_T}\\
&- \langle F^\varepsilon(t,x,\bar{u}^{N,h}_{\mu,m}), \nabla\zeta\rangle_{G^\varepsilon_T} + \frac{1}{\mu}\int_0^T\langle\mathcal{B}(\bar{u}^N_{\mu,m}-\kappa_D),\zeta\rangle_{V',V}dt = -\langle\varepsilon f^\varepsilon(t,x,\bar{u}^{N,h}_{\mu,m}),\zeta\rangle_{\Gamma^\varepsilon_T},
\end{aligned} \tag{25}$$

for $\zeta \in L^2(0,T;V_m)$, where $\bar{u}^{N,h}_{\mu,m}(t,x) = \bar{u}^N_{\mu,m}(t-h,x)$ for $t\in[h,T]$ and $\bar{u}^{N,h}_{\mu,m}(t,x) = u^0_{\mu,m}(x)$ for $t\in[0,h]$ and $x\in G^\varepsilon$.

A priori estimates (24) imply that there exist $u_\mu \in H^1(0,T;H^1(G^\varepsilon))$ and $\Lambda \in L^2(0,T;V')$ such that, up to a subsequence,

$$\begin{aligned}
\bar{u}^N_{\mu,m} &\rightharpoonup u_\mu && \text{weakly in } L^2(0,T;H^1(G^\varepsilon)),\ \text{strongly in } L^2(0,T;H^\sigma(G^\varepsilon)),\ 1/2<\sigma<1,\\
u^N_{\mu,m} &\rightharpoonup u_\mu && \text{weakly-}* \text{ in } L^\infty(0,T;H^1(G^\varepsilon)),\ \text{strongly in } L^2(0,T;H^\sigma(G^\varepsilon)),\\
\partial_t u^N_{\mu,m} &\rightharpoonup \partial_t u_\mu && \text{weakly in } L^2(0,T;H^1(G^\varepsilon)),\\
\mathcal{B}(\bar{u}^N_{\mu,m}-\kappa_D) &\rightharpoonup \Lambda && \text{weakly in } L^2(0,T;V'),
\end{aligned} \tag{26}$$

as $N, m \to \infty$, and

$$\|\bar u^N_{\mu,m} - \bar u^{N,h}_{\mu,m}\|_{L^2(0,T:H^1(G^\varepsilon))} \le \frac{C}{\sqrt{N}}.$$

Using a priori estimates (24) we also obtain

$$\begin{aligned}\|b'_\delta(\bar u^N_{\mu,m})\partial_t u^N_{\mu,m}\|^2_{L^2(G^\varepsilon_T)} &\le C_1 \int_0^T \big(\|\bar u^N_{\mu,m}\|^4_{L^6(G^\varepsilon)} + 1 + \delta^{4(\alpha-1)}\big)\|\partial_t u^N_{\mu,m}\|^2_{L^6(G^\varepsilon)} dt \\ &\le C_2\big(\|\nabla \bar u^N_{\mu,m}\|^4_{L^\infty(0,T;L^2(G^\varepsilon))} + C_\delta\big)\|\partial_t \nabla u^N_{\mu,m}\|^2_{L^2(G^\varepsilon_T)} \le C.\end{aligned} \tag{27}$$

Taking in (25) the limit as $N, m \to \infty$ and using convergence results in (26), together with the continuity of nonlinear functions, we obtain

$$\begin{aligned}&\langle b'_\delta(u_\mu)\partial_t u_\mu, \zeta\rangle_{G^\varepsilon_T} + \langle A^\varepsilon(x) k_\delta(u_\mu)[P_{c,\delta}(u_\mu)\nabla u_\mu + \partial_t \nabla u_\mu], \nabla\zeta\rangle_{G^\varepsilon_T} \\ &\quad - \langle F^\varepsilon(t,x,u_\mu), \nabla \zeta\rangle_{G^\varepsilon_T} + \frac{1}{\mu}\int_0^T \langle \Lambda, \zeta\rangle_{V',V} dt = -\varepsilon\langle f^\varepsilon(t,x,u_\mu), \zeta\rangle_{\Gamma^\varepsilon_T},\end{aligned} \tag{28}$$

for $\zeta \in L^2(0,T;V)$. To show strong convergence of $\bar u^N_{\mu,m}$ to u_μ in $L^2(0,T;H^1(G^\varepsilon))$ we consider $\int_{u_\mu}^{\bar u^N_{\mu,m}} \frac{1}{k_\delta(\xi)} d\xi$ as a test function in (25) and obtain

$$\begin{aligned}&\big\langle A^\varepsilon(x)\nabla(\bar u^N_{\mu,m} - u_\mu)(s), \nabla(\bar u^N_{\mu,m} - u_\mu)(s)\big\rangle_{G^\varepsilon} + \big\langle A^\varepsilon(x) P_{c,\delta}(\bar u^N_{\mu,m})\nabla(\bar u^N_{\mu,m} - u_\mu), \nabla(\bar u^N_{\mu,m} - u_\mu)\big\rangle_{G^\varepsilon_s} \\ &\quad + \frac{1}{\mu}\int_0^s \Big\langle \mathcal{B}(\bar u^N_{\mu,m} - \kappa_D) - \mathcal{B}(u_\mu - \kappa_D), \int_{u_\mu}^{\bar u^N_{\mu,m}} \frac{1}{k_\delta(\xi)} d\xi\Big\rangle_{V',V} dt \\ &= \big\langle A^\varepsilon(x)\nabla(u^0_{\mu,m} - u_0), \nabla(u^0_{\mu,m} - u_0)\big\rangle_{G^\varepsilon} - \big\langle A^\varepsilon(x)\partial_t\nabla u_\mu, \nabla(\bar u^N_{\mu,m} - u_\mu)\big\rangle_{G^\varepsilon_s} \\ &\quad - \Big\langle A^\varepsilon(x)[P_{c,\delta}(\bar u^N_{\mu,m})\nabla \bar u^N_{\mu,m} + \partial_t \nabla u^N_{\mu,m}], \Big[1 - \frac{k_\delta(\bar u^N_{\mu,m})}{k_\delta(u_\mu)}\Big]\nabla u_\mu\Big\rangle_{G^\varepsilon_s} - \Big\langle \partial_t b_\delta(\bar u^N_{\mu,m}), \int_{u_\mu}^{\bar u^N_{\mu,m}} \frac{1}{k_\delta(\xi)} d\xi\Big\rangle_{G^\varepsilon_s} \\ &\quad - \langle A^\varepsilon(x) P_{c,\delta}(\bar u^N_{\mu,m})\nabla u_\mu, \nabla \bar u^N_{\mu,m} - \nabla u_\mu\rangle_{G^\varepsilon_s} + \Big\langle F^\varepsilon(t,x,\bar u^{N,h}_{\mu,m}), \frac{1}{k_\delta(\bar u^N_{\mu,m})}\nabla \bar u^N_{\mu,m} - \frac{1}{k_\delta(u_\mu)}\nabla u_\mu\Big\rangle_{G^\varepsilon_s} \\ &\quad - \frac{1}{\mu}\int_0^s \Big\langle \mathcal{B}(u_\mu - \kappa_D), \int_{u_\mu}^{\bar u^N_{\mu,m}} \frac{1}{k_\delta(\xi)} d\xi\Big\rangle_{V',V} dt - \varepsilon\Big\langle f^\varepsilon(t,x,\bar u^{N,h}_{\mu,m}), \int_{u_\mu}^{\bar u^N_{\mu,m}} \frac{1}{k_\delta(\xi)} d\xi\Big\rangle_{\Gamma^\varepsilon_s},\end{aligned}$$

for $s \in (0,T]$. Then using the following estimate for the penalty operator $\mathcal{B}$

$$\int_0^s \Big\langle \mathcal{B}(\bar u^N_{\mu,m} - \kappa_D) - \mathcal{B}(u_\mu - \kappa_D), \int_{u_\mu}^{\bar u^N_{\mu,m}} \frac{1}{k_\delta(\xi)} d\xi\Big\rangle_{V',V} dt \ge 0, \tag{29}$$

for $s \in (0,T]$, shown below, the strong convergence of $u^N_{\mu,m}$ in $L^2(G^\varepsilon_T)$ and weak convergence in $H^1(0,T;H^1(G^\varepsilon))$ as $m, N \to \infty$, together with the continuity of nonlinear functions and assumptions on A^ε and P_c, imply

$$\sup_{(0,T)} \|\nabla(\bar u^N_{\mu,m} - u_\mu)\|_{L^2(G^\varepsilon)} \to 0 \quad \text{as } m, N \to \infty.$$

To show (29) we consider

$$\frac{1}{k(\delta)}\int_0^T \Big\langle \mathcal{B}(\bar u^N_{\mu,m}-\kappa_D)-\mathcal{B}(u_\mu-\kappa_D), k(\delta)\int_{u_\mu}^{\bar u^N_{\mu,m}}\frac{1}{k_\delta(\xi)}d\xi\Big\rangle_{V',V}dt$$

$$= \frac{1}{k(\delta)}\int_0^T \Big\langle \mathcal{B}(\bar u^N_{\mu,m}-\kappa_D)-\mathcal{B}(u_\mu-\kappa_D), (\bar u^N_{\mu,m}-P_{\mathcal{K}^\varepsilon}(\bar u^N_{\mu,m}-\kappa_D))-(u_\mu-P_{\mathcal{K}^\varepsilon}(u_\mu-\kappa_D))\Big\rangle_{V',V}dt$$

$$+\frac{1}{k(\delta)}\int_0^T \Big\langle \mathcal{B}(\bar u^N_{\mu,m}-\kappa_D)-\mathcal{B}(u_\mu-\kappa_D), P_{\mathcal{K}^\varepsilon}(\bar u^N_{\mu,m}-\kappa_D)-P_{\mathcal{K}^\varepsilon}(u_\mu-\kappa_D)$$

$$-(\bar u^N_{\mu,m}-u_\mu)+k(\delta)\int_{u_\mu}^{\bar u^N_{\mu,m}}\frac{d\xi}{k_\delta(\xi)}\Big\rangle_{V',V}dt,$$

where $k(\delta)>0$. The monotonicity of $\mathcal{B}$ ensures

$$\int_0^T \Big\langle \mathcal{B}(\bar u^N_{\mu,m}-\kappa_D)-\mathcal{B}(u_\mu-\kappa_D), (\bar u^N_{\mu,m}-P_{\mathcal{K}^\varepsilon}(\bar u^N_{\mu,m}-\kappa_D))-(u_\mu-P_{\mathcal{K}^\varepsilon}(u_\mu-\kappa_D))\Big\rangle_{V',V}dt\geq 0.$$

For the second term due to the properties of the projection operator we have

$$\int_0^T \Big\langle \mathcal{B}(\bar u^N_{\mu,m}-\kappa_D), P_{\mathcal{K}^\varepsilon}(\bar u^N_{\mu,m}-\kappa_D)-\Big[P_{\mathcal{K}^\varepsilon}(u_\mu-\kappa_D)+(\bar u^N_{\mu,m}-u_\mu)-k(\delta)\int_{u_\mu}^{\bar u^N_{\mu,m}}\frac{1}{k_\delta(\xi)}d\xi\Big]\Big\rangle_{V',V}dt\geq 0$$

and

$$\int_0^T \Big\langle \mathcal{B}(u_\mu-\kappa_D), P_{\mathcal{K}^\varepsilon}(u_\mu-\kappa_D)-\Big[P_{\mathcal{K}^\varepsilon}(\bar u^N_{\mu,m}-\kappa_D)+k(\delta)\int_{u_\mu}^{\bar u^N_{\mu,m}}\frac{1}{k_\delta(\xi)}d\xi-(\bar u^N_{\mu,m}-u_\mu)\Big]\Big\rangle_{V',V}dt\geq 0,$$

if $P_{\mathcal{K}^\varepsilon}(u_\mu-\kappa_D)+(\bar u^N_{\mu,m}-u_\mu)-k(\delta)\int_{u_\mu}^{\bar u^N_{\mu,m}}k_\delta(\xi)^{-1}d\xi\in\mathcal{K}^\varepsilon-\kappa_D$ and $P_{\mathcal{K}^\varepsilon}(\bar u^N_{\mu,m}-\kappa_D)+k(\delta)\int_{u_\mu}^{\bar u^N_{\mu,m}}k_\delta(\xi)^{-1}d\xi-(\bar u^N_{\mu,m}-u_\mu)\in\mathcal{K}^\varepsilon-\kappa_D$, respectively. Notice that if $u_\mu\leq 0$ and $\bar u^N_{\mu,m}\leq 0$, then $k(\delta)\int_{u_\mu}^{\bar u^N_{\mu,m}}k_\delta(\xi)^{-1}d\xi=\bar u^N_{\mu,m}-u_\mu$. If $\bar u^N_{\mu,m}>u_\mu$ then $(\bar u^N_{\mu,m}-u_\mu)-k(\delta)\int_{u_\mu}^{\bar u^N_{\mu,m}}k_\delta(\xi)^{-1}d\xi\geq 0$ and if $\bar u^N_{\mu,m}<u_\mu$ and $u_\mu>0$, then for $\bar u^N_{\mu,m}\leq 0$ we have $k(\delta)\int_0^{\bar u^N_{\mu,m}}k_\delta(\xi)^{-1}d\xi=\bar u^N_{\mu,m}$ and hence $\bar u^N_{\mu,m}-k(\delta)\int_{u_\mu}^{\bar u^N_{\mu,m}}k_\delta(\xi)^{-1}d\xi\geq 0$ and also $P_{\mathcal{K}^\varepsilon}(u_\mu-\kappa_D)=u_\mu-\kappa_D$. Thus combining these considerations yields $P_{\mathcal{K}^\varepsilon}(u_\mu-\kappa_D)+(\bar u^N_{\mu,m}-u_\mu)-k(\delta)\int_{u_\mu}^{\bar u^N_{\mu,m}}k_\delta(\xi)^{-1}d\xi\in\mathcal{K}^\varepsilon-\kappa_D$. For the second term, if $u_\mu>u^N_{\mu,m}$ then $k(\delta)\int_{u_\mu}^{u^N_{\mu,m}}k_\delta(\xi)^{-1}d\xi-(u^N_{\mu,m}-u_\mu)>0$ and if $u_\mu<\bar u^N_{\mu,m}$ and $\bar u^N_{\mu,m}>0$, then since for $u_\mu<0$ we have $k(\delta)\int_{u_\mu}^0 k_\delta(\xi)^{-1}d\xi=-u_\mu$ and hence $k(\delta)\int_{u_\mu}^{\bar u^N_{\mu,m}}k_\delta(\xi)^{-1}d\xi+u_\mu\geq 0$, we obtain $P_{\mathcal{K}^\varepsilon}(\bar u^N_{\mu,m}-\kappa_D)+k(\delta)\int_{u_\mu}^{\bar u^N_{\mu,m}}k_\delta(\xi)^{-1}d\xi-(\bar u^N_{\mu,m}-u_\mu)\in\mathcal{K}^\varepsilon-\kappa_D$. Notice that for $\bar u^N_{\mu,m}>0$ we have $P_{\mathcal{K}^\varepsilon}(\bar u^N_{\mu,m}-\kappa_D)=\bar u^N_{\mu,m}-\kappa_D$. Thus inequality (29) follows.

The strong convergence of $\bar u^N_{\mu,m}$ in $L^2(0,T;H^1(G^\varepsilon))$ implies $\mathcal{B}(\bar u^N_{\mu,m}-\kappa_D)\rightharpoonup\mathcal{B}(u_\mu-\kappa_D)$ in $L^2(0,T;V')$ as $m,N\to\infty$, and hence $\Lambda=\mathcal{B}(u_\mu-\kappa_D)$. Therefore we obtain that u_μ is a weak solution of problem (13).

To prove the existence of a solution of variational inequality (12) we need to take in (13) the limit as $\mu \to 0$. Notice that a priori estimates (24) and (27) are uniform in μ. Hence taking the limit as $N, m \to \infty$ and using lower semicontinuity of a norm we obtain the corresponding estimates for u_μ in $H^1(0,T; H^1(G^\varepsilon))$ and that there exists $u \in H^1(0,T; H^1(G^\varepsilon))$ such that, up to a subsequence, $u_\mu \rightharpoonup u$ in $H^1(0,T; H^1(G^\varepsilon))$ as $\mu \to 0$. Assumptions on b, k, and P_c and strong convergence of $u_\mu \to u$ in $L^{r_1}(G^\varepsilon_T)$, for $1 < r_1 < 6$, ensure strong convergence $b_\delta(u_\mu) \to b_\delta(u)$ in $L^{r_2}(G^\varepsilon_T)$, for $1 < r_2 < 2$, $k_\delta(u_\mu) \to k_\delta(u)$, $k_\delta(u_\mu)P_{c,\delta}(u_\mu) \to k_\delta(u)P_{c,\delta}(u)$ in $L^q(G^\varepsilon_T)$, for $1 < q < \infty$, as $\mu \to 0$, and $\partial_t b_\delta(u) \in L^2(G^\varepsilon_T)$. From equation (28) follows

$$\int_0^T \langle \mathcal{B}(u_\mu - \kappa_D), v\rangle_{V',V} dt = \mu \int_0^T \Big[\big\langle F^\varepsilon(t,x,u_\mu) - A^\varepsilon(x)\, k_\delta(u_\mu)\big[P_{c,\delta}(u_\mu)\nabla u_\mu + \partial_t \nabla u_\mu\big], \nabla v\big\rangle_{G^\varepsilon} \\ - \varepsilon\langle f^\varepsilon(t,x,u_\mu), v\rangle_{\Gamma^\varepsilon} - \langle \partial_t b_\delta(u_\mu), v\rangle_{G^\varepsilon}\Big] dt$$

for all $v \in L^2(0,T;V)$. Then boundedness of u_μ in $H^1(0,T; H^1(G^\varepsilon))$ yields

$$\mathcal{B}(u_\mu - \kappa_D) \rightharpoonup 0 \quad \text{weakly in } \; L^2(0,T;V') \quad \text{as } \; \mu \to 0. \tag{30}$$

The monotonicity of $\mathcal{B}$ ensures

$$\int_0^T \langle \mathcal{B}(v), u_\mu - \kappa_D - v\rangle_{V',V} dt \leq \int_0^T \langle \mathcal{B}(u_\mu - \kappa_D), u_\mu - \kappa_D - v\rangle_{V',V} dt$$

for $v \in L^2(0,T;V)$. Considering $\mu \to 0$ and using weak convergence of $u_\mu \rightharpoonup u$ in $L^2(0,T; H^1(G^\varepsilon))$ as $\mu \to 0$, convergence of $\mathcal{B}(u_\mu - \kappa_D)$, see (30), and the fact that

$$\int_0^T \langle \mathcal{B}(u_\mu - \kappa_D), u_\mu - \kappa_D\rangle_{V',V} dt \leq C\mu$$

imply

$$\int_0^T \langle \mathcal{B}(v), u - \kappa_D - v\rangle_{V',V} dt \leq 0.$$

Taking $v = u - \kappa_D - \lambda w$ for $\lambda > 0$ and $w \in L^2(0,T;V)$, passing to the limit as $\lambda \to 0$, and using hemicontinuity of $\mathcal{B}$ we obtain

$$\int_0^T \langle \mathcal{B}(u - \kappa_D), w\rangle_{V',V} dt \leq 0$$

for all $w \in L^2(0,T;V)$ and hence $\mathcal{B}(u - \kappa_D) = 0$ and $u(t) \in \mathcal{K}^\varepsilon$ for a.a. $t \in (0,T)$.

To show that u is a solution of variational inequality (12) we consider $\zeta = v - u - k(\delta)\int_u^{u_\mu} \frac{1}{k_\delta(\xi)} d\xi$ as a test function in (28) and obtain

$$\begin{aligned}
&\Big\langle \partial_t b_\delta(u_\mu), v-u-k(\delta)\int\limits_u^{u_\mu}\frac{d\xi}{k_\delta(\xi)}\Big\rangle_{G^\varepsilon_T} - \Big\langle F^\varepsilon(t,x,u_\mu), \nabla(v-u) - \frac{k(\delta)}{k_\delta(u_\mu)}\nabla u_\mu + \frac{k(\delta)}{k_\delta(u)}\nabla u\Big\rangle_{G^\varepsilon_T}\\
&\quad + \Big\langle A^\varepsilon(x)k_\delta(u_\mu)[P_{c,\delta}(u_\mu)\nabla u_\mu + \partial_t\nabla u_\mu], \nabla(v-u) - \frac{k(\delta)}{k_\delta(u_\mu)}\nabla u_\mu + \frac{k(\delta)}{k_\delta(u)}\nabla u\Big\rangle_{G^\varepsilon_T}\\
&\quad + \varepsilon\Big\langle f^\varepsilon(t,x,u_\mu), v-u-k(\delta)\int\limits_u^{u_\mu}\frac{d\xi}{k_\delta(\xi)}\Big\rangle_{\Gamma^\varepsilon_T} = \frac{1}{\mu}\int\limits_0^T\Big\langle \mathcal{B}(u_\mu-\kappa_D), u + k(\delta)\int\limits_u^{u_\mu}\frac{d\xi}{k_\delta(\xi)} - v\Big\rangle_{V',V}dt
\end{aligned} \tag{31}$$

for $v \in L^2(0,T;\mathcal{K}^\varepsilon)$. In order to pass to the limit as $\mu \to 0$ we need to show that

$$\int\limits_0^T\Big\langle \mathcal{B}(u_\mu-\kappa_D), u + k(\delta)\int\limits_u^{u_\mu}\frac{d\xi}{k_\delta(\xi)} - v\Big\rangle_{V',V}dt \geq 0$$

for $v \in L^2(0,T;\mathcal{K}^\varepsilon)$. Since $\mathcal{B}(v-\kappa_D)=0$ we can rewrite the left had side in the last inequality as

$$\begin{aligned}
&\int\limits_0^T\Big\langle \mathcal{B}(u_\mu-\kappa_D) - \mathcal{B}(v-\kappa_D), (u_\mu-\kappa_D)-(v-\kappa_D)\Big\rangle_{V',V}dt\\
&\quad + \int\limits_0^T\Big\langle \mathcal{B}(u_\mu-\kappa_D), P_{\mathcal{K}^\varepsilon}(u_\mu-\kappa_D) - \Big[P_{\mathcal{K}^\varepsilon}(u_\mu-\kappa_D) + (u_\mu-u) - k(\delta)\int\limits_u^{u_\mu}\frac{d\xi}{k_\delta(\xi)}\Big]\Big\rangle_{V',V}dt.
\end{aligned} \tag{32}$$

The first term in (32) is nonnegative due to the monotonicity of $\mathcal{B}$, whereas the second term is nonnegative if $q_\mu = P_{\mathcal{K}^\varepsilon}(u_\mu-\kappa_D) + (u_\mu-u) - k(\delta)\int_u^{u_\mu}k_\delta(\xi)^{-1}d\xi \in \mathcal{K}^\varepsilon-\kappa_D$. First notice that $u \in \mathcal{K}^\varepsilon$ and hence $u \geq 0$ on Γ^ε. If $u_\mu \geq u$ then due to assumptions on k we have $(u_\mu-u) - k(\delta)\int_u^{u_\mu}k_\delta(\xi)^{-1}d\xi \geq 0$ and hence $q_\mu \in \mathcal{K}^\varepsilon-\kappa_D$. If $u=u_\mu=0$ on Γ^ε or if $u=0$ and $u_\mu \leq 0$ on Γ^ε we obtain $u_\mu - k(\delta)\int_0^{u_\mu}k_\delta(\xi)^{-1}d\xi = 0$ and $q_\mu = P_{\mathcal{K}^\varepsilon}(u_\mu-\kappa_D) \in \mathcal{K}^\varepsilon-\kappa_D$. If $u>0$ and $u_\mu<u$ on Γ^ε, then, since $u_\mu \to u$ in $L^2(\Gamma^\varepsilon_T)$ as $\mu \to 0$, there exists such $\mu>0$ that $0<u_\mu \leq u$ and $|u-u_\mu| \leq u_\mu$ a.e. on $(0,T)\times\Gamma^\varepsilon$, and thus $q_\mu \geq u_\mu-\kappa_D-(u-u_\mu) \geq -\kappa_D$ and $q_\mu \in \mathcal{K}^\varepsilon-\kappa_D$.

Considering the limit as $\mu \to 0$ in (31) and integration by parts in $\langle A^\varepsilon(x)\partial_t\nabla u_\mu, \nabla u_\mu\rangle_{G^\varepsilon_T}$, combined with strong convergence of u_μ in $L^p((0,T)\times G^\varepsilon)$ for any $1<p<6$, positivity of functions k_δ and $P_{c,\delta}$, continuity of nonlinear functions and lower semicontinuity of a norm, yield

$$\begin{aligned}
&\big\langle \partial_t b_\delta(u), v-u\big\rangle_{G^\varepsilon_T} + \big\langle A^\varepsilon(x)k_\delta(u)[P_{c,\delta}(u)\nabla u + \partial_t\nabla u], \nabla(v-u)\big\rangle_{G^\varepsilon_T} - \big\langle F^\varepsilon(t,x,u), \nabla(v-u)\big\rangle_{G^\varepsilon_T}\\
&\quad + \varepsilon\big\langle f^\varepsilon(t,x,u), v-u\big\rangle_{\Gamma^\varepsilon_T} \geq 0.
\end{aligned} \tag{33}$$

Thus we obtain that $u^\varepsilon_\delta = u$ is a solution of variational inequality (12).

To show the uniqueness of a solution of variational inequality (12) we assume that there are two solutions $u^\varepsilon_{\delta,1}$ and $u^\varepsilon_{\delta,2}$ and consider $v = u^\varepsilon_{\delta,2}$ and $v = u^\varepsilon_{\delta,1}$ as test functions in variational inequalities for $u^\varepsilon_{\delta,1}$ and $u^\varepsilon_{\delta,2}$, respectively,

$$\begin{aligned}
&\big\langle A^\varepsilon(x)\big(k_\delta(u^\varepsilon_{\delta,1})[P_{c,\delta}(u^\varepsilon_{\delta,1})\nabla u^\varepsilon_{\delta,1} + \partial_t\nabla u^\varepsilon_{\delta,1}] - k_\delta(u^\varepsilon_{\delta,2})[P_{c,\delta}(u^\varepsilon_{2,\delta})\nabla u^\varepsilon_{2,\delta} + \partial_t\nabla u^\varepsilon_{2,\delta}]\big), \nabla(u^\varepsilon_{\delta,1}-u^\varepsilon_{\delta,2})\big\rangle_{G^\varepsilon_\tau}\\
&\quad + \big\langle \partial_t(b_\delta(u^\varepsilon_{\delta,1}) - b_\delta(u^\varepsilon_{\delta,2})), u^\varepsilon_{\delta,1}-u^\varepsilon_{\delta,2}\big\rangle_{G^\varepsilon_\tau} - \big\langle F^\varepsilon(t,x,u^\varepsilon_{\delta,1}) - F^\varepsilon(t,x,u^\varepsilon_{\delta,2}), \nabla(u^\varepsilon_{\delta,1}-u^\varepsilon_{\delta,2})\big\rangle_{G^\varepsilon_\tau}\\
&\quad + \varepsilon\big\langle f^\varepsilon(t,x,u^\varepsilon_{\delta,1}) - f^\varepsilon(t,x,u^\varepsilon_{\delta,2}), u^\varepsilon_{\delta,1}-u^\varepsilon_{\delta,2}\big\rangle_{\Gamma^\varepsilon_\tau} \leq 0,
\end{aligned} \tag{34}$$

for $\tau \in (0, T]$. Rearranging terms in (34) implies

$$\frac{1}{2}\int\limits_{G^\varepsilon_\tau} A^\varepsilon(x)\partial_t\big(k_\delta(u^\varepsilon_{\delta,1})|\nabla u^\varepsilon_{\delta,1} - \nabla u^\varepsilon_{\delta,2}|^2\big)dxdt - \frac{1}{2}\int\limits_{G^\varepsilon_\tau} A^\varepsilon(x)\partial_t k_\delta(u^\varepsilon_{\delta,1})|\nabla u^\varepsilon_{\delta,1} - \nabla u^\varepsilon_{\delta,2}|^2 dxdt$$
$$+ \big\langle \partial_t(b_\delta(u^\varepsilon_{\delta,1}) - b_\delta(u^\varepsilon_{\delta,2})), u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2}\big\rangle_{G^\varepsilon_\tau} + \big\langle A^\varepsilon(x)(k_\delta(u^\varepsilon_{\delta,1}) - k_\delta(u^\varepsilon_{\delta,2}))\partial_t\nabla u^\varepsilon_{\delta,2}, \nabla(u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2})\big\rangle_{G^\varepsilon_\tau}$$
$$+ \big\langle A^\varepsilon(x)k_\delta(u^\varepsilon_{\delta,1})P_{c,\delta}(u^\varepsilon_{\delta,1})\nabla(u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2}), \nabla(u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2})\big\rangle_{G^\varepsilon_\tau}$$
$$+ \big\langle A^\varepsilon(x)(k_\delta(u^\varepsilon_{\delta,1})P_{c,\delta}(u^\varepsilon_{\delta,1}) - k(u^\varepsilon_{\delta,2})P_{c,\delta}(u^\varepsilon_{\delta,2}))\nabla u^\varepsilon_{\delta,2}, \nabla(u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2})\big\rangle_{G^\varepsilon_\tau}$$
$$- \big\langle F^\varepsilon(t,x,u^\varepsilon_{\delta,1}) - F^\varepsilon(t,x,u^\varepsilon_{\delta,2}), \nabla(u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2})\big\rangle_{G^\varepsilon_\tau} + \varepsilon\big\langle f^\varepsilon(t,x,u^\varepsilon_{\delta,1}) - f^\varepsilon(t,x,u^\varepsilon_{\delta,2}), u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2}\big\rangle_{\Gamma^\varepsilon_\tau} \le 0, \tag{35}$$

for $\tau \in (0, T]$. Using regularity assumptions on $\partial_t u^\varepsilon_{\delta,1}$, the Lipschitz continuity of k and boundedness of A^ε, the second term in (35) can be estimated as

$$\Big|\int\limits_{G^\varepsilon_\tau} A^\varepsilon(x)\partial_t k_\delta(u^\varepsilon_{\delta,1})|\nabla u^\varepsilon_{\delta,1} - \nabla u^\varepsilon_{\delta,2}|^2 dxdt\Big| \le C \sup_{(0,\tau)} \|\nabla u^\varepsilon_{\delta,1} - \nabla u^\varepsilon_{\delta,2}\|^2_{L^2(G^\varepsilon)}\tau^{\frac{1}{2}}\|\partial_t u^\varepsilon_{\delta,1}\|_{L^2(0,T;L^\infty(G^\varepsilon))}.$$

The third term in (35) is estimated as

$$\begin{aligned}
\langle \partial_t(b_\delta(u^\varepsilon_{\delta,1}) - b_\delta(u^\varepsilon_{\delta,2})), u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2}\rangle_{G^\varepsilon_\tau} &= \big\langle b'_\delta(u^\varepsilon_{\delta,1})\partial_t(u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2}), u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2}\big\rangle_{G^\varepsilon_\tau} \\
&+ \big\langle (b'_\delta(u^\varepsilon_{\delta,1}) - b'_\delta(u^\varepsilon_{\delta,2}))\,\partial_t u^\varepsilon_{\delta,2}, u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2}\big\rangle_{G^\varepsilon_\tau} \\
&\ge \frac{1}{2}C_\delta\|u^\varepsilon_{\delta,1}(\tau) - u^\varepsilon_{\delta,2}(\tau)\|^2_{L^2(G^\varepsilon)} - C_1\tau^{\frac{1}{2}}\sup_{(0,\tau)}\|\nabla(u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2})\|^2_{L^2(G^\varepsilon)}.
\end{aligned} \tag{36}$$

Here we used the fact that the continuous embedding $H^1(G^\varepsilon) \subset L^6(G^\varepsilon)$ for $n \le 3$ and regularity $\partial_t u^\varepsilon_{\delta,j} \in L^2(0, T; L^6(G^\varepsilon))$ and $u^\varepsilon_{\delta,j} \in L^\infty(0, T; L^6(G^\varepsilon))$, for $j = 1, 2$, together with assumptions on b, ensure

$$\begin{aligned}
&\int\limits_{G^\varepsilon_\tau} |\partial_t b'_\delta(u^\varepsilon_{\delta,1})||u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2}|^2 dxdt + \int\limits_{G^\varepsilon_\tau} |b'_\delta(u^\varepsilon_{\delta,1}) - b'_\delta(u^\varepsilon_{\delta,2})||\partial_t u^\varepsilon_{\delta,2}||u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2}|dxdt \\
&\le C_1\Big(\int\limits_{G^\varepsilon_\tau} \big[|b''_\delta(u^\varepsilon_{\delta,1})| + |b''_\delta(u^\varepsilon_{\delta,2})|\big]^{\frac{3}{2}}\big[|\partial_t u^\varepsilon_{\delta,1}| + |\partial_t u^\varepsilon_{\delta,2}|\big]^{\frac{3}{2}} dxdt\Big)^{\frac{2}{3}}\Big(\int\limits_{G^\varepsilon_\tau} |u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2}|^6 dxdt\Big)^{\frac{1}{3}} \\
&\le C_2\tau^{\frac{1}{2}}\big(\|\nabla u^\varepsilon_{\delta,1}\|_{L^\infty(0,T;L^2(G^\varepsilon))} + \|\nabla u^\varepsilon_{\delta,2}\|_{L^\infty(0,T;L^2(G^\varepsilon))} + 1\big)\big(\|\partial_t u^\varepsilon_{\delta,1}\|_{L^2(G^\varepsilon_T)} \\
&\qquad + \|\partial_t u^\varepsilon_{\delta,2}\|_{L^2(G^\varepsilon_T)}\big)\sup_{(0,\tau)}\|\nabla(u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2})\|^2_{L^2(G^\varepsilon)}.
\end{aligned} \tag{37}$$

Notice that $u^\varepsilon_{\delta,1}$ and $u^\varepsilon_{\delta,2}$ satisfy Dirichlet boundary condition on ∂G. Lipschitz continuity of k and regularity assumptions on $\partial_t u^\varepsilon_{\delta,2}$ ensure

$$\begin{aligned}
&\big|\big\langle A^\varepsilon(x)(k_\delta(u^\varepsilon_{\delta,1}) - k_\delta(u^\varepsilon_{\delta,2}))\,\partial_t\nabla u^\varepsilon_{\delta,2}, \nabla(u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2})\big\rangle_{G^\varepsilon_\tau}\big| \le C_{1,\tau}\|\nabla(u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2})\|^2_{L^2(G^\varepsilon_\tau)} \\
&\quad + C_2\tau^{\frac{1}{2}}\|\partial_t\nabla u^\varepsilon_{\delta,2}\|^2_{L^2(0,\tau;L^p(G^\varepsilon))}\sup_{(0,\tau)}\|u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2}\|^2_{L^{\frac{2p}{p-2}}(G^\varepsilon)} \\
&\le C_\tau\|\nabla(u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2})\|^2_{L^2(G^\varepsilon_\tau)} + \tau^{\frac{1}{2}}\sup_{(0,\tau)}\|\nabla(u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2})\|^2_{L^2(G^\varepsilon)},
\end{aligned}$$

for $p \ge n$. Using assumptions on k and P_c we also obtain

$$\begin{aligned}&\big|\left\langle A^\varepsilon(x)(k_\delta(u^\varepsilon_{\delta,1})P_{c,\delta}(u^\varepsilon_{\delta,1}) - k(u^\varepsilon_{\delta,2})P_{c,\delta}(u^\varepsilon_{\delta,2}))\nabla u^\varepsilon_{\delta,2}, \nabla(u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2})\right\rangle_{G^\varepsilon_\tau}\big| \\ &\leq C\|\nabla u^\varepsilon_{\delta,2}\|^2_{L^\infty(0,\tau;L^p(G^\varepsilon))}\|\nabla(u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2})\|^2_{L^2(G^\varepsilon_\tau)} + \|\nabla(u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2})\|^2_{L^2(G^\varepsilon_\tau)},\end{aligned}$$

for $p \geq n$. The last two terms in (35) are estimated using Lipschitz continuity of F^ε and f^ε and the trace estimate. Then integrating by parts in the first term in (35), using the fact that $k_\delta(u^\varepsilon_{\delta,1}) \geq \delta > 0$, choosing a sufficiently small $\tau > 0$ and applying the Gronwall inequality we obtain

$$\sup_{(0,\tau)} \|\nabla(u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2})\|^2_{L^2(G^\varepsilon)} \leq 0.$$

Using the Poincaré inequality and iterating over $\tau > 0$, which depends on the coefficients in the variational inequality and is independent of a solution of (12), yield $u^\varepsilon_{\delta,1}(t,x) = u^\varepsilon_{\delta,2}(t,x)$ a.e. in $(0,T) \times G^\varepsilon$, and hence the uniqueness of a solution of variational inequality (12).

If $k(\xi) = \text{const}$, for two solutions $u^\varepsilon_{\delta,1}$ and $u^\varepsilon_{\delta,2}$ of (12) we have

$$\begin{aligned}&\frac{1}{2}\int_{G^\varepsilon} A^\varepsilon(x)|\nabla(u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2})(\tau)|^2 dx + \int_{G^\varepsilon_\tau} A^\varepsilon(x)P_{c,\delta}(u^\varepsilon_{\delta,1})|\nabla(u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2})|^2 dxdt \\ &\quad + \frac{1}{2}\int_{G^\varepsilon} b'_\delta(u^\varepsilon_{\delta,1})|u^\varepsilon_{\delta,1}(\tau) - u^\varepsilon_{\delta,2}(\tau)|^2 dx + \int_{G^\varepsilon_\tau}(b'_\delta(u^\varepsilon_{\delta,1}) - b'_\delta(u^\varepsilon_{\delta,2}))\partial_t u^\varepsilon_{\delta,2}(u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2})dxdt \\ &\quad - \frac{1}{2}\int_{G^\varepsilon_\tau} \partial_t b'_\delta(u^\varepsilon_{\delta,1})|u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2}|^2 dxdt + \left\langle A^\varepsilon(x)(P_{c,\delta}(u^\varepsilon_{\delta,1}) - P_{c,\delta}(u^\varepsilon_{\delta,2}))\nabla u^\varepsilon_{\delta,2}, \nabla(u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2})\right\rangle_{G^\varepsilon_\tau} \\ &\leq \langle F^\varepsilon(t,x,u^\varepsilon_{\delta,1}) - F^\varepsilon(t,x,u^\varepsilon_{\delta,2}), \nabla(u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2})\rangle_{G^\varepsilon_\tau} - \varepsilon\langle f^\varepsilon(t,x,u^\varepsilon_{\delta,1}) - f^\varepsilon(t,x,u^\varepsilon_{\delta,2}), u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2}\rangle_{\Gamma^\varepsilon_\tau}.\end{aligned} \tag{38}$$

The fourth and fifth terms on the left-hand side in (38) are estimates as in (37). For the sixth term on the left-hand side, using Lipschitz continuity of $P_{c,\delta}$ and regularity assumption on $u^\varepsilon_{\delta,2}$ we have

$$\begin{aligned}&\left\langle A^\varepsilon(x)(P_{c,\delta}(u^\varepsilon_{\delta,1}) - P_{c,\delta}(u^\varepsilon_{\delta,2}))\nabla u^\varepsilon_{\delta,2}, \nabla(u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2})\right\rangle_{G^\varepsilon_\tau} \\ &\leq C_1\tau^{\frac{1}{2}}\sup_{(0,\tau)}\|u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2}\|^2_{L^{\frac{2p}{p-2}}(G^\varepsilon)}\|\nabla u^\varepsilon_{\delta,2}\|^2_{L^2(0,\tau;L^p(G^\varepsilon))} + C_{2,\tau}\|\nabla(u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2})\|^2_{L^2(G^\varepsilon_\tau)} \\ &\leq \tau^{\frac{1}{2}}\sup_{(0,\tau)}\|\nabla(u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2})\|^2_{L^2(G^\varepsilon)} + C_\tau\|\nabla(u^\varepsilon_{\delta,1} - u^\varepsilon_{\delta,2})\|^2_{L^2(G^\varepsilon_\tau)}.\end{aligned}$$

Lipschitz continuity of F^ε and f^ε ensures the corresponding estimates for the terms on the right-hands side of (38). Combining these estimates, applying Gronwall inequality, and iterating over $\tau > 0$ yield uniqueness of a solution of variational inequality (12) if $k = \text{const}$. Notice that if both k and P_c are constant the uniqueness result is obtained without additional regularity assumptions on solutions of variational inequality (12). □

Remark. By extending the L^p-theory for parabolic equations to pseudoparabolic equations and variational inequalities it may be possible to prove higher regularity for solutions of variational inequality (12). However this nontrivial analysis will not be considered here and will be the topic of future research.

To prove existence of a solution of the original problem (6) and to derive macroscopic variational inequality we first derive a priori estimates for solutions of regularised problem (12) uniformly in δ and ε.

Lemma 3.2. *Under Assumption 2.1 and if $\beta \geq \lambda > 4 + \alpha$ for $n = 3$ and $\beta \geq \lambda > 3 + \alpha + 4/(q-2)$ for $n = 2$ and any $q > 2$, solutions of variational inequality (12) are non-negative and satisfy the following a priori estimates*

$$
\begin{aligned}
&\|(u^\varepsilon_\delta+\delta)^{1+\alpha-\beta}\|_{L^\infty(0,T;L^1(G^\varepsilon))} + \|\sqrt{P_{c,\delta}(u^\varepsilon_\delta)}\nabla u^\varepsilon_\delta\|_{L^2((0,T)\times G^\varepsilon)} \le C,\\
&\|\nabla u^\varepsilon_\delta\|_{L^\infty(0,T;L^2(G^\varepsilon))} + \|b_\delta(u^\varepsilon_\delta)\|_{L^\infty(0,T;L^2(G^\varepsilon))} \le C,\\
&\|\sqrt{k_\delta(u^\varepsilon_\delta)}\partial_t \nabla u^\varepsilon_\delta\|_{L^2((0,T)\times G^\varepsilon)} + \|\sqrt{b'_\delta(u^\varepsilon_\delta)}\partial_t u^\varepsilon_\delta\|_{L^2((0,T)\times G^\varepsilon)} \le C,\\
&\|\partial_t b_\delta(u^\varepsilon_\delta)\|_{L^2(0,T;L^r(G^\varepsilon))} + \|\nabla \partial_t u^\varepsilon_\delta\|_{L^p((0,T)\times G^\varepsilon)} \le C,
\end{aligned}
\tag{39}
$$

for $1 < p < 2$ defined in (50), $1 < r < 3/2$ for $n = 3$ and $1 < r < 4/3$ for $n = 2$, and the constant $C > 0$ is independent of ε and δ.

Proof. To show that solutions of (12) are non-negative we consider $v^\varepsilon_\delta = u^\varepsilon_\delta - \widetilde{h}((u^\varepsilon_\delta)^-)$ as a test function in (12), where $u^- = \min\{u, 0\}$ and

$$
\widetilde{h}(w) = \int_0^w \frac{1}{k_\delta(\xi)} d\xi.
$$

Notice that $v^\varepsilon_\delta(t,x) = \kappa_D \ge 0$ on ∂G and $v^\varepsilon_\delta(t,x) \ge 0$ on Γ^ε for $t \in (0,T)$. The definition of $\widetilde{h}$ implies that $\widetilde{h}((u^\varepsilon_\delta)^-) = 0$ if $u^\varepsilon_\delta \ge 0$ and $\widetilde{h}((u^\varepsilon_\delta)^-) < 0$ for $u^\varepsilon_\delta < 0$, and hence $\widetilde{h}((u^\varepsilon_\delta)^-) = (u^\varepsilon_\delta)^-/k_\delta(\delta)$. Thus we obtain

$$
\begin{aligned}
&\langle \partial_t b_\delta(u^\varepsilon_\delta), \widetilde{h}((u^\varepsilon_\delta)^-)\rangle_{G^\varepsilon_\tau} + \langle A^\varepsilon(x)(P_{c,\delta}(u^\varepsilon_\delta)\nabla u^\varepsilon_\delta + \partial_t \nabla u^\varepsilon_\delta), \nabla (u^\varepsilon_\delta)^-\rangle_{G^\varepsilon_\tau}\\
&\quad - \langle F^\varepsilon(t,x,u^\varepsilon_\delta), \nabla \widetilde{h}((u^\varepsilon_\delta)^-)\rangle_{G^\varepsilon_\tau} + \varepsilon\langle f^\varepsilon(t,x,u^\varepsilon_\delta), \widetilde{h}((u^\varepsilon_\delta)^-)\rangle_{\Gamma^\varepsilon_\tau} \le 0,
\end{aligned}
\tag{40}
$$

for $\tau \in (0,T]$. Using the definition of $\widetilde{h}$ and properties of f^ε, for the boundary integral we have

$$
\langle \varepsilon f^\varepsilon(t,x,u^\varepsilon_\delta), \widetilde{h}((u^\varepsilon_\delta)^-)\rangle_{\Gamma^\varepsilon_\tau} = \langle \varepsilon f^\varepsilon(t,x,u^\varepsilon_\delta), \widetilde{h}((u^\varepsilon_\delta)^-)\chi_{u^\varepsilon_\delta \le 0}\rangle_{\Gamma^\varepsilon_\tau} \ge 0.
$$

Assumptions on F^ε and the boundary conditions on ∂G^ε imply

$$
\langle F^\varepsilon(t,x,u^\varepsilon_\delta), \nabla \widetilde{h}((u^\varepsilon_\delta)^-)\rangle_{G^\varepsilon_\tau} = \langle g, \nabla (u^\varepsilon_\delta)^-\rangle_{G^\varepsilon_\tau} + \int_0^\tau \int_{G^\varepsilon} \nabla \cdot\ \widetilde{H}^\varepsilon_\delta(t,x,(u^\varepsilon_\delta)^-)\, dxdt = 0,
$$

where $\widetilde{H}^\varepsilon_\delta(t,x,v) = Q^\varepsilon(t,x)\int_0^v H(\xi)/k_\delta(\xi)\, d\xi$. Assumptions on b, the definition of $\widetilde{h}$, and the non-negativity of initial data ensure

$$
\langle \partial_t b_\delta(u^\varepsilon_\delta), \widetilde{h}((u^\varepsilon_\delta)^-)\rangle_{G^\varepsilon_\tau} = \langle \partial_t b_\delta((u^\varepsilon_\delta)^-), \widetilde{h}((u^\varepsilon_\delta)^-)\rangle_{G^\varepsilon_\tau} = \int_{G^\varepsilon} \int_0^{(u^\varepsilon_\delta(\tau))^-} b'_\delta(\xi) \int_0^\xi \frac{d\eta}{k_\delta(\eta)} d\xi dx \ge 0,
$$

for $\tau \in (0,T]$. Then the non-negativity of initial conditions, i.e. $u_0(x) \ge 0$ in G, and assumptions on A yield

$$
\sup_{(0,T)} \|\nabla (u^\varepsilon_\delta)^-\|_{L^2(G^\varepsilon)} = 0,
$$

and using the non-negativity of u^ε_δ on $(0,T)\times\partial G^\varepsilon$ we conclude $u^\varepsilon_\delta(t,x) \ge 0$ a.e. in $(0,T)\times G^\varepsilon$.

To derive a priori estimates in (39), we first consider $v^\varepsilon_\delta = u^\varepsilon_\delta - h_\delta(u^\varepsilon_\delta)$ as a test function in (12), where

$$
h_\delta(v) = \theta \int_{\kappa_D}^v \frac{1}{k_\delta(\xi)} d\xi \quad \text{and} \quad \theta = \min_{z \ge \kappa_D} k(z) > 0,
$$

and obtain

$$\begin{aligned}&\langle \partial_t b_\delta(u^\varepsilon_\delta), h_\delta(u^\varepsilon_\delta)\rangle_{G^\varepsilon_s} + \theta\langle A^\varepsilon(x)(P_{c,\delta}(u^\varepsilon_\delta)\nabla u^\varepsilon_\delta + \partial_t \nabla u^\varepsilon_\delta), \nabla u^\varepsilon_\delta\rangle_{G^\varepsilon_s} \\ &\quad - \langle F^\varepsilon(t,x,u^\varepsilon_\delta), \nabla h_\delta(u^\varepsilon_\delta)\rangle_{G^\varepsilon_s} + \langle \varepsilon f^\varepsilon(t,x,u^\varepsilon_\delta), h_\delta(u^\varepsilon_\delta)\rangle_{\Gamma^\varepsilon_s} \leq 0\end{aligned} \tag{41}$$

for $s \in (0,T]$. Notice that $h_\delta(v) < 0$ for $v < \kappa_D$, $h_\delta(\kappa_D) = 0$, and $0 < h_\delta(v) \leq v$ for $v > \kappa_D$. Thus we obtain that $v^\varepsilon_\delta(t) \in \mathcal{K}^\varepsilon$ for $u^\varepsilon_\delta(t) \in \mathcal{K}^\varepsilon$, since $v^\varepsilon_\delta(t) \geq 0$ on Γ^ε if $u^\varepsilon_\delta(t) \geq 0$ on Γ^ε and $v^\varepsilon_\delta(t) = \kappa_D$ on ∂G if $u^\varepsilon_\delta(t) = \kappa_D$ on ∂G.

We shall estimate each term in (41) separately. The boundary integral can be written as

$$\langle \varepsilon f^\varepsilon(t,x,u^\varepsilon_\delta), h_\delta(u^\varepsilon_\delta)\rangle_{\Gamma^\varepsilon_s} = \langle \varepsilon f^\varepsilon(t,x,u^\varepsilon_\delta), h_\delta(u^\varepsilon_\delta)\chi_{u^\varepsilon_\delta<\kappa_D}\rangle_{\Gamma^\varepsilon_s} + \langle \varepsilon f^\varepsilon(t,x,u^\varepsilon_\delta), h_\delta(u^\varepsilon_\delta)\chi_{u^\varepsilon_\delta\geq\kappa_D}\rangle_{\Gamma^\varepsilon_s}.$$

Assumptions on f^ε imply

$$\begin{aligned}&\langle \varepsilon f^\varepsilon(t,x,u^\varepsilon_\delta), h_\delta(u^\varepsilon_\delta)\chi_{u^\varepsilon_\delta\geq\kappa_D}\rangle_{\Gamma^\varepsilon_s} \geq 0, \\ &\big|\langle \varepsilon f^\varepsilon(t,x,u^\varepsilon_\delta), h_\delta(u^\varepsilon_\delta)\chi_{u^\varepsilon_\delta<\kappa_D}\rangle_{\Gamma^\varepsilon_s}\big| \leq C,\end{aligned}$$

where the constant C is independent of δ and ε. To estimate the third term in (41) we use the properties of Q^ε and H and obtain

$$\langle F^\varepsilon(t,x,u^\varepsilon_\delta), \nabla h_\delta(u^\varepsilon_\delta)\rangle_{G^\varepsilon_s} = \theta\langle g, \nabla u^\varepsilon_\delta\rangle_{G^\varepsilon_s} + \int_0^s\int_{G^\varepsilon} \nabla\cdot \mathcal{H}^\varepsilon_\delta(t,x,u^\varepsilon_\delta)dxdt,$$

where $\mathcal{H}^\varepsilon_\delta(t,x,v) = \theta\, Q^\varepsilon(t,x)\int_{\kappa_D}^v H(\xi)[k_\delta(\xi)]^{-1}d\xi$. Using $Q^\varepsilon(t,x)\cdot\nu = 0$ on Γ^ε and $\mathcal{H}^\varepsilon_\delta(t,x,\kappa_D) = 0$ yields

$$\int_0^s\int_{G^\varepsilon} \nabla\cdot\mathcal{H}^\varepsilon_\delta(t,x,u^\varepsilon_\delta)dxdt = \int_0^s\int_{\partial G^\varepsilon} \mathcal{H}^\varepsilon_\delta(t,x,u^\varepsilon_\delta)\cdot\nu\, d\gamma_x dt = 0.$$

The first term in (41) can be written as

$$\begin{aligned}\langle \partial_t b_\delta(u^\varepsilon_\delta), h_\delta(u^\varepsilon_\delta)\rangle_{G^\varepsilon_s} &= \int_{G^\varepsilon_s} \partial_t \int_{\kappa_D}^{u^\varepsilon_\delta} b'_\delta(\xi)h_\delta(\xi)\, d\xi dx dt \\ &= \int_{G^\varepsilon}\int_{\kappa_D}^{u^\varepsilon_\delta(s)} b'_\delta(\xi)h_\delta(\xi)\, d\xi dx - \int_{G^\varepsilon}\int_{\kappa_D}^{u^\varepsilon_\delta(0)} b'_\delta(\xi)h_\delta(\xi)\, d\xi dx.\end{aligned}$$

The definition of h_δ and properties of function b ensure that for $u^\varepsilon_\delta \leq \kappa_D$

$$\int_{G^\varepsilon}\int_{\kappa_D}^{u^\varepsilon_\delta(s)} b'_\delta(\xi)h_\delta(\xi)\, d\xi dx = \int_{G^\varepsilon}\int_{u^\varepsilon_\delta(s)}^{\kappa_D} b'_\delta(\xi)\int_\xi^{\kappa_D}\frac{d\eta}{k_\delta(\eta)}\, d\xi dx \geq C_1\int_{G^\varepsilon}|u^\varepsilon_\delta + \delta|^{(1+\alpha-\beta)}dx - C_2,$$

for $s \in (0,T]$ and positive constants C_1 and C_2, which are independent of δ and ε. For $u^\varepsilon_\delta > \kappa_D$, the monotonicity of b and nonnegativity of k ensure

$$\int_{G^\varepsilon}\int_{\kappa_D}^{u^\varepsilon_\delta(s)} b'_\delta(\xi)h_\delta(\xi)\, d\xi dx = \theta\int_{G^\varepsilon}\int_{\kappa_D}^{u^\varepsilon_\delta(s)} b'_\delta(\xi)\int_{\kappa_D}^{\xi}\frac{1}{k_\delta(\eta)}d\eta\, d\xi\, dx \geq 0$$

for $s \in (0,T]$. Then integrating in (41) by parts with respect to time variable yields

$$\int_{G^\varepsilon} \Big[\int_{\kappa_D}^{u^\varepsilon_\delta(s)} b'_\delta(\xi) h_\delta(\xi) d\xi \, \chi_{u^\varepsilon_\delta \le \kappa_D} + \int_{\kappa_D}^{u^\varepsilon_\delta(s)} b'_\delta(\xi) h_\delta(\xi) d\xi \, \chi_{u^\varepsilon_\delta \ge \kappa_D} \Big] dx + \int_{G^\varepsilon} |\nabla u^\varepsilon_\delta(s)|^2 dx$$
$$+ \int_{G^\varepsilon_s} P_{c,\delta}(u^\varepsilon_\delta) |\nabla u^\varepsilon_\delta|^2 dx dt \le C_1 + C_2 \int_{G^\varepsilon} \int_{\kappa_D}^{u^\varepsilon_\delta(0)} b'_\delta(\xi) h_\delta(\xi)\, d\xi dx + C_3 \int_{G^\varepsilon} |\nabla u^\varepsilon_\delta(0)|^2 dx \tag{42}$$

for $s \in (0,T]$, where the constants C_j, with $j = 1, 2, 3$, are independent of ε and δ. Hence assumptions on u_0 ensure

$$\sup_{(0,T)} \int_{G^\varepsilon} |u^\varepsilon_\delta + \delta|^{1+\alpha-\beta} \chi_{u^\varepsilon_\delta \le \kappa_D} dx + \sup_{(0,T)} \int_{G^\varepsilon} |\nabla u^\varepsilon_\delta|^2 dx + \int_0^T \int_{G^\varepsilon} P_{c,\delta}(u^\varepsilon_\delta) |\nabla u^\varepsilon_\delta|^2 dx dt \le C, \tag{43}$$

with a positive constant C independent of ε and δ.

To derive an estimate for $\sqrt{k_\delta(u^\varepsilon_\delta)}\partial_t \nabla u^\varepsilon_\delta$ we need to use the equation with the penalty operator (13). Testing equation (13) by $v^\varepsilon = \partial_t u^\varepsilon_{\delta,\mu}$ yields

$$\langle \partial_t b_\delta(u^\varepsilon_{\delta,\mu}), \partial_t u^\varepsilon_{\delta,\mu} \rangle_{G^\varepsilon_s} + \langle A^\varepsilon(x) k_\delta(u^\varepsilon_{\delta,\mu}) [P_{c,\delta}(u^\varepsilon_{\delta,\mu}) \nabla u^\varepsilon_{\delta,\mu} + \partial_t \nabla u^\varepsilon_{\delta,\mu}], \partial_t \nabla u^\varepsilon_{\delta,\mu} \rangle_{G^\varepsilon_s} - \langle F^\varepsilon(t,x,u^\varepsilon_{\delta,\mu}), \nabla \partial_t u^\varepsilon_{\delta,\mu} \rangle_{G^\varepsilon_s}$$
$$+ \langle \varepsilon f^\varepsilon(t,x,u^\varepsilon_{\delta,\mu}), \partial_t u^\varepsilon_{\delta,\mu} \rangle_{\Gamma^\varepsilon_s} + \frac{1}{\mu} \int_0^s \langle \mathcal{B}(u^\varepsilon_{\delta,\mu} - \kappa_D), \partial_t u^\varepsilon_{\delta,\mu} \rangle_{V',V} dt = 0, \tag{44}$$

for $s \in (0,T]$. Using the property of the projection operator (19) for the difference quotient of $P_{\mathcal{K}^\varepsilon} u$ with respect to the time variable we obtain

$$0 \le \frac{1}{h} \langle J(u - P_{\mathcal{K}^\varepsilon} u), P_{\mathcal{K}^\varepsilon} u - P_{\mathcal{K}^\varepsilon} u(\cdot - h) \rangle_{V',V}.$$

Then, the last inequality, together with the regularity $\partial_t u^\varepsilon_{\delta,\mu} \in L^2(0,T;V)$ and the fact that $u_0, \kappa_D \in \mathcal{K}^\varepsilon$, yields

$$\int_0^s \langle \mathcal{B}(\widetilde{u}^\varepsilon_{\delta,\mu}), \partial_t u^\varepsilon_{\delta,\mu} \rangle_{V',V} dt = \lim_{h\to 0} \sum_{j=1}^N \big\langle \mathcal{B}(\widetilde{u}^\varepsilon_{\delta,\mu}(t_j)), u^\varepsilon_{\delta,\mu}(t_j) - u^\varepsilon_{\delta,\mu}(t_{j-1}) \big\rangle_{V',V}$$
$$= \lim_{h\to 0} \sum_{j=1}^N \Big[\Big\langle J(\widetilde{u}^\varepsilon_{\delta,\mu} - P_{\mathcal{K}^\varepsilon} \widetilde{u}^\varepsilon_{\delta,\mu})(t_j), (\widetilde{u}^\varepsilon_{\delta,\mu} - P_{\mathcal{K}^\varepsilon} \widetilde{u}^\varepsilon_{\delta,\mu}) \Big|_{t_{j-1}}^{t_i} \Big\rangle_{V',V}$$
$$+ \big\langle J(\widetilde{u}^\varepsilon_{\delta,\mu} - P_{\mathcal{K}^\varepsilon} \widetilde{u}^\varepsilon_{\delta,\mu})(t_j), P_{\mathcal{K}^\varepsilon} \widetilde{u}^\varepsilon_{\delta,\mu}(t_j) - P_{\mathcal{K}^\varepsilon} \widetilde{u}^\varepsilon_{\delta,\mu}(t_{j-1}) \big\rangle_{V',V} \Big]$$
$$\ge \frac{1}{2} \int_{G^\varepsilon} \Big[|(\widetilde{u}^\varepsilon_{\delta,\mu} - P_{\mathcal{K}^\varepsilon} \widetilde{u}^\varepsilon_{\delta,\mu})(s)|^2 + |\nabla(\widetilde{u}^\varepsilon_{\delta,\mu} - P_{\mathcal{K}^\varepsilon} \widetilde{u}^\varepsilon_{\delta,\mu})(s)|^2 \Big] dx \ge 0,$$

where $\widetilde{u}^\varepsilon_{\delta,\mu} = u^\varepsilon_{\delta,\mu} - \kappa_D$ and $t_j = jh$ for $j = 1, \dots, N$, and $N \in \mathbb{N}$, with $t_N = Nh = s$. Using assumptions on the functions k and P_c and applying the Hölder inequality yield

$$\langle A^\varepsilon(x) k_\delta(u^\varepsilon_{\delta,\mu}) P_{c,\delta}(u^\varepsilon_{\delta,\mu}) \nabla u^\varepsilon_{\delta,\mu}, \partial_t \nabla u^\varepsilon_{\delta,\mu} \rangle_{G^\varepsilon_s} \le \sigma \| \sqrt{k_\delta(u^\varepsilon_{\delta,\mu})} \partial_t \nabla u^\varepsilon_{\delta,\mu} \|_{L^2(G^\varepsilon_s)}$$
$$+ C_\sigma \| k_\delta(u^\varepsilon_{\delta,\mu}) P_{c,\delta}(u^\varepsilon_{\delta,\mu}) \|_{L^\infty(G^\varepsilon_s)} \| \sqrt{P_{c,\delta}(u^\varepsilon_{\delta,\mu})} \nabla u^\varepsilon_{\delta,\mu} \|_{L^2(G^\varepsilon_s)},$$

for some $0 < \sigma \le a_0/8$. The boundary term can be written as

$$\langle \varepsilon f^\varepsilon(t,x,u^\varepsilon_{\delta,\mu}), \partial_t u^\varepsilon_{\delta,\mu}\rangle_{\Gamma^\varepsilon_s} = \varepsilon \int\limits_{\Gamma^\varepsilon_s} \partial_t \int\limits_{\kappa_D}^{u^\varepsilon_{\delta,\mu}} f^\varepsilon(t,x,\xi)\, d\xi d\gamma dt - \varepsilon \int\limits_{\Gamma^\varepsilon_s} \int\limits_{\kappa_D}^{u^\varepsilon_{\delta,\mu}} \partial_t f^\varepsilon(t,x,\xi)\, d\xi d\gamma dt.$$

Hence assumptions on f^ε imply

$$\big|\langle \varepsilon f^\varepsilon(t,x,u^\varepsilon_{\delta,\mu}), \partial_t u^\varepsilon_{\delta,\mu}\rangle_{\Gamma^\varepsilon_s}\big| \le \sigma\varepsilon\Big[\int\limits_{\Gamma^\varepsilon} |u^\varepsilon_{\delta,\mu}(s)|^2 d\gamma + \int\limits_{\Gamma^\varepsilon_s} |u^\varepsilon_{\delta,\mu}|^2 d\gamma dt\Big] + C_\sigma,$$

with some constant C_σ independent of μ, ε and δ, and an arbitrary fixed $\sigma > 0$. Then the trace estimate

$$\varepsilon \|v\|^2_{L^2(\Gamma^\varepsilon)} \le C\big[\|v\|^2_{L^2(G^\varepsilon)} + \varepsilon^2 \|\nabla v\|^2_{L^2(G^\varepsilon)}\big],$$

which follows from the definition of G^ε and Γ^ε, the standard trace estimate for $v \in H^1(Y^*)$, and a scaling argument, combined with the properties of an extension of $u^\varepsilon_{\delta,\mu}$ from G^ε into G, see Remark 3.3, and the Dirichlet boundary condition on ∂G, ensures

$$\big|\langle \varepsilon f^\varepsilon(t,x,u^\varepsilon_{\delta,\mu}), \partial_t u^\varepsilon_{\delta,\mu}\rangle_{\Gamma^\varepsilon_s}\big| \le \sigma_1\big[\|\nabla u^\varepsilon_{\delta,\mu}(s)\|^2_{L^2(G^\varepsilon)} + \|\nabla u^\varepsilon_{\delta,\mu}\|^2_{L^2(G^\varepsilon_s)}\big] + C,$$

with $s \in (0,T]$. The assumptions on F^ε and k and the fact that $\partial_t u^\varepsilon_{\delta,\mu}(t,x) = 0$ on $(0,T) \times \partial G$ yield

$$\begin{aligned}&\langle F^\varepsilon(t,x,u^\varepsilon_{\delta,\mu}), \nabla\partial_t u^\varepsilon_{\delta,\mu}\rangle_{G^\varepsilon_s}\\ &= \langle g\, k_\delta(u^\varepsilon_{\delta,\mu}), \nabla\partial_t u^\varepsilon_{\delta,\mu}\rangle_{G^\varepsilon_s} - \langle Q^\varepsilon(t,x) H'(u^\varepsilon_{\delta,\mu})[b'_\delta(u^\varepsilon_{\delta,\mu})]^{-\frac12}\nabla u^\varepsilon_{\delta,\mu}, \sqrt{b'_\delta(u^\varepsilon_{\delta,\mu})}\partial_t u^\varepsilon_{\delta,\mu}\rangle_{G^\varepsilon_s}.\end{aligned}$$

Applying the Hölder inequality and using assumptions on H and Q^ε we obtain

$$\begin{aligned}|\langle F^\varepsilon(t,x,u^\varepsilon_{\delta,\mu}), \nabla\partial_t u^\varepsilon_{\delta,\mu}\rangle_{G^\varepsilon_s}| &\le \sigma_1 \|\sqrt{k_\delta(u^\varepsilon_{\delta,\mu})}\nabla\partial_t u^\varepsilon_{\delta,\mu}\|^2_{L^2(G^\varepsilon_s)}\\ &+ \sigma_2 \|\sqrt{b'_\delta(u^\varepsilon_{\delta,\mu})}\partial_t u^\varepsilon_{\delta,\mu}\|^2_{L^2(G^\varepsilon_s)} + C_1\|\nabla u^\varepsilon_{\delta,\mu}\|^2_{L^2(G^\varepsilon_s)} + C_2,\end{aligned}$$

for $0 < \sigma_1 \le a_0/8$, $0 < \sigma_2 \le 1/4$ and constants $C_1, C_2 > 0$ are independent of μ, ε, and δ.

Using the estimate for $\nabla u^\varepsilon_{\delta,\mu}$ in $L^\infty(0,T;L^2(G^\varepsilon))$ and $\sqrt{P_{c,\delta}(u^\varepsilon_{\delta,\mu})}\nabla u^\varepsilon_{\delta,\mu}$ in $L^2((0,T)\times G^\varepsilon)$, which can be derived in a similar way as the corresponding estimates for $\nabla u^\varepsilon_\delta$ and $\sqrt{P_{c,\delta}(u^\varepsilon_\delta)}\nabla u^\varepsilon_\delta$ in (43), by deriving estimates for the penalty operator $\mathcal{B}$ similar to those obtained in the derivation of inequality (29), we obtain

$$\|\sqrt{b'_\delta(u^\varepsilon_{\delta,\mu})}\partial_t u^\varepsilon_{\delta,\mu}\|_{L^2(G^\varepsilon_s)} + \|\sqrt{k_\delta(u^\varepsilon_{\delta,\mu})}\partial_t \nabla u^\varepsilon_{\delta,\mu}\|_{L^2(G^\varepsilon_s)} \le C,$$

for any $s \in (0,T]$ and a constant C independent of μ, ε and δ. Notice that assumptions on k and definition of θ imply that $u^\varepsilon_{\delta,\mu} - \kappa_D - \theta\int_{\kappa_D}^{u^\varepsilon_{\delta,\mu}}[k_\delta(\xi)]^{-1} d\xi \ge 0$. Considering $\mu \to 0$ and using continuity and strict positivity of k_δ and b'_δ, together with the strong convergence of $u^\varepsilon_{\delta,\mu}$ in $L^2(G^\varepsilon_T)$, as $\mu \to 0$, and lower-semicontinuity of a norm, we obtain the third estimate in (39).

If b is Lipschitz continuous we also have

$$\|\partial_t b_\delta(u^\varepsilon_\delta)\|^2_{L^2(G^\varepsilon_T)} \le \sup_{(t,x)\in G^\varepsilon_T} |b'_\delta(u^\varepsilon_\delta)|\, \|\sqrt{b'_\delta(u^\varepsilon_\delta)}\partial_t u^\varepsilon_\delta\|^2_{L^2(G^\varepsilon_T)} \le C.$$

Otherwise, we can consider

$$\begin{aligned}\|\partial_t b_\delta(u^\varepsilon_\delta)\|_{L^2(0,T;L^r(G^\varepsilon))} &= \|b'_\delta(u^\varepsilon_\delta)\partial_t u^\varepsilon_\delta\|_{L^2(0,T;L^r(G^\varepsilon))} \\ &\leq \sup_{(0,T)} \|\sqrt{b'_\delta(u^\varepsilon_\delta)}\|^{\frac{2-r}{r}}_{L^{\frac{2r}{2-r}}(G^\varepsilon)} \|\sqrt{b'_\delta(u^\varepsilon_\delta)}\partial_t u^\varepsilon_\delta\|^2_{L^2(G^\varepsilon_T)},\end{aligned}$$

for some $1 < r < 2$. Then the first estimate in (39) for $0 \leq u^\varepsilon_\delta(t,x) \leq 1$ and if $0 < \alpha < 1$, and assumptions on b' for $u^\varepsilon_\delta(t,x) \geq 1$, combined with the uniform boundedness of $\|u^\varepsilon_\delta\|_{L^\infty(0,T;H^1(G^\varepsilon))}$, ensure

$$\sup_{(0,T)} \left\|\sqrt{b'_\delta(u^\varepsilon_\delta)}\right\|^{\frac{2-r}{r}}_{L^{\frac{2r}{2-r}}(G^\varepsilon)} \leq C,$$

where $1 < r < 3/2$ for $n = 3$ and $1 < r < 4/3$ for $n = 2$.

From assumptions on b and the estimate for u^ε_δ in $L^\infty(0,T;H^1(G^\varepsilon))$, we also obtain the boundedness of $b_\delta(u^\varepsilon_\delta)$ in $L^\infty(0,T;L^2(G^\varepsilon))$, uniformly in ε and δ.

To derive the estimate for $\nabla\partial_t u^\varepsilon_\delta$ in $L^p((0,T)\times G^\varepsilon)$, with some $p > 1$, we follow the same ideas as in [31]. Using assumptions on P_c together with $u^\varepsilon_\delta \geq 0$ we can rewrite

$$\sqrt{P_{c,\delta}(u^\varepsilon_\delta)}\nabla u^\varepsilon_\delta = \nabla\Big(\int_0^{u^\varepsilon_\delta} \sqrt{P_{c,\delta}(\xi)}d\xi\Big),$$

where

$$\int_0^{u^\varepsilon_\delta} \sqrt{P_{c,\delta}(\xi)}d\xi = C_1\left[(u^\varepsilon_\delta+\delta)^{1-\lambda/2} - \delta^{1-\lambda/2}\right] + C_2,$$

with some constants C_1 and C_2 independent of ε and δ. Then the estimate for $P_{c,\delta}(u^\varepsilon_\delta)|\nabla u^\varepsilon_\delta|^2$, together with the Dirichlet boundary condition on ∂G, implies that $(u^\varepsilon_\delta+\delta)^{1-\lambda/2} \in L^2(0,T;H^1(G^\varepsilon))$. Considering an extension $\overline{(u^\varepsilon_\delta+\delta)}^{1-\frac{\lambda}{2}}$ of $(u^\varepsilon_\delta+\delta)^{1-\frac{\lambda}{2}}$ from G^ε into G, see Remark 3.3 applied to $v^\varepsilon = (u^\varepsilon_\delta+\delta)^{1-\frac{\lambda}{2}}$, we obtain

$$\begin{aligned}\|\nabla\overline{(u^\varepsilon_\delta+\delta)}^{1-\lambda/2}\|_{L^2((0,T)\times G)} &\leq C_1\|\nabla(u^\varepsilon_\delta+\delta)^{1-\lambda/2}\|_{L^2((0,T)\times G^\varepsilon)} \leq C_2, \\ \|(u^\varepsilon_\delta+\delta)^{1-\lambda/2}\|_{L^2((0,T)\times G^\varepsilon)} &\leq \|\overline{(u^\varepsilon_\delta+\delta)}^{1-\lambda/2}\|_{L^2((0,T)\times G)} \\ &\leq C_3\|\nabla\overline{(u^\varepsilon_\delta+\delta)}^{1-\lambda/2}\|_{L^2((0,T)\times G)} + C_4 \leq C_5,\end{aligned}$$

where the constants C_j, with $j = 1,\dots,5$, are independent of δ and ε. Notice that the extension $\overline{(u^\varepsilon_\delta+\delta)}^{1-\lambda/2}$ satisfies the same Dirichlet boundary condition on ∂G as the original function $(u^\varepsilon_\delta+\delta)^{1-\lambda/2}$. Then the Sobolev embedding theorem ensures

$$\begin{aligned}&\|\overline{(u^\varepsilon_\delta+\delta)}^{1-\lambda/2}\|_{L^2(0,T;L^{q_1}(G))} \leq C, \qquad q_1 \in (2,+\infty) \qquad \text{for } n = 2, \\ &\|\overline{(u^\varepsilon_\delta+\delta)}^{1-\lambda/2}\|_{L^2(0,T;L^{q_2}(G))} \leq C, \qquad q_2 = \frac{2n}{n-2} \qquad \text{for } n \geq 3,\end{aligned} \tag{45}$$

with a constant $C > 0$ independent of ε and δ.

For θ and θ_1 such that $(1-\lambda/2)\theta + (1+\alpha-\beta)\theta_1 = -\gamma\beta$, where $\gamma > 1$ and β is as in the assumption on k, we obtain

$$\begin{aligned}\int\limits_{G^\varepsilon}(u^\varepsilon_\delta+\delta)^{-\gamma\beta}dx&=\int\limits_{G^\varepsilon}(u^\varepsilon_\delta+\delta)^{(1-\lambda/2)\theta}(u^\varepsilon_\delta+\delta)^{(1+\alpha-\beta)\theta_1}dx\\&\leq\Big(\int\limits_{G^\varepsilon}(u^\varepsilon_\delta+\delta)^{(1-\lambda/2)p}dx\Big)^{\theta/p}\Big(\int\limits_{G^\varepsilon}(u^\varepsilon_\delta+\delta)^{(1+\alpha-\beta)\theta_1p_1}dx\Big)^{1/p_1}\\&\leq\Big(\int\limits_{G}(\overline{u^\varepsilon_\delta+\delta})^{(1-\lambda/2)p}dx\Big)^{\theta/p}\Big(\int\limits_{G^\varepsilon}(u^\varepsilon_\delta+\delta)^{(1+\alpha-\beta)\theta_1p_1}dx\Big)^{1/p_1}.\end{aligned}\tag{46}$$

For $n=3$ we have $p=6$ and $p_1=6/(6-\theta)$. Then the estimate for $(u^\varepsilon_\delta+\delta)^{(1+\alpha-\beta)}$ in $L^1((0,T)\times G^\varepsilon)$ yields $\theta_1=1-\theta/6$ and the integrability of $\overline{(u^\varepsilon_\delta+\delta)}^{1-\lambda/2}$ with respect to the time variable implies $\theta=2$. Hence $-\gamma\beta=2-\lambda+\frac{2}{3}(1+\alpha-\beta)$ and in order to ensure that $\gamma>1$ we require

$$-\frac{1}{\beta}\Big(\frac{8}{3}+\frac{2}{3}\alpha-\lambda-\frac{2}{3}\beta\Big)>1\quad\Longleftrightarrow\quad\frac{8}{3}+\frac{2}{3}\alpha+\frac{\beta}{3}<\lambda.\tag{47}$$

If $n=2$ the Hölder exponents in (46) are $p=q_1/\theta$ and $1/p_1=1-\theta/q_1$, for any $q_1>2$. Thus we obtain $\theta=2$, $\theta_1=1-2/q_1$ and

$$-\gamma\beta=(2-\lambda)+(1+\alpha-\beta)(1-2/q_1)\quad\text{ and }\quad\gamma>1\Longleftrightarrow3-\frac{2}{q_1}+\alpha\big(1-\frac{2}{q_1}\big)+\frac{2}{q_1}\beta<\lambda.\tag{48}$$

Then, combining (45), third estimate in (39), and (46), we obtain the following estimate

$$\begin{aligned}\int\limits_0^T\int\limits_{G^\varepsilon}|\nabla\partial_tu^\varepsilon_\delta|^pdxdt&=\int\limits_0^T\int\limits_{G^\varepsilon}|k_\delta(u^\varepsilon_\delta)^{\frac12}\nabla\partial_tu^\varepsilon_\delta|^p|k_\delta(u^\varepsilon_\delta)|^{-\frac{p}{2}}dxdt\\&\leq\Big(\int\limits_0^T\int\limits_{G^\varepsilon}k_\delta(u^\varepsilon_\delta)|\nabla\partial_tu^\varepsilon_\delta|^2dxdt\Big)^{\frac{p}{2}}\Big(\int\limits_0^T\int\limits_{G^\varepsilon}|k_\delta(u^\varepsilon_\delta)|^{-\frac{p}{2-p}}dxdt\Big)^{1-\frac{p}{2}}\\&\leq C_1\Big(\int\limits_0^T\int\limits_{G^\varepsilon}|k_\delta(u^\varepsilon_\delta)|^{-\frac{p}{2-p}}dxdt\Big)^{1-\frac{p}{2}}\end{aligned}\tag{49}$$

for some $1<p<2$. Assumptions on k and conditions on α, β and λ, specified in the formulation of the lemma, together with the first estimate in (39), ensure that there exists such $p=p(\beta,\lambda,\alpha,n)>1$ that

$$\||k_\delta(u^\varepsilon_\delta)|^{-\frac{p}{2-p}}\|_{L^1((0,T)\times G^\varepsilon)}\leq C_2,$$

where C_2 is independent of ε and δ and the exponent p is defined as

$$\begin{aligned}p&=\frac{2(3\lambda+2\beta-2\alpha-8)}{3\lambda+5\beta-2\alpha-8}&&\text{for } n=3\text{ and }\beta\geq\lambda>4+\alpha,\\p&=\frac{2[2(1+\alpha-\beta)+q_1(\lambda+\beta-3-\alpha)]}{2(1+\alpha-\beta)+q_1(\lambda+2\beta-3-\alpha)}&&\text{for } n=2,\text{ any }q_1>2,\text{ and }\beta\geq\lambda>3+\alpha+4/(q_1-2),\end{aligned}\tag{50}$$

and additionally inequalities in (47) and (48) are satisfied. This implies the last estimate in (39). □

Remark 3.3. To ensure that in the derivation of a priori estimates the embedding and Poincaré constants are independent of ε, we considered an extension of u^ε_δ and of $(u^\varepsilon_\delta+\delta)^{1-\lambda/2}$ from G^ε to G with the following properties: There exists an extension $\overline{v}^\varepsilon$ of v^ε from $L^p(0,T;W^{1,p}(G^\varepsilon))$ into $L^p(0,T;W^{1,p}(G))$ such that

$$\|\overline{v}^\varepsilon\|_{L^p(G_T)} \leq C\|v^\varepsilon\|_{L^p(G_T^\varepsilon)}, \quad \|\nabla\overline{v}^\varepsilon\|_{L^p(G_T)} \leq C\|\nabla v^\varepsilon\|_{L^p(G_T^\varepsilon)}, \tag{51}$$

where $1 \leq p < \infty$ and the constant $C > 0$ is independent of ε. The existence of an extension $\overline{v}^\varepsilon$ satisfying estimates (51) follows from the assumptions on the geometry of G^ε and a standard extension operator, see e.g. [1,10].

A priori estimates (39) ensure the following convergence results for a subsequence of $\{u^\varepsilon_\delta\}$ as $\delta \to 0$:

Lemma 3.4. *Under assumptions in Lemma 3.2, there exists a function $u^\varepsilon \in L^2(0,T;H^1(G^\varepsilon))$, with $\partial_t u^\varepsilon \in L^p(0,T;W^{1,p}(G^\varepsilon))$, such that, up to a subsequence,*

$$\begin{aligned}
&u^\varepsilon_\delta \to u^\varepsilon && \text{strongly in } L^2(0,T;H^\sigma(G^\varepsilon)) \text{ and } L^{r_1}((0,T)\times G^\varepsilon) \ \text{ for } 1 < r_1 < 6,\\
&b_\delta(u^\varepsilon_\delta) \to b(u^\varepsilon) && \text{strongly in } L^{r_2}((0,T)\times G^\varepsilon) \ \text{ for } 1 < r_2 < 2,\\
&k_\delta(u^\varepsilon_\delta) \to k(u^\varepsilon) && \text{strongly in } L^{q}((0,T)\times G^\varepsilon) \ \text{ for } 1 < q < \infty,\\
&b_\delta(u^\varepsilon_\delta) \rightharpoonup b(u^\varepsilon) && \text{weakly-}* \text{ in } L^\infty(0,T;L^2(G^\varepsilon)),\\
&u^\varepsilon_\delta \rightharpoonup u^\varepsilon && \text{weakly-}* \text{ in } L^\infty(0,T;H^1(G^\varepsilon)),
\end{aligned} \tag{52}$$

where $1/2 < \sigma < 1$, and

$$\begin{aligned}
&\partial_t b_\delta(u^\varepsilon_\delta) \rightharpoonup \partial_t b(u^\varepsilon) && \text{weakly in } L^2(0,T;L^r(G^\varepsilon)),\\
&\partial_t u^\varepsilon_\delta \rightharpoonup \partial_t u^\varepsilon && \text{weakly in } L^p(0,T;W^{1,p}(G^\varepsilon)),\\
&\sqrt{k_\delta(u^\varepsilon_\delta)}\nabla\partial_t u^\varepsilon_\delta \rightharpoonup \sqrt{k(u^\varepsilon)}\nabla\partial_t u^\varepsilon && \text{weakly in } L^2((0,T)\times G^\varepsilon),\\
&\sqrt{k_\delta(u^\varepsilon_\delta)P_{c,\delta}(u^\varepsilon_\delta)}\nabla u^\varepsilon_\delta \rightharpoonup \sqrt{k(u^\varepsilon)P_c(u^\varepsilon)}\nabla u^\varepsilon && \text{weakly in } L^2((0,T)\times G^\varepsilon),
\end{aligned} \tag{53}$$

as $\delta \to 0$, where $1 < p < 2$ is defined in (50), $1 < r < 3/2$ for $n = 3$ and $1 < r < 4/3$ for $n = 2$. Due to the lower semicontinuity of a norm we also have

$$\begin{aligned}
&\|\nabla u^\varepsilon\|_{L^\infty(0,T;L^2(G^\varepsilon))} + \|\sqrt{k(u^\varepsilon)}\partial_t\nabla u^\varepsilon\|_{L^2(G^\varepsilon_T)} + \|\nabla\partial_t u^\varepsilon\|_{L^p(G^\varepsilon_T)}\\
&\quad + \|b(u^\varepsilon)\|_{L^\infty(0,T;L^2(G^\varepsilon))} + \|\partial_t b(u^\varepsilon)\|_{L^2(0,T;L^r(G^\varepsilon))} \leq C,
\end{aligned} \tag{54}$$

with a constant $C > 0$ independent of ε, and $u^\varepsilon(t,x) \geq 0$ in $(0,T)\times G^\varepsilon$.

Proof. Weak-$*$ convergence of u^ε_δ in $L^\infty(0,T;H^1(G^\varepsilon))$ and weak convergence of $\partial_t u^\varepsilon_\delta$ in $L^p(0,T;W^{1,p}(G^\varepsilon))$ follow directly from the a priori estimates (39), combined with the Dirichlet boundary condition on ∂G and the Poincaré inequality. Then using Lions–Aubin compactness lemma [24] and the fact that the embeddings $H^1(G^\varepsilon) \subset H^\sigma(G^\varepsilon)$ and $H^1(G^\varepsilon) \subset L^{r_1}(G^\varepsilon)$, for $1 \leq r_1 < 6$ and $1/2 < \sigma < 1$, are compact, we obtain the strong convergence of u^ε_δ in $L^{r_1}((0,T)\times G^\varepsilon)$ and $L^2(0,T;H^\sigma(G^\varepsilon))$.

Continuity of b_δ, $P_{c,\delta}$ and k_δ and the strong convergence of u^ε_δ imply point-wise convergence $b_\delta(u^\varepsilon_\delta) \to b(u^\varepsilon)$, $k_\delta(u^\varepsilon_\delta) \to k(u^\varepsilon)$, $k_\delta(u^\varepsilon_\delta)P_{c,\delta}(u^\varepsilon_\delta) \to k(u^\varepsilon)P_c(u^\varepsilon)$ a.e. in $(0,T)\times G^\varepsilon$ as $\delta \to 0$. Assumptions on b yield $\|b_\delta(u^\varepsilon_\delta)\|_{L^{r_2}(G^\varepsilon_T)} \leq C_1(1 + \|u^\varepsilon_\delta\|^3_{L^{3r_2}(G^\varepsilon_T)})$, where $3 \leq 3r_2 < 6$. Then the strong convergence of u^ε_δ together with the Lebesgue dominated convergence theorem implies the strong convergence of $b(u^\varepsilon_\delta)$ in $L^{r_2}(G^\varepsilon_T)$ for $1 < r_2 < 2$. Assumptions on functions k and P_c, stated in Assumption 2.1, ensure that $|k_\delta(u^\varepsilon_\delta)| \leq C$ and $|k_\delta(u^\varepsilon_\delta)P_{c,\delta}(u^\varepsilon_\delta)| \leq C$ a.e. in G^ε_T independently of δ. Then applying the Lebesgue dominated convergence theorem implies strong convergence of $k_\delta(u^\varepsilon_\delta)$ and $k_\delta(u^\varepsilon_\delta)P_{c,\delta}(u^\varepsilon_\delta)$ in $L^q((0,T)\times G^\varepsilon)$ for any $1 < q < \infty$.

Estimates for $\partial_t b_\delta(u^\varepsilon_\delta)$ together with the convergence $b_\delta(u^\varepsilon_\delta) \to b(u^\varepsilon)$ in $L^{r_2}(G^\varepsilon_T)$ ensure weak convergence of $\partial_t b_\delta(u^\varepsilon_\delta) \rightharpoonup \partial_t b(u^\varepsilon)$ in $L^2(0,T;L^r(G^\varepsilon))$. Weak convergence $\partial_t u^\varepsilon_\delta$ in $L^p(0,T;W^{1,p}(G^\varepsilon))$ and strong convergence and boundedness of $k_\delta(u^\varepsilon_\delta)$ ensure weak convergence of $\sqrt{k_\delta(u^\varepsilon_\delta)}\partial_t \nabla u^\varepsilon_\delta \rightharpoonup \sqrt{k(u^\varepsilon)}\partial_t \nabla u^\varepsilon$ in $L^{p_1}(G^\varepsilon_T)$ for $1 < p_1 < p$, as $\delta \to 0$. A priori estimates (39) imply $\sqrt{k_\delta(u^\varepsilon_\delta)}\partial_t \nabla u^\varepsilon_\delta \rightharpoonup w$ in $L^2(G^\varepsilon_T)$. Hence $w = \sqrt{k(u^\varepsilon)}\partial_t \nabla u^\varepsilon \in L^2(G^\varepsilon_T)$. Similar arguments imply the last convergence in (53). □

Theorem 3.5. *Under assumptions in Lemma 3.2, for every fixed $\varepsilon > 0$ there exists a nonnegative solution of variational inequality (6). If $k(\xi)$ is non-degenerate, $\partial_t u^\varepsilon \in L^2(0,T;W^{1,p_2}(G^\varepsilon))$ and $u_0 \in W^{1,p_2}(G^\varepsilon)$ for $p_2 > n$, or if $k(\xi) = \text{const}$, $P_c(\xi)$ is Lipschitz continuous for $\xi \geq 0$ and $u^\varepsilon \in L^2(0,T;W^{1,p_2}(G^\varepsilon))$, then solution of (6) is unique.*

Proof. Using the convergence results in Lemma 3.4, together with assumptions on k, P_c, b, H, f_0, and f_1, stated in Assumption 2.1, and taking $\delta \to 0$ in the regularised problem (12), we obtain that u^ε satisfies variational inequality (6). The regularity of u^ε implies $u^\varepsilon \in C([0,T];L^2(G^\varepsilon))$ and $u^\varepsilon(t) \to u_0$ in $L^2(G^\varepsilon)$ as $t \to 0$. The weak convergence of u^ε_δ in $L^2(0,T;H^1(G^\varepsilon))$ and non-negativity of u^ε_δ in G^ε_T, together with $u^\varepsilon_\delta \in \mathcal{K}^\varepsilon$, ensure that $u^\varepsilon(t,x) \geq 0$ in $(0,T)\times G^\varepsilon$ and on $(0,T)\times\Gamma^\varepsilon$, as well as $u^\varepsilon(t,x) = \kappa_D$ on $(0,T)\times\partial G$. Hence $u^\varepsilon(t) \in \mathcal{K}^\varepsilon$ for $t \in [0,T]$.

The proof of the uniqueness result in the case k is nondegenerate or $k(\xi) = \text{const}$ for $\xi \geq 0$ follows the same steps as the corresponding proof for the regularised problem (12) in Lemma 3.1. □

4. Derivation of macroscopic obstacle problem

Using estimates (54) and compactness theorems for the two-scale convergence, see e.g. [2,34,35] or Appendix for more details, we obtain the following convergence results for a subsequence of the sequence $\{u^\varepsilon\}$ of solutions of the microscopic problem (6), as $\varepsilon \to 0$.

Lemma 4.1. *Under assumptions in Lemma 3.2, there exist functions $u \in L^2(0,T;H^1(G))$ and $w \in L^2(G_T;H^1_{\text{per}}(Y^*)/\mathbb{R})$, with $\partial_t u \in L^p(0,T;W^{1,p}(G))$ and $\partial_t w \in L^p(G_T;W^{1,p}_{\text{per}}(Y^*)/\mathbb{R})$, such that, up to a subsequence,*

$$
\begin{aligned}
& u^\varepsilon \to u && \textit{strongly in } L^{r_1}((0,T)\times G) \quad \textit{for } 1 < r_1 < 6,\\
& b(u^\varepsilon) \to b(u) && \textit{strongly in } L^{r_2}((0,T)\times G) \quad \textit{for } 1 < r_2 < 2,\\
& k(u^\varepsilon) \to k(u) && \textit{strongly in } L^{q}((0,T)\times G) \quad \textit{for } 1 < q < \infty,\\
& b(u^\varepsilon) \rightharpoonup b(u) && \textit{weakly-}* \textit{ in } L^\infty(0,T;L^2(G)),\\
& \partial_t b(u^\varepsilon) \rightharpoonup \partial_t b(u) && \textit{weakly in } L^2(0,T;L^r(G)),
\end{aligned}
\tag{55}
$$

for $1 < r < 3/2$ for $n = 3$ and $1 < r < 4/3$ for $n = 2$, where u^ε is equated with its extension from G^ε into G, as in Remark 3.3, and

$$
\begin{aligned}
& \nabla u^\varepsilon \rightharpoonup \nabla u + \nabla_y w && \textit{two-scale},\\
& \nabla \partial_t u^\varepsilon \rightharpoonup \nabla \partial_t u + \nabla_y \partial_t w && \textit{two-scale},\\
& k(u^\varepsilon)\nabla \partial_t u^\varepsilon \rightharpoonup k(u)(\nabla \partial_t u + \nabla_y \partial_t w) && \textit{two-scale},\\
& k(u^\varepsilon)P_c(u^\varepsilon)\nabla u^\varepsilon \rightharpoonup k(u)P_c(u)(\nabla u + \nabla_y w) && \textit{two-scale},\\
& \varepsilon \|u^\varepsilon\|^2_{L^2((0,T)\times\Gamma^\varepsilon)} \to |Y|^{-1}\|u\|^2_{L^2((0,T)\times G\times\Gamma)},
\end{aligned}
\tag{56}
$$

as $\varepsilon \to 0$, where exponent p is defined in (50).

Proof. The estimate for $\nabla\partial_t u^\varepsilon$ in (54), combined with the Dirichlet boundary condition on ∂G and the Poincaré and Sobolev inequalities, ensures that $\partial_t u^\varepsilon$ and its extension $\partial_t \overline{u}^\varepsilon$, see Remark 3.3, satisfy the following estimate

$$\|\partial_t u^\varepsilon\|_{L^p(0,T;W^{1,p}(G^\varepsilon))} + \|\partial_t \overline{u}^\varepsilon\|_{L^p(0,T;W^{1,p}(G))} + \|\partial_t u^\varepsilon\|_{L^p(0,T;L^{q_2}(G^\varepsilon))} + \|\partial_t \overline{u}^\varepsilon\|_{L^p(0,T;L^{q_2}(G))} \leq C,$$

for $1 < p < 2$ as in (50), $q_2 = np/(n-p)$, and a constant $C > 0$ independent of ε. Then using Lions–Aubin compactness lemma [24] we obtain strong convergence of u^ε in $L^{r_1}((0,T)\times G)$, for $1 < r_1 < 6$. Strong convergence of u^ε, continuity of k and b, boundedness of $k(u^\varepsilon)$, and estimates for $b(u^\varepsilon)$ and $\partial_t b(u^\varepsilon)$ ensure the strong convergence of $\{k(u^\varepsilon)\}$ and $\{b(u^\varepsilon)\}$ and weak convergence of $\{\partial_t b(u^\varepsilon)\}$. A priori estimates (54), the strong convergence of u^ε, continuity and boundedness of $k(\xi)$ and $k(\xi)P_c(\xi)$ for $\xi \geq 0$, together with the compactness theorems for the two-scale convergence, see e.g. [2,34,35], imply the first four convergence results in (56). The last convergence in (56) follows from the compactness of the embedding $H^1(G) \subset H^\sigma(G)$ for $1/2 < \sigma < 1$ and the estimate

$$\varepsilon\|v\|^2_{L^2(\Gamma^\varepsilon)} \leq C\|v\|^2_{H^\sigma(G^\varepsilon)} \qquad \text{for} \quad \sigma > 1/2,$$

with a constant $C > 0$ independent of ε, see e.g. [25] for the proof. □

Theorem 4.2. *Under assumptions in Lemma 3.2, a subsequence of $\{u^\varepsilon\}$, denoted again by $\{u^\varepsilon\}$, where u^ε are solutions of problem (6), convergences to a function $u \in \kappa_D + L^2(0,T;H^1_0(G))$, with $\partial_t u \in L^p(0,T;W^{1,p}(G))$, $\sqrt{k(u)}\partial_t\nabla u \in L^2(G_T)$, $\partial_t b(u) \in L^2(0,T;L^r(G))$, where $1 < r < 3/2$ for $n = 3$ and $1 < r < 4/3$ for $n = 2$, and $p > 1$ is defined in (50), and $u(t) \in \mathcal{K}$ for $t \in [0,T]$, satisfying macroscopic variational inequality*

$$\begin{aligned}\langle \partial_t b(u), v - u\rangle_{G_T} + \big\langle A_{\text{hom}} k(u)[P_c(u)\nabla u + \partial_t \nabla u], \nabla(v-u)\big\rangle_{G_T} \\ - \langle F_{\text{hom}}(t,x,u), \nabla(v-u)\rangle_{G_T} + \langle f_{\text{hom}}(t,u), v-u\rangle_{G_T} \geq 0\end{aligned} \tag{57}$$

for $v - \kappa_D \in L^2(0,T;H^1_0(G))$, with $v(t) \in \mathcal{K}$, where $\mathcal{K}$ is defined in (7),

$$F_{\text{hom}}(t,x,u) = \fint_{Y^*} Q(t,x,y)\,dy\,H(u) + k(u)g,$$

$$f_{\text{hom}}(t,u) = \fint_{Y^*} f_0(t,y)\,dy\,f_1(u),$$

and matrix A_{hom} is defined in (63).

If $k(\xi) = \text{const}$, $P_c(\xi)$ is Lipschitz continuous for $\xi \geq 0$, and $u \in L^2(0,T;W^{1,p_2}(G))$ for $p_2 > n$ or if $k(u) \geq \delta > 0$ for $u \geq 0$, $\partial_t u \in L^2(0,T;W^{1,p_2}(G))$ and $u_0 \in W^{1,p_2}(G)$, then variational inequality (57) has a unique solution and the whole sequence of microscopic solutions $\{u^\varepsilon\}$ converges to the solution of (57).

Proof. To derive macroscopic inequality (57) we consider

$$v^\varepsilon(t,x) = u^\varepsilon(t,x) + \phi(t,x) + \sigma(\varepsilon)\varphi(t,x) + \varepsilon\psi(t,x,x/\varepsilon)$$

as a test function in (6), where $\psi \in C^1_0(G_T, C^1_{\text{per}}(Y))$, $\phi, \varphi \in H^1_0((0,T)\times G)$, with $\phi(t,x) + u(t,x) \geq 0$ and $\varphi(t,x) \geq 0$ in $(0,T)\times G$, and $\sigma(\varepsilon) \to 0$ as $\varepsilon \to 0$. Notice that since $u^\varepsilon \to u$ strongly two-scale on $(0,T)\times\Gamma^\varepsilon$ as $\varepsilon \to 0$, there exist such functions φ and $\sigma(\varepsilon) > 0$ that $v^\varepsilon(t,x) \geq 0$ on $(0,T)\times\Gamma^\varepsilon$ for sufficiently small $\varepsilon > 0$. We also have that $v^\varepsilon(t,x) = \kappa_D$ on $(0,T)\times\partial G$. Then using the convergence results in (55) and (56) and taking in (6) the limit as $\varepsilon \to 0$ yield

$$|Y^*|\int\limits_{G_T}\partial_t b(u)\phi\, dxdt+\int\limits_{G_T}\int\limits_{Y^*}A(y)k(u)\big[\partial_t(\nabla u+\nabla_y w)+P_c(u)(\nabla u+\nabla_y w)\big]\big(\nabla\phi+\nabla_y\psi\big)dydxdt$$
$$-\int\limits_{G_T}\int\limits_{Y^*}F(t,x,y,u)(\nabla\phi+\nabla_y\psi)dydxdt+\int\limits_{G_T}\int\limits_{\Gamma}f(t,y,u)\,\phi\, d\gamma_y dxdt\geq 0.$$

Assumptions on F^ε, i.e. $\nabla\cdot Q^\varepsilon(t,x)=0$ in G^ε_T and $Q^\varepsilon(t,x)\cdot\nu=0$ on Γ^ε_T, which imply that $\nabla_y\cdot Q(t,x,y)=0$ in $G_T\times Y^*$, $Q(t,x,y)\cdot\nu=0$ on $G_T\times\Gamma$, and Q is Y-periodic, and the fact that u is independent of y ensure

$$\int\limits_{G_T}\int\limits_{Y^*}F(t,x,y,u)\nabla_y\psi\, dydxdt=0.$$

By choosing $\phi=0$ and $\psi=0$, respectively, we obtain

$$\int\limits_{G_T}\int\limits_{Y^*}A(y)k(u)\left[\partial_t(\nabla u+\nabla_y w)+P_c(u)(\nabla u+\nabla_y w)\right]\nabla_y\psi\, dydxdt\geq 0 \tag{58}$$

and

$$\begin{aligned}&\int\limits_{G_T}\partial_t b(u)\,\phi\, dxdt+\int\limits_{G_T}\fint\limits_{Y^*}A(y)k(u)\left[\partial_t(\nabla u+\nabla_y w)+P_c(u)(\nabla u+\nabla_y w)\right]\nabla\phi\, dydxdt\\&-\int\limits_{G_T}\fint\limits_{Y^*}F(t,x,y,u)dy\,\nabla\phi\, dxdt+\int\limits_{G_T}\frac{1}{|Y^*|}\int\limits_{\Gamma}f(t,y,u)\, d\gamma_y\,\phi\, dxdt\geq 0.\end{aligned} \tag{59}$$

Considering $\pm\psi$ in (58) yields

$$\int\limits_{G_T}\int\limits_{Y^*}A(y)k(u)\left[\partial_t(\nabla u+\nabla_y w)+P_c(u)(\nabla u+\nabla_y w)\right]\nabla_y\psi\, dydxdt=0,$$

for all $\psi\in C^1_0(G_T;C^1_{\rm per}(Y))$. For a given $u\in L^2(0,T;H^1(G))$, the last equation is a pseudoparabolic equation for w with respect to microscopic variables y:

$$\begin{aligned}\nabla_y\cdot\big(A(y)k(u)[\partial_t(\nabla u+\nabla_y w)+P_c(u)(\nabla u+\nabla_y w)]\big)&=0\quad \text{in } Y^*_T,\\ A(y)k(u)[\partial_t(\nabla u+\nabla_y w)+P_c(u)(\nabla u+\nabla_y w)]\cdot\nu&=0\quad \text{on } \Gamma_T,\\ w\qquad & Y\text{-periodic},\end{aligned} \tag{60}$$

for $x\in G$, where $Y^*_T=(0,T)\times Y^*$. Using a regularisation of k and P_c, in a similar way as for (6), we can show the existence of a solution of problem (60), see also the existence proof for (66) in Lemma 4.3. To prove the existence of a solution of (60), with regularized k and P_c, we apply the Rothe method, use the Lax–Milgram theorem for the resulting linear elliptic problem, and consider $w\,[k(u+\delta)]^{-1}$ and $\partial_t w$ as test functions to derive the corresponding a priori estimates. We also use the fact that $\nabla u\in L^2((0,T)\times G)$, $k(u)\partial_t\nabla u\in L^2((0,T)\times G)$, and $k(u)P_c(u)$ is bounded. Considering the equation for the difference of two solutions w_1 and w_2 of (60), taking $\psi=(w_1-w_2)\,[k(u+\delta)]^{-1}$, with $\delta>0$, as a test function, using assumptions on A, and letting $\delta\to 0$, yield

$$\|\nabla_y(w_1-w_2)\|_{L^\infty(0,T;L^2(G\times Y^*))}=0.$$

Hence a solution of (60) is defined uniquely up to an additive function independent of y. The structure of (60) suggests that w is of the form

$$w(t,x,y) = \sum_{j=1}^{n} \partial_{x_j} u(t,x)\, \omega^j(y) + \overline{w}(t,x), \tag{61}$$

where ω^j, for $j = 1, \dots, n$, satisfy the following 'unit cell' problems

$$\begin{aligned} &\mathrm{div}_y(A(y)(\nabla_y \omega^j + e_j)) = 0 && \text{in } Y^*, && \int_{Y^*} \omega^j(y) dy = 0, \\ &A(y)(\nabla_y \omega^j + e_j) \cdot \nu = 0 && \text{on } \Gamma, && \omega^j \;\; Y\text{-periodic}, \end{aligned} \tag{62}$$

with $\{e_j\}_{j=1,\dots,n}$ being the standard basis of $\mathbb{R}^n$. Notice that the well-posedness of (62) follows directly from the assumptions on A in Assumption 2.1.

Substituting expression (61) for w into (59) determines the matrix $A_{\mathrm{hom}} = (A^{ij}_{\mathrm{hom}})_{i,j=1,\dots,n}$, with

$$A^{ij}_{\mathrm{hom}} = \fint_{Y^*} A(y) \left(\delta_{ij} + \frac{\partial \omega^j}{\partial y_i} \right) dy. \tag{63}$$

For any $\psi \in C_0(G_T, C_{\mathrm{per}}(\Gamma))$, with $\psi(t,x,y) \geq 0$ in $(0,T) \times G \times \Gamma$, using the non-negativity and two-scale convergence of u^ε on Γ^ε, we obtain

$$0 \leq \lim_{\varepsilon \to 0} \varepsilon \langle u^\varepsilon(t,x), \psi(t,x,x/\varepsilon) \rangle_{\Gamma^\varepsilon_T} = |Y|^{-1} \langle u(t,x), \psi(t,x,y) \rangle_{G_T \times \Gamma} = \langle u(t,x), \overline{\psi}(t,x) \rangle_{G_T},$$

where

$$\overline{\psi}(t,x) = \frac{1}{|Y|} \int_{\Gamma} \psi(t,x,y) d\gamma_y \geq 0 \quad \text{ in } \; (0,T) \times G.$$

Hence $u(t,x) \geq 0$ in $(0,T) \times G$. The weak convergence in $L^2(0,T; H^1(G))$ of the extension $\overline{u}^\varepsilon$ of u^ε, see Remark 3.3, ensures that $u(t,x) = \kappa_D$ on $(0,T) \times \partial G$. Thus we have that $u(t) \in \mathcal{K}$ for $t \in [0,T]$.

Considering $\phi = v - u$, for any $v \in \kappa_D + L^2(0,T; H^1_0(G))$ with $v(t,x) \geq 0$ in $(0,T) \times G$, as a test function in (59) yields the macroscopic variational inequality (57).

The proof of the uniqueness result follows the same steps as the proof of the uniqueness result for the regularised problem (12) in Lemma 3.1. □

Remark. Notice that if in pseudoparabolic and elliptic terms we have two different functions depending on microscopic variables y, i.e. $A(y)k(u)\nabla \partial_t u$ and $B(y)k(u)P_c(u)\nabla u$, with $0 < a_0 \leq A(y) \leq A_0 < \infty$ and $0 < b_0 \leq B(y) \leq B_0 < \infty$, we need to consider a modified form for function w, i.e.

$$w(t,x,y) = \sum_{j=1}^{n} \frac{\partial u(t,x)}{\partial x_j} \vartheta^j(y) + \sum_{j=1}^{n} \int_0^t \frac{\partial^2 u(s,x)}{\partial s \partial x_j} \chi^j(t-s,x,y) ds + \overline{w}(t,x), \tag{64}$$

instead of (61), where ϑ^j and χ^j satisfy the following 'unit cell' problems:

$$\begin{aligned} &\mathrm{div}_y(B(y)(\nabla_y \vartheta^j + e_j)) = 0 && \text{in } Y^*, && \int_{Y^*} \vartheta^j(y) dy = 0, \\ &B(y)(\nabla_y \vartheta^j + e_j) \cdot \nu = 0 && \text{on } \Gamma, && \vartheta^j \;\; Y\text{-periodic}, \end{aligned} \tag{65}$$

and

$$
\begin{aligned}
\operatorname{div}_y\big(k(u(t+s))\,\big[A(y)\nabla_y\partial_t\chi^j + B(y)P_c(u(t+s))\nabla_y\chi^j\big]\big) = 0 \qquad & \text{in } Y^*_{T-s}, \\
k(u(t+s))[A(y)\nabla_y\partial_t\chi^j + B(y)P_c(u(t+s))\nabla_y\chi^j]\cdot\nu = 0 \qquad & \text{on } \Gamma_{T-s}, \\
\chi^j \qquad & Y\text{-periodic}, \\
\chi^j(0,x,y) = \omega^j(y) - \vartheta^j(y) \qquad & \text{in } Y^*, \quad \int_{Y^*}\chi^j(t,x,y)dy = 0,
\end{aligned}
\tag{66}
$$

for $s \in [0,T)$, $x \in G$, and $j = 1,\dots,n$, with ω^j satisfying (62).

The well-posedness of (62) and (65) follows from the strict positivity and boundedness of functions A and B. To show the well-posedness of (66) we first consider the regularised problem

$$
\begin{aligned}
&\operatorname{div}_y\Big(k(u+\delta)\Big[A(y)\nabla_y\partial_t\chi^j_\delta + B(y)P_c(u+\delta)\nabla_y\chi^j_\delta\Big]\Big) = 0 && \text{in } Y^*_{T-s}, \\
&k(u+\delta)[A(y)\nabla_y\partial_t\chi^j_\delta + B(y)P_c(u+\delta)\nabla_y\chi^j_\delta]\cdot\nu = 0 && \text{on } \Gamma_{T-s}, \\
&\chi^j_\delta && Y\text{-periodic}, \\
&\chi^j_\delta(0,x,y) = \omega^j(y) - \vartheta^j(y) && \text{in } Y^*, \quad \int_{Y^*}\chi^j_\delta(t,x,y)dy = 0.
\end{aligned}
\tag{67}
$$

Lemma 4.3. *Under assumptions on A and B and on nonlinear functions k and P_c, see Assumption 2.1, there exists a unique solution $\chi^j \in L^\infty((0,T-s)\times G; H^1_{\rm per}(Y^*))$ of (66), with $\sqrt{k(u)}\,\partial_t\chi^j \in L^2((0,T-s)\times G; H^1_{\rm per}(Y^*))$, for each $j = 1,\dots,n$ and $s \in [0,T)$.*

Proof. First we consider the regularised problem (67). To show existence of a solution of (67) we consider the discretisation in time of (67) and obtain

$$
\begin{aligned}
&\operatorname{div}_y\Big(k(u(t_m+s)+\delta)\Big[A(y)\frac{1}{h}\nabla_y(\chi^j_{\delta,m} - \chi^j_{\delta,m-1}) + B(y)P_c(u(t_m+s)+\delta)\nabla_y\chi^j_{\delta,m}\Big]\Big) = 0 \quad \text{in } Y^*, \\
&k(u(t_m+s)+\delta)[A(y)\frac{1}{h}\nabla_y(\chi^j_{\delta,m} - \chi^j_{\delta,m-1}) + B(y)P_c(u(t_m+s)+\delta)\nabla_y\chi^j_{\delta,m}]\cdot\nu = 0 \quad \text{on } \Gamma, \\
&\int_{Y^*}\chi^j_{\delta,m}(x,y)dy = 0, \qquad \chi^j_{\delta,m} \quad Y\text{-periodic},
\end{aligned}
\tag{68}
$$

where $\chi^j_{\delta,0}(x,y) = \omega^j(y) - \vartheta^j(y)$ in Y^*, with $\chi^j_{\delta,0}(x,\cdot) \in H$ for $x \in G$, and $t_m = hm$ for $h = (T-s)/N$, $s \in [0,T)$, $m = 1,\dots,N$, and $N \in \mathbb{N}$. Here $H = \{v \in H^1_{\rm per}(Y^*) \,:\, \int_{Y^*} v(y)dy = 0\}$.

A weak solution of problem (68) is a function $\chi^j_{\delta,m} \in H$ satisfying

$$
\begin{aligned}
&\Big\langle k(u(t_m+s)+\delta)\Big[A(y)\frac{1}{h}\nabla_y\chi^j_{\delta,m} + B(y)P_c(u(t_m+s)+\delta)\nabla_y\chi^j_{\delta,m}\Big], \nabla_y\varphi\Big\rangle_{Y^*} \\
&= \frac{1}{h}\big\langle k(u(t_m+s)+\delta)A(y)\nabla_y\chi^j_{\delta,m-1}, \nabla_y\varphi\big\rangle_{Y^*}
\end{aligned}
\tag{69}
$$

for $x \in G$, $\varphi \in H^1_{\rm per}(Y^*)$, and a given $\chi^j_{\delta,m-1} \in H$. Assumptions on A, B, k, and P_c ensure that problem (68) is uniformly elliptic and the bilinear map $a : H \times H \to \mathbb{R}$ defined as

$$
a(\chi^j_{\delta,m},\varphi) = \int_{Y^*} k(u(t_m+s)+\delta)\Big[A(y)\frac{1}{h}\nabla_y\chi^j_{\delta,m} + B(y)P_c(u(t_m+s)+\delta)\nabla_y\chi^j_{\delta,m}\Big]\nabla_y\varphi\,dy
$$

is coercive and bounded, $F \in (H^1_{\rm per}(Y^*))'$ given by

$$\langle F, \varphi\rangle_{(H^1_{\rm per}(Y^*))', H^1_{\rm per}(Y^*)} = \frac{1}{h}\int\limits_{Y^*} k(u(t_m+s)+\delta)A(y)\nabla_y \chi^j_{\delta,m-1}\nabla_y\varphi dy$$

is bounded, and $\langle F, 1\rangle_{(H^1_{\rm per}(Y^*))', H^1_{\rm per}(Y^*)} = 0$. Thus applying the Lax–Milgram theorem yields existence of a unique solution $\chi^j_{\delta,m} \in H$ of (68) for $x \in G$ and $s \in [0, T)$.

Considering first $\chi^j_{\delta,m} - \chi^j_{\delta,m-1}$ and then $\chi^j_{\delta,m}$ as test functions in (69), summing over $m = 1, \dots, l$, for $1 < l \le N$, and using assumptions on functions A, B k, and P_c yield the following a priori estimates

$$\sum_{m=1}^{l} h\Big\|\frac{\nabla_y(\chi^j_{\delta,m} - \chi^j_{\delta,m-1})}{h}\Big\|^2_{L^2(Y^*)} + \sum_{m=1}^{l} h\Big\|\nabla_y\chi^j_{\delta,m}\Big\|^2_{L^2(Y^*)} \le C$$

for $x \in G$. Here we used discrete Gronwall and Hölder inequalities and the fact that

$$\sum_{m=1}^{l} h\|\nabla_y\chi^j_{\delta,m}\|^2_{L^2(Y^*)} \le C\sum_{m=1}^{l} h\sum_{i=1}^{m} h\Big\|\frac{\nabla_y(\chi^j_{\delta,i} - \chi^j_{\delta,i-1})}{h}\Big\|^2_{L^2(Y^*)} + \|\nabla_y\chi^j_{\delta,0}\|^2_{L^2(Y^*)}.$$

Then for piecewise linear and piecewise constant interpolations given by

$$\begin{aligned}
\hat\chi^j_{\delta,N}(t,x,y) &= \chi^j_{\delta,m-1}(x,y) + (t-t_{m-1})\frac{\chi^j_{\delta,m}(x,y) - \chi^j_{\delta,m-1}(x,y)}{h} \quad \text{for } \ t \in (t_{m-1}, t_m],\\
\bar\chi^j_{\delta,N}(t,x,y) &= \chi^j_{\delta,m}(x,y) \quad \text{for } \ t \in (t_{m-1}, t_m], \quad m = 1, \dots, N,
\end{aligned}$$

for $x \in G$ and $y \in Y^*$, using the zero-mean value of $\chi^j_{\delta,m}$ and the Poincaré inequality, we obtain

$$\|\partial_t\hat\chi^j_{\delta,N}\|_{L^2(Y^*_{T-s})} + \|\partial_t\nabla_y\hat\chi^j_{\delta,N}\|_{L^2(Y^*_{T-s})} + \|\bar\chi^j_{\delta,N}\|_{L^2(Y^*_{T-s})} + \|\nabla_y\bar\chi^j_{\delta,N}\|_{L^2(Y^*_{T-s})} \le C,$$

for $x \in G$ and a constant C independent of N and $x \in G$. Last estimates ensure that there exists a function $\chi^j_\delta \in H$, with $\partial_t\chi^j_\delta \in H$, such that

$$\begin{aligned}
\bar\chi^j_{\delta,N} &\rightharpoonup \chi^j_\delta \quad && \text{weakly}^* \text{ in } \ L^2(0, T-s; L^\infty(G; H^1(Y^*))),\\
\partial_t\hat\chi^j_{\delta,N} &\rightharpoonup \partial_t\chi^j_\delta \quad && \text{weakly}^* \text{ in } \ L^2(0, T-s; L^\infty(G; H^1(Y^*))),
\end{aligned}$$

as $N \to \infty$. Using continuity of u with respect to time variable, integrating (69) with respect to t and x, and taking the limit as $N \to \infty$ yield that χ^j_δ is a weak solution of the regularised 'unit cell' problem (67). The linearity of the problem and properties of A, B, k, and P_c ensure the uniqueness of a solution of (67).

Now we shall derive a priori estimates for χ^j_δ, uniformly in δ. Considering $\chi^j_\delta/k(u+\delta)$ and $\partial_t\chi^j_\delta$ as test functions in the weak formulation of (67) we obtain

$$\|\nabla_y\chi^j_\delta\|_{L^\infty(0,T-s;L^2(Y^*))} + \|\sqrt{P_c(u+\delta)}\nabla_y\chi^j_\delta\|_{L^2((0,T-s)\times Y^*)} + \|\sqrt{k(u+\delta)}\nabla_y\partial_t\chi^j_\delta\|_{L^2((0,T-s)\times Y^*)} \le C, \tag{70}$$

for $x \in G$ and a constant $C > 0$ independent of δ and $x \in G$. Assumptions on k and P_c in Assumption 2.1, together with the additional assumption that k is continuously differentiable for $z \ge 0$, combined with the regularity $\partial_t u \in L^p(0, T; L^{q_1}(G))$, where $q_1 = pn/(n-p)$ and $1 < p < 2$, imply

$$\langle k(u+\delta)\nabla_y\partial_t\chi^j_\delta, \nabla_y\psi\rangle_{Y^*_{T-s}} = -\langle k'(u+\delta)\partial_t u\nabla_y\chi^j_\delta, \nabla_y\psi\rangle_{Y^*_{T-s}} - \langle k(u+\delta)\nabla_y\chi^j_\delta, \nabla_y\partial_t\psi\rangle_{Y^*_{T-s}},$$

for $\psi \in C_0^1(G_{T-s} \times Y^*)$ and $x \in G$. Taking in the last equality the limit as $\delta \to 0$ and considering estimates in (70) yield

$$\sqrt{k(u+\delta)}\nabla_y \partial_t \chi_\delta^j \rightharpoonup \sqrt{k(u)}\nabla_y \partial_t \chi^j \qquad \text{weakly in } \; L^2(G_{T-s} \times Y^*).$$

Then, using the continuity of k and P_c, regularity of $\partial_t u$ and the estimate for $\nabla_y \chi_\delta^j$ in $L^\infty(G_{T-s}; L^2(Y^*))$, we can pass to the limit as $\delta \to 0$ in the weak formulation of (67) and obtain that the limit function $\chi^j \in L^\infty(G_{T-s}; H^1_{\text{per}}(Y^*))$, with $\sqrt{k(u)}\partial_t \chi^j \in L^2(G_{T-s}; H^1_{\text{per}}(Y^*))$, is a solution of (66).

To prove the uniqueness result for (66) we assume that there are two solutions χ_1^j and χ_2^j of (66) and consider $[k(u+\delta)]^{-1}(\chi_1^j - \chi_2^j)$ as a test function in the weak formulation of the equations for the difference $(\chi_1^j - \chi_2^j)$ to obtain

$$\int_{G_{\tau-s}} \int_{Y^*} \frac{k(u(t+s))}{k(u(t+s)+\delta)} \Big(A(y)\partial_t \nabla_y(\chi_1^j - \chi_2^j)\nabla_y(\chi_1^j - \chi_2^j) + B(y)P_c(u(t+s))|\nabla_y(\chi_1^j - \chi_2^j)|^2 \Big) dy dx dt = 0,$$

for $\tau \in (s, T]$. Using the properties of functions k, P_c, A, B, and the nonnegativity of u, considering the nonnegativity of the second term, integrating by parts in the first term, and taking $\delta \to 0$ imply

$$\sup_{(0,T-s)} \|\nabla_y(\chi_1^j - \chi_2^j)\|_{L^2(G\times Y^*)} \leq 0.$$

Then Poincaré inequality and the fact that the mean value of χ_l^j, for $l = 1, 2$, is zero ensure $\chi_1^j = \chi_2^j$ a.e. in $G \times Y^*_{T-s}$, for $s \in [0, T)$ and $j = 1, \dots, n$. □

Considering the expression (64) for w in (59) and choosing $\phi = v - u$ yield the corresponding macroscopic variational inequality

$$\begin{aligned} &\langle \partial_t b(u), v-u\rangle_{G_T} + \big\langle k(u)[A_{\text{hom}}\partial_t \nabla u + B_{\text{hom}} P_c(u)\nabla u], \nabla(v-u)\big\rangle_{G_T} \\ &\quad + \Big\langle \int_0^t K_{\text{hom}}(t-s,x)\partial_s \nabla u \, ds, \nabla(v-u)\Big\rangle_{G_T} - \langle F_{\text{hom}}(t,x,u), \nabla(v-u)\rangle_{G_T} + \langle f_{\text{hom}}(t,u), v-u\rangle_{G_T} \geq 0, \end{aligned}$$

for $v \in L^2(0,T;\mathcal{K})$, where A_{hom}, F_{hom} and f_{hom} are defined in Theorem 4.2, and matrices $B_{\text{hom}} = (B^{ij}_{\text{hom}})$ and $K_{\text{hom}}(t,x) = (K^{ij}_{\text{hom}}(t,x))$ are determined by

$$\begin{aligned} B^{ij}_{\text{hom}} &= \fint_{Y^*} B(y)\left(\delta_{ij} + \frac{\partial \vartheta^j}{\partial y_i}\right) dy, \\ K^{ij}_{\text{hom}}(t,x) &= \fint_{Y^*} k(u(t+s,x))[A(y)\partial_t \partial_{y_i}\chi^j + B(y)P_c(u(t+s,x))\partial_{y_i}\chi^j] dy. \end{aligned}$$

Appendix A

Definition A.1. [2,35] A sequence $\{u^\varepsilon\} \subset L^p(G)$ converges two-scale to u, with $u \in L^p(G \times Y)$, iff for any $\phi \in L^q(G, C_{\text{per}}(Y))$ we have

$$\lim_{\varepsilon\to 0} \int_G u^\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right) dx = \int_G \fint_Y u(x,y)\phi(x,y) dx dy,$$

where $1/p + 1/q = 1$.

Definition A.2. [2,34] A sequence $\{u^\varepsilon\} \subset L^2(\Gamma^\varepsilon)$ converges two-scale to u, with $u \in L^2(G \times \Gamma)$, iff for $\psi \in C_0(G, L^2_{\text{per}}(\Gamma))$ there holds

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\Gamma^\varepsilon} u^\varepsilon(x) \psi\Big(x, \frac{x}{\varepsilon}\Big)\, d\gamma_x = \frac{1}{|Y|} \int_G \int_\Gamma u(x,y)\psi(x,y)\, dx d\gamma_y.$$

Theorem A.3 *(Compactness). [2,35] Let $\{u^\varepsilon\}$ be a bounded sequence in $H^1(G)$, which converges weakly to $u \in H^1(G)$. Then there exists $u_1 \in L^2(G, H^1_{\text{per}}(Y))$ such that, up to a subsequence, u^ε two-scale converges to u and ∇u^ε two-scale converges to $\nabla u + \nabla_y u_1$.*

Let $\{u_\varepsilon\}$ and $\{\varepsilon \nabla u^\varepsilon\}$ be bounded sequences in $L^2(G)$. Then there exists $u_0 \in L^2(G, H^1_{\text{per}}(Y))$ such that, up to a subsequence, u^ε and $\varepsilon \nabla u^\varepsilon$ two-scale converge to u_0 and $\nabla_y u_0$, respectively.

Let $\{\sqrt{\varepsilon} u_\varepsilon\}$ be a bounded sequence in $L^2(\Gamma^\varepsilon)$. Then there exists $u_0 \in L^2(G \times \Gamma)$ such that, up to a subsequence, u^ε two-scale converge to u_0.

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