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Nonlinear pseudoparabolic equations as singular limit of reaction–diffusion equations

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In this article, a solution of a nonlinear pseudoparabolic equation is constructed as a singular limit of a sequence of solutions of quasilinear hyperbolic equations. If a system with cross diffusion, modelling the reaction and diffusion of two biological, chemical, or physical substances, is reduced then such an hyperbolic equation is obtained. For regular solutions even uniqueness can be shown, although the needed regularity can only be proved in two dimensions.

Keywords: Pseudoparabolic equation; Reaction-diffusion equations; Galerkin's method

AMS Subject Classifications: 35K50; 35K57; 35K55; 35K60; 35K70

1. Introduction

In this work, we consider existence and uniqueness of solution of the nonlinear pseudoparabolic equation

$$b(t,x,u_t) - \nabla\cdot a(t,x,\nabla u_t) - \nabla\cdot(d(t,x)h(u)\nabla u) = f(t,x,u).$$

Pseudoparabolic equations are used to model fluid flow in fissured porous media [1], two-phase flow in porous media with dynamical capillary pressure [7,10], and heat conduction in two-temperature systems [6].

We consider a reaction system with diffusion of one of the substances:

$$\begin{cases} \varepsilon\partial_t v = \nabla\cdot a(t,x,\nabla v) + \nabla\cdot(d(t,x)\nabla w) + \tilde{f}(t,x,w) - b(t,x,v), \\ \partial_t w = h(w)v, \end{cases}$$

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where the function h satisfies $0 < h_0 \leq h(w) \leq h_1$. After a change to a new variable $u = H(w)$, where $H(w) = \int_0^w (1/h(s))\mathrm{d}s$, we obtain $\partial_t u = v$. Hereby the system is reduced to the single equation

$$\varepsilon u_{tt} = \nabla \cdot a(t, x, \nabla u_t) + \nabla \cdot (d(t, x)h(u)\nabla u) + \tilde{f}(t, x, H^{-1}(u)) - b(t, x, u_t).$$

The pseudoparabolic equation describes the reaction and diffusion of the faster evolving substance.

This article is organised in the following way: First, the existence of a solution of a quasilinear hyperbolic equation is shown using Galerkin's approximation. To obtain *a priori* estimates the monotonicity and the growth assumptions on the nonlinear functions are used. Second, the convergence of the sequence of solutions to a solution of the pseudoparabolic equation is shown. The regularity of this solution is proved in two dimensions. The uniqueness follows from the strong monotonicity of the nonlinear functions.

The question of regularity of solutions of linear and quasilinear pseudoparabolic equations is considered in [2–5,12], where it is shown that regularity or singularity of the initial data is preserved.

2. Existence

In this section, we show at first the existence of a weak solution of the equation

$$\varepsilon u_{tt} + b(t, x, u_t) - \nabla \cdot a(t, x, \nabla u_t) - \nabla \cdot (d(t, x)h(u)\nabla u) = f(t, x, u) \tag{1}$$

in Q_T accompanied by the initial conditions $u(0) = u_0$ and $u_t(0) = 0$. Here, $\Omega \subset \mathbb{R}^N$ is a bounded domain and $Q_T = (0, T) \times \Omega$. In a second step, we prove the convergence of a (sub)sequence of solutions $\{u^\varepsilon\}$ as $\varepsilon \to 0$ to a solution of the pseudoparabolic equation

$$b(t, x, u_t) - \nabla \cdot a(t, x, \nabla u_t) - \nabla \cdot (d(t, x)h(u)\nabla u) = f(t, x, u) \tag{2}$$

with initial condition $u(0) = u_0$. Both initial value problems are completed by posing spatial boundary conditions. Here, we choose a closed subspace V_0, $H_0^1(\Omega) \subset V_0 \subset H^1(\Omega)$, densely and continuously embedded in $L^2(\Omega)$.

The existence of a solution will be ensured by the following assumption.

Assumption 2.1

A1 The function $b : (0, T) \times \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable in t and x, continuous in ξ, elliptic in ξ, i.e. $b_0 > 0$, $b(t, x, \xi)\xi \geq b_0|\xi|^p$ for $\xi \in \mathbb{R}$ and a.a. $(t, x) \in Q_T$, and strongly monotone, i.e. $b_1 > 0$, $(b(t, x, \xi_1) - b(t, x, \xi_2))(\xi_1 - \xi_2) \geq b_1|\xi_1 - \xi_2|^p$, for $\xi_1, \xi_2 \in \mathbb{R}$ and a.a. $(t, x) \in Q_T$, $p \geq 2$, and satisfies a growth assumption, i.e. $b^0 < \infty$, $|b(t, x, \xi)| \leq b^0(1 + |\xi|^{p-1})$ for $\xi \in \mathbb{R}$ and a.a. $(t, x) \in Q_T$.

A2 The function $a : (0, T) \times \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is measurable in t and x, continuous in η, elliptic in η, i.e. $a_0 > 0$, $a(t, x, \eta)\eta \geq a_0|\eta|^2$ for $\eta \in \mathbb{R}^N$ and $(t, x) \in Q_T$, strongly monotone, i.e. $a_1 > 0$, $(a(t, x, \eta_1) - a(t, x, \eta_2))(\eta_1 - \eta_2) \geq a_1|\eta_1 - \eta_2|^2$ for $\eta_1, \eta_2 \in \mathbb{R}^N$ and a.a. $(t, x) \in Q_T$, and satisfies a growth assumption, i.e. $a^0 < \infty$, $|a(t, x, \eta)| \leq a^0(1 + |\eta|)$ for $\eta \in \mathbb{R}^N$ and a.a. $(t, x) \in Q_T$.

A3 The matrix field $d \in L^{\infty}(Q_T)^{N \times N}$, i.e. $|d(t,x)| \le d_1$ for a.a. $(t,x) \in Q_T$, $h : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $0 < h_0 \le h(\xi) \le h_1 < \infty$ for $\xi \in \mathbb{R}$.

A4 The function $f : (0,T) \times \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable in t and x, continuous in ξ, and sublinear, i.e. $f_1 < \infty$, $|f(t,x,\xi)| \le f_1(1 + |\xi|)$ for $\xi \in \mathbb{R}$ and a.a. $(t,x) \in Q_T$.

A5 The initial condition u_0 is in V_0.

2.1. *Existence of a weak solution of hyperbolic equation*

Definition 2.2 A function $u : Q_T \to \mathbb{R}$ is called *a weak solution* of (1) if

(i) $u_t \in C([0,T]; L^2(\Omega))$, $u_t \in L^p(Q_T) \cap L^2(0,T;V_0)$, $u \in C([0,T];V_0)$,

(ii) u satisfies the initial condition, i.e. $u(t) \to u_0$ in V_0, $u_t(t) \to 0$ in $L^2(\Omega)$ for $t \to 0$, and

$$\int_{Q_T} \Big[-\varepsilon u_t v_t + b(t,x,u_t)v + a(t,x,\nabla u_t)\nabla v + d(t,x)h(u)\nabla u \, \nabla v \Big] \mathrm{d}x \, \mathrm{d}t$$

$$+ \varepsilon \int_{\Omega} u_t(T)v(T)\mathrm{d}x = \int_{Q_T} f(t,x,u)v \, \mathrm{d}x \, \mathrm{d}t \tag{3}$$

for all $v \in L^p(Q_T) \cap L^2(0,T;V_0)$, s.t. $v_t \in L^2(Q_T)$, $v \in C([0,T];L^2(\Omega))$.

THEOREM 2.3 *There exists a weak solution u^{ε} of the problem* (1).

The existence of a solution of (1) is proved using Galerkin's method: let $\{\phi^k\}_{k=1}^{\infty} \subset V_0 \cap L^p(\Omega)$ be a basis of the spaces V_0 and $L^p(\Omega)$. We consider the sequence of the functions $\{u^m\}$ of the form $u^m(t,x) = \sum_{k=1}^{m} z_k^m(t)\phi^{(k)}(x)$, $m = 1,2,\ldots$, such that u^m is a solution of the Cauchy problem

$$\varepsilon \int_{\Omega} u_{tt}^m \phi^{(k)} \, \mathrm{d}x + \int_{\Omega} b(t,x,u_t^m)\phi^{(k)} \, \mathrm{d}x + \int_{\Omega} a(t,x,\nabla u_t^m)\nabla\phi^{(k)} \, \mathrm{d}x$$

$$+ \int_{\Omega} d(t,x)h(u^m)\nabla u^m \, \nabla\phi^{(k)} \, \mathrm{d}x = \int_{\Omega} f(t,x,u^m)\phi^{(k)} \, \mathrm{d}x, \tag{4}$$

$$u^m(0,x) = u_0^m(x), \quad u_t^m(0,x) = 0, \tag{5}$$

where $\{u_0^m\}$ is an approximation of u_0 in the space V_0. Due to the generalisation of Peano's theorem for Carathéodory functions [8], there exists a local solution of this problem in $[0, t_{0m}]$. The following lemma allows an extension of the solutions to the whole interval $[0,T]$.

LEMMA 2.4 *The estimates*

$$\varepsilon \|u_t^m(t)\|_{L^2(\Omega)} \le C, \quad t \in [0, t_{0m}], \ \|u_t^m\|_{L^p(Q_{t_{0m}})} \le C, \quad \|\nabla u_t^m\|_{L^2(Q_{t_{0m}})} \le C \tag{6}$$

hold uniformly with respect to m and ε.

Proof We multiply the equation (4) by z^m_{kt}, sum up over k from 1 to m, and integrate over $[0,\tau]$, where $0<\tau\leq t_{0m}$

$$\int_{Q_\tau}\Big[\varepsilon u^m_{tt}u^m_t + b(t,x,u^m_t)u^m_t + a(t,x,\nabla u^m_t)\nabla u^m_t + d(t,x)h(u^m)\nabla u^m\,\nabla u^m_t\Big]\mathrm{d}x\,\mathrm{d}t$$
$$=\int_{Q_\tau} f(t,x,u^m)u^m_t\,\mathrm{d}x\,\mathrm{d}t. \tag{7}$$

Due to $\partial_t u(0)=0$ and Assumption 2.1 the first three terms in (7) are bounded from below by

$$\frac{\varepsilon}{2}\int_\Omega |u^m_t(\tau)|^2\mathrm{d}x + \int_{Q_\tau}\big(b_0|u^m_t|^p + a_0|\nabla u^m_t|^2\big)\mathrm{d}x\,\mathrm{d}t.$$

For the fourth term, we have

$$\int_{Q_\tau} d(t,x)h(u^m)\nabla u^m\,\nabla u^m_t\,\mathrm{d}x\,\mathrm{d}t \leq \frac{d_1^2h_1^2}{2\delta}\int_{Q_\tau}|\nabla u^m|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\delta}{2}\int_{Q_\tau}|\nabla u^m_t|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq c_1\int_0^\tau\int_{Q_t}|\nabla u^m_t|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\delta}{2}\int_{Q_\tau}|\nabla u^m_t|^2\,\mathrm{d}x\,\mathrm{d}t + c_2.$$

Due to the assumption on f, we have

$$\int_{Q_\tau} f(t,x,u^m)u^m_t\,\mathrm{d}x\,\mathrm{d}t \leq \frac{\delta}{p}\int_{Q_\tau}|u^m_t|^p\mathrm{d}x\,\mathrm{d}t + c_3\int_0^\tau\int_{Q_t}|u^m_t|^p\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t + c_4.$$

Applying Gronwall's lemma to (7) implies the assertion. ■

Remark Since the constant C is independent of t_{0m}, the solution u^m may be assumed to be the maximal solution, i.e. the one that exists for all $t\in[0,T]$. Furthermore, since the estimates of the last lemma are independent of m, they are satisfied by every u^m_t for all $t\in[0,T]$.

From the estimates for u^m_t we obtain the estimate for u^m. Due to (6), $u_0\in V_0$, and $p\geq 2$ we have

$$\int_\Omega\Big(|u^m(\tau)|^2 + |\nabla u^m(\tau)|^2\Big)\mathrm{d}x$$
$$\leq \int_{Q_\tau}\Big(|u^m_t|^2 + |\nabla u^m_t|^2 + |u^m|^2 + |\nabla u^m|^2\Big)\mathrm{d}x\,\mathrm{d}t$$
$$+\int_\Omega\Big(|u^m_0|^2 + |\nabla u^m_0|^2\Big)\mathrm{d}x \leq c_2 + \int_{Q_\tau}\Big(|u^m|^2 + |\nabla u^m|^2\Big)\mathrm{d}x\,\mathrm{d}t.$$

Then Gronwall's lemma implies

$$\|u^m(\tau)\|_{V_0} \leq C, \quad \tau\in[0,T]. \tag{8}$$

Proof (of Theorem 2.3) The growth assumptions on a and b imply

$$\left|\int_{Q_T} b(t,x,u_t^m)v\,\mathrm{d}x\,\mathrm{d}t\right| \le C\Big(1+\|u_t^m\|_{L^p(Q_T)}^{p/q}\Big)\|v\|_{L^p(Q_T)},$$

$$\left|\int_{Q_T} a(t,x,\nabla u_t^m)\nabla v\,\mathrm{d}x\,\mathrm{d}t\right| \le C\big(1+\|u_t^m\|_{L^2(0,T;V_0)}\big)\|v\|_{L^2(0,T;V_0)}$$

for all $v\in L^p(Q_T)\cap L^2(0,T;V_0)$. Hence, the estimates, (6) and (8), imply the existence of a subsequence of $\{u^m\}$, again denoted by $\{u^m\}$, such that

$$\begin{aligned}
u^m &\to u^\varepsilon \quad \text{weakly-}*\ \text{in } L^\infty(0,T;V_0),\\
u_t^m &\to u_t^\varepsilon \quad \text{weakly in } L^p(Q_T)\cap L^2(0,T;V_0),\\
u_t^m &\to u_t^\varepsilon \quad \text{weakly-}*\ \text{in } L^\infty(0,T;L^2(\Omega)),\\
b(t,x,u_t^m) &\to \beta^\varepsilon \quad \text{weakly in } L^q(Q_T),\\
a(t,x,\nabla u_t^m) &\to \eta^\varepsilon \quad \text{weakly in } L^2(Q_T)^N,
\end{aligned}$$

as $m\to\infty$. Using Aubin–Lions's Compactness Lemma [11], yields $u^m\to u^\varepsilon$ strongly in $L^2(Q_T)$; therefore $u^m\to u^\varepsilon$ a.e. in Q_T. The continuity of h and f implies $h(u^m)\to h(u^\varepsilon)$ and $f(t,x,u^m)\to f(t,x,u^\varepsilon)$ a.e. in Q_T. From the assumptions it follows that $h(u^m)$, $h(u^\varepsilon)\in L^\infty(Q_T)$ and $f(t,x,u^m)$, $f(t,x,u^\varepsilon)\in L^2(Q_T)$. Then by Egorov's Theorem, $h(u^m)\to h(u^\varepsilon)$ uniformly a.e. in Q_T and by the Dominated Convergence Theorem $f(t,x,u^m)\to f(t,x,u^\varepsilon)$ strongly in $L^2(Q_T)$. The sum of all but the first term of (4) defines a functional $w\in L^q(Q_T)+L^2(0,T;V_0^*)$

$$\varepsilon\langle w,\tilde v\rangle = \int_\Omega f(t,x,u^\varepsilon)\tilde v\,\mathrm{d}x - \int_\Omega\Big(\beta^\varepsilon\tilde v+\eta^\varepsilon\,\nabla\tilde v + d(t,x)h(u^\varepsilon)\nabla u^\varepsilon\,\nabla\tilde v\Big)\mathrm{d}x$$

in $L^q(0,T)+L^2(0,T)$ for $\tilde v\in L^p(\Omega)\cap V_0$. Since $u_t^m\to u_t^\varepsilon$ weakly in $L^p(Q_T)$, we obtain $\langle u_{tt}^m,\tilde v\rangle = (\mathrm{d}/\mathrm{d}t)\langle u_t^m,\tilde v\rangle \to \langle u_{tt}^\varepsilon,\tilde v\rangle$ in $\mathcal{D}'(0,T)$ as $m\to\infty$ for $\tilde v\in L^p(\Omega)$. Hence, $w=u_{tt}^\varepsilon$ in $\mathcal{D}'(0,T,L^q(\Omega)+V_0^*)$. Since $w\in L^q(Q_T)+L^2(0,T;V_0^*)$ we may assume $u_{tt}^\varepsilon\in L^q(Q_T)+L^2(0,T;V_0^*)$. Thus, [9, Theorem IV.1.17], it may be assumed that $u_t^\varepsilon\in C([0,T];L^2(\Omega))$ and the integration by parts formula

$$\int_{t_1}^{t_2}\langle u_{tt}^\varepsilon,u_t^\varepsilon\rangle\mathrm{d}t = \frac12\int_\Omega |u_t^\varepsilon(t_2)|^2\,\mathrm{d}x - \frac12\int_\Omega |u_t^\varepsilon(t_1)|^2\,\mathrm{d}x$$

holds for all $0\le t_1<t_2\le T$. Now we will show that u^ε satisfies the initial condition. Since all u_t^m and u_t^ε are in $C([0,T];L^2(\Omega))$, and $u_t^m\to u_t^\varepsilon$ weakly-$*$ in $L^\infty(0,T,L^2(\Omega))$, we obtain

$$\int_\Omega u_t^m(0)\tilde v\,\mathrm{d}x \to \int_\Omega u_t^\varepsilon(0)\tilde v\,\mathrm{d}x \quad\text{and}\quad \int_\Omega u_t^m(T)\tilde v\,\mathrm{d}x \to \int_\Omega u_t^\varepsilon(T)\tilde v\,\mathrm{d}x,$$

as $m \to \infty$ for $\tilde{v} \in L^p(\Omega)$. Then we have $u_t^\varepsilon(0) = 0$ in $L^2(\Omega)$ because of $u_t^m(0) = 0$ in $L^2(\Omega)$. Since $u^\varepsilon \in L^\infty(0, T; V_0)$ and $u_t^\varepsilon \in L^2(0, T; V_0)$ it may be assumed that $u^\varepsilon \in C([0, T]; V_0)$ [11], and $u^m(0) \to u^\varepsilon(0)$ strongly in $L^2(\Omega)$ as $m \to \infty$. Thus, $u^\varepsilon(0) = u_0$.

Integrating in the equation (4) the first term by part, passing to the limit as $m \to \infty$ and using the fact that the set of all functions of the form $\sum_{l<\infty} d_l \phi^l$, where $d_l \in C^1([0, T])$, is dense in $L^p(Q_T)$, $L^2(0, T; V_0)$, $C([0, T]; L^2(\Omega))$, and $H^1(0, T; L^2(\Omega))$ yields

$$-\varepsilon \int_{Q_T} u_t^\varepsilon v_t \, dx\, dt + \int_{Q_T} (\beta^\varepsilon v + \eta^\varepsilon \nabla v) dx\, dt + \int_{Q_T} d(t, x) h(u^\varepsilon) \nabla u^\varepsilon \, \nabla v \, dx\, dt$$
$$+ \varepsilon \int_\Omega u_t^\varepsilon(T) v(T) dx = \int_{Q_T} f(t, x, u^\varepsilon) v \, dx\, dt$$

for all $v \in L^p(Q_T) \cap L^2(0, T; V_0)$, s.t. $v_t \in L^2(Q_T)$ and $v \in C([0, T]; L^2(\Omega))$.

To complete the proof, we have to show $\beta^\varepsilon = b(t, x, u_t^\varepsilon)$ and $\eta^\varepsilon = a(t, x, \nabla u_t^\varepsilon)$. For this we show the strong convergence of $\{u_t^m\}$ to u_t^ε in $L^p(Q_T) \cap L^2(0, T; V_0)$. We choose $u_t^m - u_t^\varepsilon$ as a test function in (4), integrate over $[0, \tau]$ and obtain

$$\varepsilon \int_0^\tau \langle u_{tt}^m, u_t^m - u_t^\varepsilon \rangle dt + \int_{Q_\tau} (b(t, x, u_t^m) - b(t, x, u_t^\varepsilon))(u_t^m - u_t^\varepsilon) dx\, dt$$
$$+ \int_{Q_\tau} (a(t, x, \nabla u_t^m) - a(t, x, \nabla u_t^\varepsilon)) \nabla(u_t^m - u_t^\varepsilon) dx\, dt = \int_{Q_\tau} b(t, x, u_t^\varepsilon)(u_t^\varepsilon - u_t^m) dx\, dt$$
$$+ \int_{Q_\tau} \Big(a(t, x, \nabla u_t^\varepsilon) \nabla(u_t^\varepsilon - u_t^m) + d(t, x)\, h(u^m) \nabla(u^m - u^\varepsilon) \nabla(u_t^\varepsilon - u_t^m) \Big) dx\, dt$$
$$+ \int_{Q_\tau} \Big(d(t, x) h(u^m) \nabla u^\varepsilon \nabla(u_t^\varepsilon - u_t^m) - f(t, x, u^m)(u_t^m - u_t^\varepsilon) \Big) dx\, dt.$$

By Fatou's lemma and weak convergence of u_{tt}^m in $L^q(Q_T) + L^2(0, T; V_0^*)$, we obtain for the first integral

$$\liminf_{m \to \infty} \int_0^\tau \langle u_{tt}^m, u_t^m - u_t^\varepsilon \rangle dt \geq \frac{1}{2} \liminf_{m \to \infty} \int_\Omega |u_t^m(\tau, x)|^2 \, dx - \frac{1}{2} \int_\Omega |u_t^\varepsilon(\tau, x)|^2 \, dx \geq 0.$$

Due to the convergences of $\{u_t^m\}$, $\{h(u^m)\}$, and $\{f(t, x, u^m)\}$, the first, second, fourth, and fifth terms on the right-hand side converge to zero as $m \to \infty$. The third term on the right hand side can be estimated by

$$\int_{Q_\tau} d(t, x) h(u^m) \, \nabla(u^m - u^\varepsilon) \nabla(u_t^m - u_t^\varepsilon) dx\, dt$$
$$\leq \frac{d_1^2 h_1^2}{2\delta} \int_{Q_\tau} |\nabla(u^m - u^\varepsilon)|^2 \, dx\, dt + \frac{\delta}{2} \int_{Q_\tau} |\nabla(u_t^m - u_t^\varepsilon)|^2 \, dx\, dt$$
$$\leq c_1 \int_{Q_\tau} |\nabla(u_0^m - u_0)|^2 \, dx\, dt + c_2 \int_0^\tau \int_{Q_s} |\nabla(u_t^m - u_t^\varepsilon)|^2 \, dx\, dt\, ds$$
$$+ \frac{\delta}{2} \int_{Q_\tau} |\nabla(u_t^m - u_t^\varepsilon)|^2 \, dx\, dt.$$

The monotonicity of b and a, and the convergence of $\{u_0^m\}$, $\{u^m\}$, and $\{u_t^m\}$ imply

$$b_1 \int_{Q_\tau} |u_t^m - u_t^\varepsilon|^p \,\mathrm{d}x\,\mathrm{d}t + \left(a_1 - \frac{\delta}{2}\right) \int_{Q_\tau} |\nabla(u_t^m - u_t^\varepsilon)|^2 \,\mathrm{d}x\,\mathrm{d}t \\ \leq \sigma\left(\frac{1}{m}\right) + c_3 \int_0^\tau \int_{Q_s} |\nabla(u_t^m - u_t^\varepsilon)|^2 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s.$$

Using Gronwall's lemma in the last inequality yields

$$\|u_t^m - u_t^\varepsilon\|_{L^p(Q_T)} + \|\nabla u_t^m - \nabla u_t^\varepsilon\|_{L^2(Q_T)} \leq C\sigma\left(\frac{1}{m}\right).$$

Thus, $u_t^m \to u_t^\varepsilon$ strongly in $L^p(Q_T) \cap L^2(0, T; V_0)$ as $m \to \infty$. The strong convergence of $\{u_t^m\}$ and the weak convergence of $\{b(t, x, u_t^m)\}$ and $\{a(t, x, \nabla u_t^m)\}$ imply $\beta^\varepsilon = b(t, x, u_t^\varepsilon)$ and $\eta^\varepsilon = a(t, x, \nabla u_t^\varepsilon)$, and the theorem is proved. ■

2.2. *Existence of a solution of a pseudoparabolic equation*

Now we show that the subsequence of solutions $\{u^\varepsilon\}$ converges as $\varepsilon \to 0$ to a solution of the initial boundary value problem for the nonlinear pseudoparabolic equation (2).

Definition 2.5 A function $u : Q_T \to \mathbb{R}$ is called *a weak solution* of (2) if

(i) $u \in C([0, T]; V_0)$, $u_t \in L^p(Q_T) \cap L^2(0, T; V_0)$,
(ii) u satisfies the initial condition, i.e., $u(t) \to u_0$ in V_0 for $t \to 0$, and

$$\int_{Q_T} [b(t, x, u_t)v + a(t, x, \nabla u_t)\nabla v + d(t, x)h(u)\nabla u\,\nabla v]\mathrm{d}x\,\mathrm{d}t \\ = \int_{Q_T} f(t, x, u)v\,\mathrm{d}x\,\mathrm{d}t \quad \text{for all } v \in L^p(Q_T) \cap L^2(0, T; V_0). \tag{9}$$

THEOREM 2.6 *There exists a weak solution of the problem* (2).

Proof We rewrite the equation (3) for $v = u_t^\varepsilon$ and obtain

$$-\varepsilon \int_{Q_T} u_t^\varepsilon u_t^\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} \Big[b(t, x, u_t^\varepsilon)\, u_t^\varepsilon + a(t, x, \nabla u_t^\varepsilon)\nabla u_t^\varepsilon\Big]\mathrm{d}x\,\mathrm{d}t \\ + \int_{Q_T} d(t, x)h(u^\varepsilon)\nabla u^\varepsilon \nabla u_t^\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \varepsilon \int_\Omega u_t^\varepsilon(T)\, u_t^\varepsilon(T)\mathrm{d}x = \int_{Q_T} f(t, x, u^\varepsilon)u_t^\varepsilon \,\mathrm{d}x\,\mathrm{d}t. \tag{10}$$

We estimate all integrals in (10) analogously to (7) and have

$$\varepsilon^{1/2}\|u_t^\varepsilon(t)\|_{L^2(\Omega)} \leq C,\ t \in [0, T], \quad \|u_t^\varepsilon\|_{L^p(Q_T)} \leq C, \quad \|\nabla u_t^\varepsilon\|_{L^2(Q_T)} \leq C,$$

where C is independent of ε. Due to the growth assumptions on b and a, and estimates for u_t^ε, we obtain

$$\|b(t, x, u_t^\varepsilon)\|_{L^q(Q_T)} \leq C, \quad \|a(t, x, \nabla u_t^\varepsilon)\|_{L^2(Q_T)^N} \leq C.$$

Similarly to (8) $\|u^\varepsilon(t)\|_{V_0} \le C$, $t \in [0, T]$ can be shown. Then there exists a subsequence of $\{u^\varepsilon\}$, again denoted by $\{u^\varepsilon\}$, such that

$$\begin{aligned}
u^\varepsilon &\to u && \text{weakly-}* \text{ in } L^\infty(0, T; V_0),\\
u^\varepsilon_t &\to u_t && \text{weakly in } L^p(Q_T) \cap L^2(0, T; V_0),\\
b(t, x, u^\varepsilon_t) &\to \beta && \text{weakly in } L^q(Q_T),\\
a(t, x, \nabla u^\varepsilon_t) &\to \eta && \text{weakly in } L^2(Q_T)^N,\\
\varepsilon u^\varepsilon_t &\to 0 && \text{weakly in } L^2(0, T; L^2(\Omega)),\\
\varepsilon u^\varepsilon_t(\cdot, T) &\to 0 && \text{weakly in } L^2(\Omega),
\end{aligned}$$

as $\varepsilon \to 0$. Using the same argument for convergence of $\{h(u^\varepsilon)\}$ and $\{f(t, x, u^\varepsilon)\}$ as in the proof of Theorem 2.3 and passing to the limit as $\varepsilon \to 0$ in (3) yields

$$\int_{Q_T} \Big(\beta v + \eta \nabla v\Big) \mathrm{d}x\,\mathrm{d}t + \int_{Q_T} d(t, x) h(u) \nabla u\, \nabla v\, \mathrm{d}x\,\mathrm{d}t = \int_{Q_T} f(t, x, u) v\, \mathrm{d}x\,\mathrm{d}t$$

for all $v \in L^p(Q_T) \cap L^2(0, T; V_0)$. Similarly as for $\{u^m_t\}$, we prove the strong convergence of $\{u^\varepsilon_t\}$ and obtain $\beta = b(t, x, u_t)$, $\eta = a(t, x, \nabla u_t)$. Using $u \in L^\infty(0, T; V_0)$, $u_t \in L^2(0, T; V_0)$ implies that $u : [0, T] \to V_0$ is continuous [11]. Due to $u^\varepsilon(0) = u_0$, we obtain $u(0) = u_0$ in V_0. Thus, u is a solution of (2). ■

3. Regularity

To prove the uniqueness of a solution of a pseudoparabolic equation additional regularity is needed.

3.1. *Regularity of solutions of hyperbolic equations*

We prove that a weak solution of a hyperbolic equation actually is in $H^1(0, T; H^2(\Omega))$ in the two dimensional case.

THEOREM 3.1 *Let Assumption* 2.1 *be satisfied,* Ω *be a* C^2*-domain,* $V_0 = H^1_0(\Omega)$, $u_0 \in H^2(\Omega)$, $a(t, \cdot, \cdot) \in C^1(\Omega \times \mathbb{R}^N)$, $d(t, \cdot) \in C^1(\Omega)^{N\times N}$ *for* $t \in (0, T)$, $h \in C^1(\mathbb{R})$, $N = 2$, $p = 2$, *and for* $\eta \in \mathbb{R}^N$, $\xi \in \mathbb{R}$,

$$\begin{aligned}
|\partial_\eta a(t, x, \eta)| \le C, &\quad |\nabla_x a(t, x, \eta)| \le a_2(1 + |\eta|),\\
|\partial_\xi h(\xi)| \le C, &\quad |\nabla_x d(t, x)| \le C.
\end{aligned}$$

Then the solution u^ε *of the problem* (1) *is in* $H^1(0, T; H^1_0(\Omega))$, *in* $H^1(0, T; H^2(\Omega))$, *and satisfies* $\varepsilon u^\varepsilon_{tt} \in L^2(Q_T)$.

Proof First we show the local regularity. We fix any open set U, and choose an open set W, such that $U \subset\subset W \subset\subset \Omega$. We choose the basis functions ϕ^k as solutions of

$$\Delta\phi^k = \lambda\phi^k \text{ in } \Omega, \qquad \phi^k = 0 \text{ on } \partial\Omega.$$

We choose $v = -\partial_{x_l}(\zeta_1^2 \partial_{x_l} u_t^m)$ as a test function in (4), where ζ_1 is the smooth cut-off function, $\zeta_1 = 1$ in U, $\zeta_1 = 0$ in $\Omega \setminus W$, $0 \le \zeta_1 \le 1$, and integrate over $t \in [0, T]$. Due to the regularity of ϕ^k, we have $v \in L^2(0, T; H_0^1(\Omega))$. Integrating by parts and summing over l implies

$$\begin{aligned}
&\varepsilon \int_{Q_T} \nabla u_{tt}^m \, \nabla u_t^m \, \zeta_1^2 \, \mathrm{d}x\, \mathrm{d}t - \sum_{l=1}^{N} \int_{Q_T} b(t, x, u_t^m) \partial_{x_l}(\zeta_1^2 \partial_{x_l} u_t^m) \mathrm{d}x\, \mathrm{d}t \\
&\quad + \sum_{l=1}^{N} \sum_{i,j=1}^{N} \int_{Q_T} \partial_{\eta_j} a^i(t, x, \nabla u_t^m) \partial_{x_j} \partial_{x_l} u_t^m \, \partial_{x_i}(\zeta_1^2 \partial_{x_l} u_t^m) \mathrm{d}x\, \mathrm{d}t \\
&\quad + \sum_{l=1}^{N} \int_{Q_T} \Big(\partial_{x_l} a(t, x, \nabla u_t^m) + \partial_{x_l}(d(t, x) h(u^m) \nabla u^m) \Big) \nabla(\zeta_1^2 \partial_{x_l} u_t^m) \mathrm{d}x\, \mathrm{d}t \\
&= - \sum_{l=1}^{N} \int_{Q_T} f(t, x, u^m) \partial_{x_l}(\zeta_1^2 \partial_{x_l} u_t^m) \mathrm{d}x\, \mathrm{d}t.
\end{aligned} \tag{11}$$

The strong monotonicity of a implies $1/\sigma(a(t, x, \tilde{\eta} + \sigma\xi) - a(\tilde{\eta}))\xi \ge a_1 |\xi|^2$ for $\eta_1 = \tilde{\eta} + \sigma\xi$, $\sigma > 0$, and $\eta_2 = \tilde{\eta}$. Taking the limit as $\sigma \to 0$ yields

$$\nabla_\eta a(t, x, \tilde{\eta}) \xi\xi \ \ge \ a_1 |\xi|^2 \quad \text{for } \tilde{\eta}, \xi \in \mathbb{R}^N. \tag{12}$$

Then we have the estimate

$$\sum_{l,i,j=1}^{N} \int_{Q_T} \partial_{\eta_j} a^i(t, x, \nabla u_t^m) \partial_{x_j x_l}^2 u_t^m \partial_{x_i x_l}^2 u_t^m \zeta_1^2 \, \mathrm{d}x\, \mathrm{d}t \ge a_1 \sum_{l,i=1}^{N} \int_{Q_T} |\partial_{x_i x_l}^2 u_t^m|^2 \zeta_1^2 \, \mathrm{d}x\, \mathrm{d}t.$$

From the equation (11), using Young's inequality, we obtain

$$\begin{aligned}
&\frac{\varepsilon}{2} \int_{\Omega} |\nabla u_t^m(T)|^2 \, \zeta_1^2 \, \mathrm{d}x + a_1 \int_{Q_T} |\nabla^2 u_t^m|^2 \, \zeta_1^2 \, \mathrm{d}x\, \mathrm{d}t \\
&\quad \le \delta_0 \int_{Q_T} |\nabla^2 u_t^m|^2 \, \zeta_1^2 \, \mathrm{d}x\, \mathrm{d}t + c_1(\delta_0) \int_{Q_T} \Big(|\nabla^2 u^m|^2 + |\nabla u^m|^4 \Big) \zeta_1^2 \, \mathrm{d}x\, \mathrm{d}t \\
&\qquad + c_2(\delta_0) \int_{Q_T} \Big(|\nabla u_t^m|^2 + |u_t^m|^2 + |\nabla u^m|^2 + |u^m|^2 \Big) \mathrm{d}x\, \mathrm{d}t + c_3(\delta_0).
\end{aligned} \tag{13}$$

For $N=2$, due to the embedding theorem, we have $\nabla u^m \in L^4(Q_T)$ and the Gagliardo–Nirenberg inequality

$$\int_{Q_T} |\nabla u^m|^4 \, \zeta_1^2 \, \mathrm{d}x \, \mathrm{d}t \le C \int_{Q_T} \Big(|\nabla^2 u^m|^2 \zeta_1^2 + |\nabla u^m|^2 \zeta_1^2 |\nabla \zeta_1|^2 \Big) \mathrm{d}x \, \mathrm{d}t \int_{Q_T} |\nabla u^m|^2 \, \mathrm{d}x \, \mathrm{d}t.$$

The estimate for ∇u^m and the assumption $u_0 \in H^2(\Omega)$ imply

$$\int_{Q_T} |\nabla u^m|^4 \zeta_1^2 \, \mathrm{d}x \, \mathrm{d}t \le c_1 \int_{Q_T} |\nabla^2 u^m|^2 \zeta_1^2 \, \mathrm{d}x \, \mathrm{d}t + c_2 \le c_3 + c_4 \int_0^T \int_{Q_t} |\nabla^2 u_\tau^m|^2 \zeta_1^2 \, \mathrm{d}x \, \mathrm{d}\tau \, \mathrm{d}t.$$

Due to the estimates (6) for u_t^m and the assumptions in the theorem, we obtain from (13) the inequality

$$\int_{Q_T} \zeta_1^2 |\nabla^2 u_t^m|^2 \, \mathrm{d}x \, \mathrm{d}t \le C_1 + C_2 \int_0^T \int_{Q_t} \zeta_1^2 |\nabla^2 u_\tau^m|^2 \, \mathrm{d}x \, \mathrm{d}\tau \, \mathrm{d}t.$$

Then Gronwall's lemma implies the estimate

$$\|\zeta_1 \nabla^2 u_t^m\|_{L^2(Q_T)} \le C. \tag{14}$$

From (13) we obtain also

$$\varepsilon \|\nabla u_t^m(\tau) \zeta_1\|_{L^\infty(0,T;L^2(\Omega))} \le C.$$

Using these extra estimates in the proof of Theorem 2.3 yields a subsequence and a limit-function $u^\varepsilon \in H^1(0,T;H_0^1(\Omega))$, which satisfies $u_t^\varepsilon \in L^2(0,T;H_{\mathrm{loc}}^2(\Omega))$ and $\varepsilon u_t^\varepsilon \in L^\infty(0,T;H_{\mathrm{loc}}^1(\Omega))$ also.

To show the regularity of u^ε up to the boundary, we need an estimate for $\nabla^2 u_t^m$ close to $\partial\Omega$. Here, we use $\phi^k = 0$ and $\Delta\phi^k = 0$ on $\partial\Omega$. In the local coordinates near the boundary Ω is of the form $B_1(0) \cap \{\mathbb{R} \times \mathbb{R}_+\}$. Hence, we consider the case $\Omega = B_1(0) \cap \{\mathbb{R} \times \mathbb{R}_+\}$ at first. We choose $v = -\partial_{x_1}(\zeta^2 \partial_{x_1} u_t^m)$ as a test function in (4), where ζ is the smooth cut-off function, $0 \le \zeta \le 1$ and $\zeta = 1$ in $B_{1/2}(0)$, $\zeta = 0$ in $\mathbb{R}^2 \setminus B_1(0)$, and ζ vanishes near the curved part of $\partial\Omega$. Integrating over t and integrating by parts imply

$$\begin{aligned}
&\varepsilon \int_{Q_T} \partial_{x_1} u_{tt}^m \partial_{x_1} u_t^m \zeta^2 \, \mathrm{d}x \, \mathrm{d}t - \int_{Q_T} b(t,x,u_t^m)(\partial_{x_1}^2 u_t^m \zeta^2 + 2\zeta \partial_{x_1}\zeta \partial_{x_1} u_t^m) \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_{Q_T} \Big(\nabla_\eta a(t,x,\nabla u_t^m) \partial_{x_1} \nabla u_t^m + \partial_{x_1} a(t,x,\nabla u_t^m) \Big) \nabla(\partial_{x_1} u_t^m \zeta^2) \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_{Q_T} \partial_{x_1}(d(t,x) h(u^m) \nabla u^m) \nabla(\partial_{x_1} u_t^m \zeta^2) \mathrm{d}x \, \mathrm{d}t \\
&= -\int_{Q_T} f(t,x,u^m) \partial_{x_1}(\partial_{x_1} u_t^m \zeta^2) \mathrm{d}x \, \mathrm{d}t.
\end{aligned} \tag{15}$$

We have $v \in L^2(0, T; H_0^1(\Omega))$ since ϕ^k is regular for all k and since ζ vanishes near the curved part of $\partial\Omega$, and $\partial_{x_1} u^m = 0$ and $\partial^2_{x_1} u^m = 0$ on $\{x_2 = 0\}$, because $u^m = \sum_{k=1}^m c_m^k \phi^k$ is zero on $\{x_2 = 0\}$ and the normal vector to this part of the boundary is $\nu = (0, -1)$.

Then starting from equation (15), we obtain, by using the strong monotonicity of a (see (12)), Young's inequality, the Gagliardo–Nirenberg inequality, the estimates (6) for u_t^m and the assumptions in the theorem, the inequality:

$$\int_{Q_T} |\partial_{x_1} \nabla u_t^m|^2 \, \zeta^2 \, \mathrm{d}x \, \mathrm{d}t \leq C_1 + \int_0^T \int_{Q_t} \Big(|\partial^2_{x_2} u_\tau^m|^2 + |\partial_{x_1} \nabla u_\tau^m|^2 \Big) \zeta^2 \, \mathrm{d}x \, \mathrm{d}\tau \, \mathrm{d}t.$$

Then Gronwall's lemma implies the estimate

$$\int_{Q_T} |\partial_{x_1} \nabla u_t^m|^2 \, \zeta^2 \, \mathrm{d}x \, \mathrm{d}t \ \leq \ C\Big(1 + \int_0^T \int_{Q_t} |\partial^2_{x_2} u_\tau^m|^2 \, \zeta^2 \, \mathrm{d}x \, \mathrm{d}\tau \, \mathrm{d}t \Big). \tag{16}$$

Similarly, we choose $v = -\partial_{x_2}(\zeta^2 \partial_{x_2} u_t^m)$ as a test function in (4) and integrate over t and integrate by parts. Using $\Delta u^m = 0$ and $\partial^2_{x_1} u^m = 0$ on $\{x_2 = 0\}$, it follows that $\partial^2_{x_2} u^m = 0$ on $\{x_2 = 0\}$, and since $\partial_{x_2} \zeta = 0$ on $\{x_2 = 0\}$ and ζ vanishes near the curved part of $\partial\Omega$, $v \in L^2(0, T; H_0^1(\Omega))$. The strong monotonicity of a for $\xi = (0, 1)$ (see (12)), yields $a^2_{\eta_2} \geq a_1$. Then, due to

$$\int_{Q_T} |\nabla u^m|^4 \zeta^2 \, \mathrm{d}x \, \mathrm{d}t \leq C + \int_0^T \int_{Q_t} |\partial_{x_1} \nabla u_\tau^m|^2 \zeta^2 \, \mathrm{d}x \, \mathrm{d}\tau \, \mathrm{d}t + \int_0^T \int_{Q_t} |\partial^2_{x_2} u_\tau^m|^2 \zeta^2 \, \mathrm{d}x \, \mathrm{d}\tau \, \mathrm{d}t,$$

that follows from Gagliardo–Nirenberg inequality, the estimate (16), and the estimates for u^m we obtain

$$\int_{Q_T} |\partial^2_{x_2} u_t^m|^2 \, \zeta^2 \, \mathrm{d}x \, \mathrm{d}t \leq C_1 + C_2 \int_0^T \int_{Q_t} |\partial^2_{x_2} u_\tau^m|^2 \, \zeta^2 \, \mathrm{d}x \, \mathrm{d}\tau \, \mathrm{d}t.$$

Hence, Gronwall's lemma implies $\|\partial^2_{x_2} u_t^m \, \zeta\|_{L^2(Q_T)} \leq C$. From this and (16) it follows that

$$\|\nabla^2 u_t^m \, \zeta\|_{L^2(Q_T)} \ \leq \ C.$$

From the preceding estimates we obtain also $\varepsilon \|\nabla u_t^m(\tau) \, \zeta\|_{L^\infty(0, T; L^2(\Omega))} \ \leq \ C$.

Using these estimates and the local estimate (14) in the proof of Theorem 2.3 yields $u_t^\varepsilon \in L^2(0, T; H_0^1(\Omega))$, $u_t^\varepsilon \in L^2(0, T; H^2(\Omega))$, and $\varepsilon u_t^\varepsilon \in L^\infty(0, T; H_0^1(\Omega))$. From $u_t^\varepsilon \in L^2(0, T; H^2(\Omega))$ and equation (1), it follows that $\varepsilon u_{tt}^\varepsilon \in L^2(Q_T)$.

All the preceding calculations are true for a general C^2 domain: for any point $x^0 \in \partial\Omega$, since $\partial\Omega$ is C^2, we may assume $\Omega \cap B(x^0, r) = \{x \in B(x^0, r), \ x_N > \gamma(x_1, \ldots, x_{N-1})\}$ for some $r > 0$ and some C^2 function $\gamma : \mathbb{R}^{N-1} \to \mathbb{R}$. We change variables to $y = \Phi(x)$, $x = \Psi(y)$ and choose $s > 0$ so small that the half-ball $\Omega' := B(0, s) \cap \{y_N > 0\}$ lies in $\Phi(\Omega \cap B(x^0, r))$. From the preceding calculations above we obtain the estimate for $\bar{u}^\varepsilon := u^\varepsilon(t, \Psi(y))$ and consequently for u^ε. ∎

3.2. *Regularity of solutions of pseudoparabolic equations*

By using the regularity of u^ε, we prove the regularity of solutions of the pseudo-parabolic equation.

THEOREM 3.2 *Let the assumptions of Theorem* 3.1 *be satisfied. Then a solution of the problem* (2) *is in* $H^1(0,T;H^1_0(\Omega))$ *and* $H^1(0,T;H^2(\Omega))$.

Proof For the proof of the local regularity, we choose $v=-\nabla\cdot(\zeta_1^2 D^\sigma u_t^\varepsilon)$ as a test function in the equation (3), where $D_i^\sigma v(x)=(1/\sigma)(u(x+\sigma e_i)-u(x))$, $i=1,\dots,N$, $D^\sigma v:=(D_1^\sigma v,\dots,D_N^\sigma v)$, and the cut-off function ζ_1 is defined in Theorem 3.1, integrating by parts and obtain

$$\begin{aligned}
&-\varepsilon\int_{Q_T} u_{tt}^\varepsilon \nabla\cdot D^\sigma u_t^\varepsilon \zeta_1^2\,\mathrm{d}x\,\mathrm{d}t-\int_{Q_T} b(t,x,u_t^\varepsilon)\nabla\cdot(\zeta_1^2 D^\sigma u_t^\varepsilon)\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_{Q_T}\nabla a(t,x,\nabla u_t^\varepsilon)\nabla(\zeta_1^2 D^\sigma u_t^\varepsilon)\mathrm{d}x\,\mathrm{d}t+\int_{Q_T}\nabla\big(d(t,x)h(u^\varepsilon)\nabla u^\varepsilon\big)\nabla(\zeta_1^2 D^\sigma u_t^\varepsilon)\mathrm{d}x\,\mathrm{d}t\\
&=-\int_{Q_T} f(t,x,u^\varepsilon)\nabla\cdot(\zeta_1^2 D^\sigma u_t^\varepsilon)\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

All integrands are integrable and uniformly bounded in σ by $L^1(Q_T)$ functions, because $u_t^\varepsilon\in L^2(0,T;H^2(\Omega))$ and $\varepsilon u_{tt}^\varepsilon\in L^2(Q_T)$. Then, due to the Dominated Convergence Theorem, we can take limits as $\sigma\to 0$ and, after integrating by parts in the first integral, obtain

$$\begin{aligned}
&\frac{\varepsilon}{2}\int_\Omega |\nabla u_t^\varepsilon(T)|^2\,\zeta_1^2\,\mathrm{d}x+\int_{Q_T}\nabla_\eta a(t,x,\nabla u_t^\varepsilon)\,\nabla^2 u_t^\varepsilon\,\nabla(\zeta_1^2\nabla u_t^\varepsilon)\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_{Q_T}\Big(\nabla_x a(t,x,\nabla u_t^\varepsilon)+\nabla\big(d(t,x)h(u^\varepsilon)\nabla u^\varepsilon\big)\Big)\nabla(\zeta_1^2\nabla u_t^\varepsilon)\mathrm{d}x\,\mathrm{d}t\\
&=-\int_{Q_T} f(t,x,u^\varepsilon)\nabla(\zeta_1^2\nabla u_t^\varepsilon)\mathrm{d}x\,\mathrm{d}t+\int_{Q_T} b(t,x,u_t^\varepsilon)\nabla(\zeta_1^2\nabla u_t^\varepsilon)\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{17}$$

Then by using in (17) the strong monotonicity of a, see (12), Young's inequality, the Gagliardo–Nirenberg inequality, the estimates for u_t^ε, and the assumptions in the theorem we obtain

$$\int_{Q_T}\zeta_1^2|\nabla^2 u_t^\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t\le C_1+C_2\int_0^T\int_{Q_t}\zeta_1^2|\nabla^2 u_\tau^\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}t.$$

Then Gronwall's lemma implies

$$\|\zeta_1\,\nabla^2 u_t^\varepsilon\|_{L^2(Q_T)}\le C.\tag{18}$$

Using this estimate in the proof of Theorem 2.6 yields a subsequence and a limit-function such that $u_t\in L^2(0,T;H^1_0(\Omega))$ and $u_t\in L^2(0,T;H^2_{\mathrm{loc}}(\Omega))$.

For the estimate near the boundary we use the same argument as for the hyperbolic equation. We can consider the equation in the half-ball, i.e., $\Omega = B_1(0) \cap \{\mathbb{R} \times \mathbb{R}_+\}$ with a straight boundary. Then we choose $v = -\partial_{x_1}(\zeta^2 D_1^\sigma u_t^\varepsilon)$ as a test function in the equation (3), where ζ is as in Theorem 3.1, and after integrating by parts and taking limits as $\sigma \to 0$ as above, we obtain

$$\begin{aligned}
&\frac{\varepsilon}{2}\int_\Omega |\partial_{x_1} u_t^\varepsilon|^2 \zeta^2 \,\mathrm{d}x - \int_{Q_T} b(t,x,u_t^\varepsilon)\partial_{x_1}(\zeta^2 \partial_{x_1} u_t^\varepsilon)\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{Q_T} \Big(\partial_{x_1} a(t,x,\nabla u_t^\varepsilon) + \partial_{x_1}(d(t,x)h(u^\varepsilon)\nabla u^\varepsilon)\Big)\nabla(\zeta^2 \partial_{x_1} u_t^\varepsilon)\mathrm{d}x\,\mathrm{d}t \\
&= -\int_{Q_T} f(t,x,u^\varepsilon)\partial_{x_1}(\zeta^2 \partial_{x_1} u_t^\varepsilon)\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

We have $v \in L^2(0,T;H_0^1(\Omega))$ since $u_t^\varepsilon \in L^2(0,T;H^2(\Omega))$ and ζ vanishes near the curved part of $\partial\Omega$, and $\partial_{x_1} u^\varepsilon = 0$ and $\partial_{x_1}^2 u^\varepsilon = 0$ on $\{x_2 = 0\}$, because the normal vector to this part of the boundary is $\nu = (0,-1)$. Similarly as for the hyperbolic equation, we obtain

$$\int_{Q_T} |\partial_{x_1} \nabla u_t^\varepsilon|^2 \, \zeta^2 \,\mathrm{d}x\,\mathrm{d}t \;\le\; C\Big(1 + \int_0^T \int_{Q_t} |\partial_{x_2}^2 u_\tau^\varepsilon|^2 \,\zeta^2 \,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}t\Big). \tag{19}$$

Since $u_t^\varepsilon \in L^2(0,T;H^2(\Omega))$, $u_t^\varepsilon \in L^2(0,T;H_0^1(\Omega))$ and $\varepsilon u_{tt}^\varepsilon \in L^2(Q_T)$ uniformly in ε, we have that u^ε satisfies (1) almost everywhere. Then we obtain

$$\begin{aligned}
\partial_{\eta_2} a^2(t,x,\nabla u_t^\varepsilon)\partial_{x_2}^2 u_t^\varepsilon &= \varepsilon u_{tt}^\varepsilon - d_{22}(t,x)h(u^\varepsilon)\partial_{x_2}^2 u^\varepsilon - d(t,x)\partial_\xi h(u^\varepsilon)\nabla u^\varepsilon\,\nabla u^\varepsilon \\
&\quad -\partial_{x_1} a^1(t,x,\nabla u_t^\varepsilon) - (\nabla_\eta a^1(t,x,\nabla u_t^\varepsilon) + \partial_{\eta_1} a^2(t,x,\nabla u_t^\varepsilon))\partial_{x_1}\nabla u_t^\varepsilon \\
&\quad -\partial_{x_1}(d^1(t,x)h(u^\varepsilon)\nabla u^\varepsilon) - \partial_{x_2} d^2(t,x)h(u^\varepsilon)\nabla u^\varepsilon + b(t,x,u_t^\varepsilon) - f(t,x,u^\varepsilon).
\end{aligned}$$

From the strong monotonicity of a for $\xi = (0,1)$, see (12), it follows that $\partial_{\eta_2} a^2 \ge a_1$. Then

$$\int_{Q_T} |\partial_{x_2}^2 u_t^\varepsilon|^2 \,\zeta^2 \,\mathrm{d}x\,\mathrm{d}t \le C + \int_0^T \int_{Q_t} |\partial_{x_2}^2 u_\tau^\varepsilon|^2 \,\zeta^2 \,\mathrm{d}x\,\mathrm{d}\tau.$$

Hence Gronwall's lemma implies the estimate $\|\partial_{x_2}^2 u_t^\varepsilon \,\zeta\|_{L^2(Q_T)} \le C$, where C is independent of ε. This, together with (19), implies

$$\|\nabla^2 u_t^\varepsilon \,\zeta\|_{L^2(Q_T)} \le C.$$

Using the last estimate and the local estimate (18) in the proof of Theorem 2.6 yields a subsequence and a limit-function such that $u \in H^1(0,T;H_0^1(\Omega))$ and $u \in H^1(0,T;H^2(\Omega))$. ∎

4. Uniqueness

In the last section, we showed regularity of weak solution in two dimension. In this section, we show that a regular solution is the unique solution in dimension $N \leq 4$.

THEOREM 4.1 *Let Assumption* 2.1, $u \in H^1(0, T; H^2(\Omega))$, *and*

$$|f(t,x,\xi_1) - f(t,x,\xi_2)| \leq C|\xi_1 - \xi_2|, \qquad |h(\xi_1) - h(\xi_2)| \leq C|\xi_1 - \xi_2|$$

for $(t,x) \in Q_T$, $\xi_1, \xi_2 \in \mathbb{R}$, *be satisfied. Then there exists at most one weak solution of* (2).

Proof Suppose u^1 and u^2 are two solutions of the problem (2). Then for $u = u^1 - u^2$ and the test function $v = u_t$, we obtain the equation

$$\int_{Q_T} (b(t,x,u_t^1) - b(t,x,u_t^2))u_t \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} (a(t,x,\nabla u_t^1) - a(t,x,\nabla u_t^2))\nabla u_t \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{Q_T} \Big(d(t,x)(h(u^1)\nabla u^1 - h(u^2)\nabla u^2)\nabla u_t - (f(t,x,u^1) - f(t,x,u^2))u_t \Big)\mathrm{d}x\,\mathrm{d}t = 0.$$

Due to the strong monotonicity of a and b, the first two integrals are estimated from below by

$$b_1 \int_{Q_T} |u_t|^2 \,\mathrm{d}x\,\mathrm{d}t + a_1 \int_{Q_T} |\nabla u_t|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

The terms of the third integral can be estimated separately:

$$\int_{Q_T} d(t,x)h(u^1)\nabla u\, \nabla u_t \,\mathrm{d}x\,\mathrm{d}t \leq \frac{c_1}{2\delta}\int_0^T \int_{Q_\tau} |\nabla u_t|^2 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau + \frac{\delta}{2}\int_{Q_T} |\nabla u_t|^2 \,\mathrm{d}x\,\mathrm{d}t,$$

since $u(0) = 0$. From the embedding theorem we have that $v \in H^1(0, T; H^2(\Omega))$ implies $\nabla v \in L^4(Q_T)$ even for Ω of the dimension $N \leq 4$. Then, due to the regularity of u^1 and u^2, we obtain $u^1, u^2 \in L^4(Q_T)$ and $\nabla u^2 \in L^4(Q_T)$. The remaining term satisfies

$$\int_{Q_T} d(t,x)(h(u^1) - h(u^2))\nabla u^2\, \nabla u_t \,\mathrm{d}x\,\mathrm{d}t$$
$$\leq c_2 \Big(\int_{Q_T} |u|^4 \,\mathrm{d}x\,\mathrm{d}t\Big)^{1/4} \Big(\int_{Q_T} |\nabla u^2|^4 \,\mathrm{d}x\,\mathrm{d}t\Big)^{1/4} \Big(\int_{Q_T} |\nabla u_t|^2 \,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}$$
$$\leq \frac{c_4}{2\delta}\int_{Q_T} \big(|u|^2 + |\nabla u|^2\big)\,\mathrm{d}x\,\mathrm{d}t + \frac{\delta}{2}\int_{Q_T} |\nabla u_t|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

The right-hand side is estimated by

$$\int_{Q_T} (f(t,x,u^1) - f(t,x,u^2))u_t \,\mathrm{d}x\,\mathrm{d}t \leq \frac{c_6}{\delta_0}\int_0^T \int_{Q_\tau} |u_t|^2 \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau + \delta_0 \int_{Q_T} |u_t|^2 \,\mathrm{d}x\,\mathrm{d}t,$$

since $u(0) = 0$. Thus, due to these estimates we obtain the inequality

$$(b_1 - \delta_0) \int_{Q_T} |u_t|^2 \mathrm{d}x\, \mathrm{d}t + (a_1 - \delta) \int_{Q_T} |\nabla u_t|^2 \mathrm{d}x\, \mathrm{d}t \leq C \int_0^T \int_{Q_\tau} \Big(|u_t|^2 + |\nabla u_t|^2 \Big) \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}\tau.$$

Using Gronwall's lemma in the last inequality implies $u(t) = u(0) = 0$. Hence, $u^1 = u^2$ a.e. in Q_T. ■

Remark The existence and uniqueness of solutions of nonlinear variational inequalities is proved also, and will be published in a forthcoming article.

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