Stochastic Homogenization

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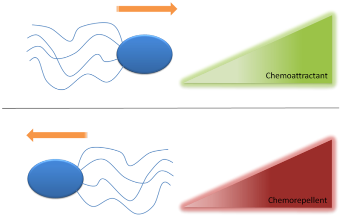
Chemotaxis

- Cells secrete chemical signal substance
- Cells partially orient their movement toward or away from increasing signal concentration



- formation of aggregations (Dictyostelium discoideum)
- tumour cell migration
- migration of immune cells into the region of a tumour
- ...

The Keller-Segel model of chemotaxis (1971)

- The Keller-Segel model

$$\begin{aligned} u_t &= \nabla \cdot (D_u(x)\nabla u - \chi(x) u \nabla v) \\ v_t &= \nabla \cdot (D_v(x)\nabla v) - \gamma v + \alpha u \end{aligned}$$

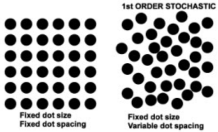
- Consider a random, heterogeneous environment

$$\begin{aligned} u_t^{\varepsilon,\omega} &= \nabla \cdot (D_u^{\omega}(x/\varepsilon)\nabla u^{\varepsilon,\omega} - \chi^{\omega}(x/\varepsilon) u^{\varepsilon,\omega} \nabla v^{\varepsilon,\omega}) \\ v_t^{\varepsilon,\omega} &= \nabla \cdot (D_v(x)\nabla v^{\varepsilon,\omega}) - \gamma v^{\varepsilon,\omega} + \alpha u^{\varepsilon,\omega} \end{aligned}$$

- $\omega \in \Omega$, where $(\Omega, \mathcal{F}, P)$ is a probability space

- $D_u^{\omega}(x/\varepsilon) = \widetilde{D}_u(\mathcal{T}(x/\varepsilon)\omega)$, $\chi^{\omega}(x/\varepsilon) = \widetilde{\chi}(\mathcal{T}(x/\varepsilon)\omega)$, where $\{\mathcal{T}(x)\}_{x\in\mathbb{R}^n}$ is a measure-preserving dynamical system

- What is the limit as $\varepsilon \to 0$ of $(u^{\varepsilon,\omega}, v^{\varepsilon,\omega})$?



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Stochastic homogenization

- Stochastic homogenization of linear elliptic eq. Papanicolaou & Varadhan 1979; Kozlov 1980; Zhikov, Kozlov, Oleinik& Ngoan 1979
- Stochastic homogenization of convex integral operators by means of $\Gamma-$convergence: Dal Maso & Modica 1986
- Quasi-linear elliptic and parabolic equations with stochastic coefficients: Bensoussan & Blankenship 1988; Castell 2001
- Hamilton-Jacobi, Hamiltonial-Jacobi-Bellman equations: Lions & Souganidis 2005, 2010; Kosygina, Rezakhanlou & Varadhan 2006; Armstrong & Soudanidis 2012
- Fully nonlinear parabolic in stationary ergodic media: Caffarelli, Souganidis, Wang 2005
- Stochastic two-scale convergence in the mean: Bourgeat, Mikelić & Wright 1994; Bourgeat, Mikelić, Piatnitski 2003
- Stochastic unfolding (in the mean): Neukamm & Varga 2018
- Stochastic two-scale convergence: Zhikov & Piatnitski 2006; Heida 2011, 2012
- ....

Heterogeneity: Dynamical system

- $(\Omega, \mathcal{F}, \mathcal{P})$ – a probability space with probability measure $\mathcal{P}$
- $\mathcal{T}(x) : \Omega \to \Omega$ dynamical system, i.e. a family $\{\mathcal{T}(x) : x \in \mathbb{R}^n\}$ of invertible maps, such that for each $x \in \mathbb{R}^n$, $\mathcal{T}(x)$ is measurable and satisfy:

  (i) $\mathcal{T}(0)$ is the identity map on Ω and $\mathcal{T}(x)$ satisfies the semigroup property:

  $$\mathcal{T}(x_1 + x_2) = \mathcal{T}(x_1)\mathcal{T}(x_2) \quad \text{for all } \; x_1,\, x_2 \in \mathbb{R}^n$$

  (ii) $\mathcal{P}$ is an invariant measure for $\mathcal{T}(x)$, i.e. for each $x \in \mathbb{R}^n$ and $F \in \mathcal{F}$ we have that

  $$\mathcal{P}(\mathcal{T}^{-1}(x)F) = \mathcal{P}(F)$$

  (iii) For each $F \in \mathcal{F}$, the set $\{(x, \omega) \in \mathbb{R}^n \times \Omega : \mathcal{T}(x)\omega \in F\}$ is a $dx \times d\mathcal{P}(\omega)$-measurable subset of $\mathbb{R}^n \times \Omega$, where dx denotes the Lebesgue measure on $\mathbb{R}^n$

- periodic case: $\Omega = [0, 1]^n$, $\mathcal{T}(x)\omega = \omega + x(\text{mod } 1)$ on Ω
- a shift: $\mathcal{T}(x)\mu(B) = \mu(B + x)$ for all Borel sets $B \subset \mathbb{R}^n$, μ- Radon measure on $\mathbb{R}^n$

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The ergodic setting

Definition A random filed $D(x,\omega)$, $x \in \mathbb{R}^d$, $\omega \in \Omega$ is *stationary* if there is a measurable function $\tilde{D}(\omega)$ on Ω

$$D(x,\omega) = \tilde{D}(\mathcal{T}(x)\omega)$$

[$x \to D(x,\omega)$ and $x \to D(x+z,\omega)$ have the same statistics for all shifts z]

Definition A measurable function f on Ω is said to be *invariant* for a dynamical system $\mathcal{T}(x)$ if for each $x \in \mathbb{R}^d$

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Definition A dynamical system $\mathcal{T}(x)$ is said to be *ergodic*, if every measurable function which is invariant for $\mathcal{T}(x)$ is $\mathcal{P}$-a.e. equal to a constant.

Ergodic Birkhoff theorem

$$\lim_{t\to\infty} \frac{1}{t^d|A|}\int_{tA} g(\mathcal{T}(x)\omega)dx = \int_{\Omega} g(\omega)d\mathcal{P} \qquad \mathcal{P}\text{-a.s.}$$

for all bounded Borel sets A with $|A| > 0$, and all $g \in C^1(\Omega)$

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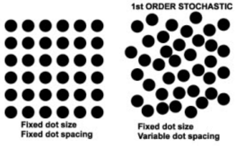
for all bounded Borel sets A with $|A| > 0$, and all $g \in C^1(\Omega)$

Ergodic environment

Poisson point process $(\Omega, \mathcal{F}, \mathcal{P})$

$\omega \in \Omega$: $\omega = \{B(\kappa_m) : m \in \mathbb{N}\}$ distribution of balls of a specific radius centered at κ_m

$N(\omega, A)$ - the number of balls the centers of which fall in the open bounded set $A \subset \mathbb{R}^n$.



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σ- algebra $\mathcal{F}$ generated by the subsets of Ω

$$\{\omega \in \Omega : N(\omega, A_1) = k_1, \dots, N(\omega, A_i) = k_i\}$$

$i, k_1, \dots, k_i \in \mathbb{N}_0$ and $A_1, \dots, A_i$ are disjoint open sets

$$\mathcal{P}(N(\omega, A_1) = k_1, \dots, N(\omega, A_i) = k_i)$$
$$= \mathcal{P}(N(\omega, A_1) = k_1) \cdot \ldots \cdot \mathcal{P}(N(\omega, A_i) = k_i)$$

with

$$\mathcal{P}(N(\omega, A) = k) = \frac{(\lambda|A|)^k}{k!} \exp(-\lambda|A|), \quad \lambda > 0$$

$$\omega \in \Omega, \quad \mathcal{T}(x)\omega = \{B(\kappa_m) + x : m \in \mathbb{N}\}, \quad x \in \mathbb{R}^n$$

Definition of Stochastic two-scale convergence

(Zhikov & Piatnitsky 2006, Heida 2011)

$\mathcal{T}(x)-$ ergodic dynamical system, $\mathcal{T}(x)\tilde{\omega}-$ "typical trajectory" (satisfy Birkhoff's theorem); realizations are typical $\mathcal{P}$-a.s.

$\{v^\varepsilon\} \subset L^2((0,\tau) \times G)$ converges stochastically two-scale to $v \in L^2((0,\tau) \times G \times \Omega)$ if

$$\limsup_{\varepsilon \to 0} \int_0^\tau \int_G |v^\varepsilon(t,x)|^2 \, dx \, dt < \infty \tag{1}$$

and

$$\lim_{\varepsilon \to 0} \int_0^\tau \int_G v^\varepsilon(t,x)\varphi(t,x) b(\mathcal{T}(x/\varepsilon)\tilde{\omega}) \, dxdt = \int_0^\tau \int_G \int_\Omega v(t,x,\omega)\varphi(t,x) b(\omega) \, d\mathcal{P}(\omega) dxdt$$

for all $\varphi \in C_0^\infty([0,\tau) \times G)$ and $b \in L^2(\Omega)$.

Theorem Every $\{v^\varepsilon\} \subset L^2(0,\tau; L^2(G))$ that satisfies (1) converges along a subsequence to some $v \in L^2(0,\tau; L^2(G \times \Omega, dx \times d\mathcal{P}(\omega)))$ in the sense of stochastic two-scale convergence.

Compactness results

Theorem $\{v^\varepsilon\} \subset H^1(G)$ satisfies

$$\|v^\varepsilon\|_{L^2(G)} \le C(\tilde\omega)\,, \qquad \|\nabla v^\varepsilon\|_{L^2(G)} \le C(\tilde\omega)$$

then $\exists\, v \in H^1(G)$ and $v_1 \in L^2(G; L^2_{\mathrm{pot}}(\Omega))$ s.t. (up to subseq.)

$$\begin{array}{lll} v^\varepsilon & \rightharpoonup & v & \text{stochastically two-scale} \\ \nabla v^\varepsilon & \rightharpoonup & \nabla v + v_1 & \text{stochastically two-scale} \end{array}$$

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$$\|v^\varepsilon\|_{L^2(G)} \le C(\tilde\omega)\,, \qquad \varepsilon\|\nabla v^\varepsilon\|_{L^2(G)} \le C(\tilde\omega)$$

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$$\begin{array}{lll} v^\varepsilon & \rightharpoonup & v & \text{stochastically two-scale} \\ \varepsilon\nabla v^\varepsilon & \rightharpoonup & \nabla_\omega v & \text{stochastically two-scale} \end{array}$$

$$\partial^j_\omega u(\omega) = \lim_{\delta\to 0} \frac{u(\mathcal{T}(\delta e_j)\omega) - u(\omega)}{\delta},\ \nabla_\omega u = (\partial^1_\omega u, \dots, \partial^n_\omega u),\ H^1(\Omega) = \{v, \nabla_\omega v \in L^2(\Omega)\}$$

$$L^2_{\mathrm{pot}}(\Omega) = \overline{\{\nabla_\omega u \,:\, u \in C^1_{\mathcal{T}}(\Omega)\}}^{L^2(\Omega)}, \quad L^2_{\mathrm{sol}}(\Omega) = L^2_{\mathrm{pot}}(\Omega)^\perp$$

Zhikov & Piatnitsky 2006

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Zhikov & Piatnitsky 2006

Weak solutions and *A priori* estimates

Weak solution $u^\varepsilon \in H^1(G_\tau)$, $v^\varepsilon \in H^1(G_\tau) \cap L^4(0,\tau; W^{1,4}(G))$:

$$\begin{aligned}
\langle u_t^\varepsilon, \phi\rangle_{G_\tau} + \langle D_u^\varepsilon(x)\nabla u^\varepsilon - \chi^\varepsilon(x) u^\varepsilon \nabla v^\varepsilon, \nabla\phi\rangle_{G_\tau} &= 0,\\
\langle v_t^\varepsilon, \psi\rangle_{G_\tau} + \langle D_v(x)\nabla v^\varepsilon, \nabla\psi\rangle_{G_\tau} + \gamma\langle v^\varepsilon, \psi\rangle_{G_\tau} &= \alpha\langle u^\varepsilon, \psi\rangle_{G_\tau},
\end{aligned}$$

for all $\phi, \psi \in L^2(0,\tau; H^1(G))$ and $\mathcal{P}$-almost surely in $\omega \in \Omega$.

Theorem For every $\varepsilon > 0$ and for $\mathcal{P}$-a.e. $\omega \in \Omega$ there exists a unique weak solution of KS, and

- $\|u^\varepsilon\|_{L^\infty(0,\tau;L^2(G))} + \|\nabla u^\varepsilon\|_{L^\infty(0,\tau;L^2(G))} + \|\partial_t u^\varepsilon\|_{L^2(G_\tau)} \le C$
- $\|v^\varepsilon\|_{L^\infty(0,\tau;H^1(G))} + \|\partial_t v^\varepsilon\|_{L^2(0,\tau;H^1(G))} + \|v^\varepsilon\|_{L^\infty(0,\tau;H^2(G))} \le C$

for some constant C that is independent of ε.

(Global solution for $n = 1$ and local in time for $n = 2$)

Macroscopic equations

$$\begin{aligned}
&\partial_t u = \nabla(D^* \nabla u - \chi^* u \nabla v) && \text{in } G_\tau, \\
&\partial_t v = \nabla(D_v(x) \nabla v) - \gamma v + \alpha u && \text{in } G_\tau, \\
&\nabla u = 0, \quad \nabla v = 0 && \text{on } \partial G \times (0,\tau), \\
&u(0,x) = u_0(x), \quad v(0,x) = v_0(x) && \text{in } G,
\end{aligned}$$

where

$$D^* \xi = \int_\Omega \widetilde{D}_u(\omega)(\bar{u}_{1,\xi} + \xi) d\mathcal{P}(\omega)$$

$$\chi^* \xi = -\int_\Omega \left(\widetilde{D}_u(\omega)\hat{u}_{1,\xi} - \widetilde{\chi}(\omega)\xi\right) d\mathcal{P}(\omega)$$

and $\bar{u}_{1,\xi}$, $\hat{u}_{1,\xi}$ are solutions of the auxiliary problems

$$\bar{u}_{1,\xi} \in L^2_{\text{pot}}(\Omega) \quad \text{such that} \quad \widetilde{D}_u(\omega)(\bar{u}_{1,\xi} + \xi) \in L^2_{\text{sol}}(\Omega),$$

$$\hat{u}_1 \in L^2_{\text{pot}}(\Omega) \quad \text{such that} \quad \widetilde{D}_u(\omega)\hat{u}_{1,\xi} - \widetilde{\chi}(\omega)\xi \in L^2_{\text{sol}}(\Omega).$$

Sketch of the Proof

$$\left.\begin{array}{l} u^\varepsilon,\ \nabla u^\varepsilon,\ \partial_t u^\varepsilon \\ v^\varepsilon,\ \nabla v^\varepsilon,\ \nabla^2 v^\varepsilon,\ \partial_t v^\varepsilon,\ \partial_t \nabla v^\varepsilon \end{array}\right\} \quad \begin{array}{c} \text{bounded in } L^2(G_\tau) \\ \text{for } \mathcal{P}\text{-a.e. } \omega \in \Omega. \end{array}$$

$$\begin{array}{lll} u^\varepsilon \rightharpoonup u & \text{stochastically two-scale,} & u \in L^2(0,\tau; H^1(G)) \\ \nabla u^\varepsilon \rightharpoonup \nabla u + u_1 & \text{stochastically two-scale,} & u_1 \in L^2(G_\tau, L^2_{pot}(\Omega)) \\ \partial_t u^\varepsilon \rightharpoonup \tilde{u} & \text{stochastically two-scale,} & \tilde{u} \in L^2(G_\tau \times \Omega) \\ v^\varepsilon \rightharpoonup v & \text{stochastically two-scale,} & v \in L^2(0,\tau; H^1(G)) \\ \partial_t v^\varepsilon \rightharpoonup \tilde{v} & \text{stochastically two-scale,} & \tilde{v} \in L^2(0,\tau; H^1(G)) \\ \nabla v^\varepsilon \rightharpoonup \hat{v} & \text{stochastically two-scale,} & \hat{v} \in L^2(0,\tau; H^1(G)) \end{array}$$

for all "typical" realizations ω.

Proof Sketch: Convergence

The stochastic two-scale limit and the strong convergence of u^ε:

$$\langle u_t, \varphi\rangle_{G_\tau} + \langle \widetilde{D}_u(\omega)(\nabla u + u_1) - \widetilde{\chi}(\omega) u \nabla v, \nabla\varphi + \varphi_1 \nabla_\omega \varphi_2(\omega)\rangle_{G_\tau,\Omega} = 0.$$

Choosing $\varphi(t,x) = 0$ for $(t,x) \in G_\tau$ we obtain

$$\langle \widetilde{D}_u(\omega)(\nabla u + u_1) - \widetilde{\chi}(\omega)\, u \nabla v, \varphi_1(t,x) \nabla_\omega \varphi_2(\omega)\rangle_{G_\tau,\Omega} = 0$$

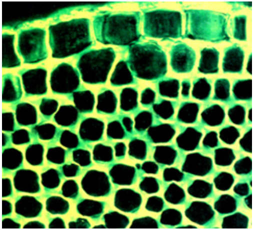
for every $\varphi_1 \in C_0^\infty(G_\tau)$ and $\varphi_2 \in C^1(\Omega)$.

$$\langle \widetilde{D}_u(\omega)(\nabla u + u_1) - \widetilde{\chi}(\omega)\, u \nabla v, \partial_\omega \varphi_2\rangle_\Omega = 0, \quad dt \times dx - \text{a.e. in} G_\tau$$

- Exists a unique solution $u_1(t,x,\cdot) \in L^2_{\text{pot}}(\Omega)$ that depends linearly on $\nabla u(t,x)$ and $u(t,x)\nabla v(t,x)$ for a.e. $(t,x) \in G_\tau$
- $$u_1(t,x,\omega) = \sum_{j=1}^{n} \partial_{x_j} u(t,x)\, \bar{u}_{1,j}(\omega) + u(t,x) \sum_{j=1}^{n} \partial_{x_j} v(t,x)\, \hat{u}_{1,j}(\omega)$$

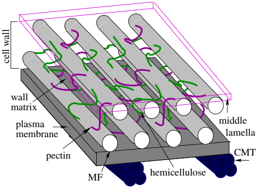
  for a.e $(t,x) \in G_\tau$ and $\mathcal{P}$-a.e. $\omega \in \Omega$
- $\bar{u}_{1,j},\ \hat{u}_{1,j} \in L^2_{\text{pot}}(\Omega)$ are solutions of the unit cell problems.

Mathematical model



Biochemistry:

- methylestrified pectin: $b_{e,1}$
- demethylestrified pectin: $b_{e,2}$
- pectin-calcium cross links: $b_{e,3}$
- calcium ions: c_e and c_f



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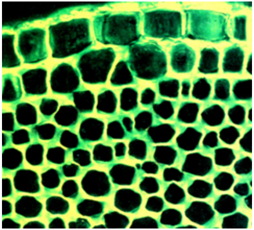

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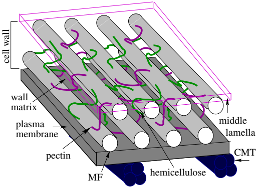
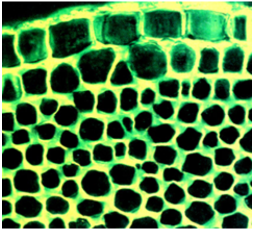
$$\begin{aligned}
\partial_t b_e - \mathrm{div}(D_b \nabla b_e) &= g_b(b_e, c_e, \mathrm{e}(\mathbf{u}_e)) && \text{in } G_e \\
\partial_t c_e - \mathrm{div}(D_e \nabla c_e) &= g_e(b_e, c_e, \mathrm{e}(\mathbf{u}_e)) && \text{in } G_e \\
\partial_t c_f - \mathrm{div}(D_f \nabla c_f - \mathcal{G}(\partial_t \mathbf{u}_f) c_f) &= g_f(c_f) && \text{in } G_f
\end{aligned}$$



Mathematical model

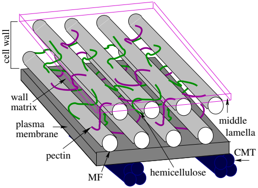
Biochemistry:

- methylestrified pectin: $b_{e,1}$
- demethylestrified pectin: $b_{e,2}$
- pectin-calcium cross links: $b_{e,3}$
- calcium ions: c_e and c_f



$$
\begin{aligned}
\partial_t b_e - \operatorname{div}(D_b \nabla b_e) &= g_b(b_e, c_e, \mathrm{e}(\mathbf{u}_e)) && \text{in } G_e \\
\partial_t c_e - \operatorname{div}(D_e \nabla c_e) &= g_e(b_e, c_e, \mathrm{e}(\mathbf{u}_e)) && \text{in } G_e \\
\partial_t c_f - \operatorname{div}(D_f \nabla c_f - \mathcal{G}(\partial_t \mathbf{u}_f) c_f) &= g_f(c_f) && \text{in } G_f
\end{aligned}
$$

Mechanics: Poroelastcity



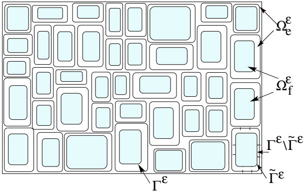
$\mathbf{u}_e$ - deformations of cell walls+middle lamella

p_e - flow pressure in cell walls+middle lamella

$\partial_t \mathbf{u}_f$ - fluid flow inside the cells

$$
\begin{aligned}
&\operatorname{div}\left(\mathbb{E}(b_e)\,\mathbf{e}(\mathbf{u}_e) - p_e I\right) = 0 && \text{in } G_e \\
&\operatorname{div}(K \nabla p_e - \partial_t \mathbf{u}_e) = 0 && \text{in } G_e \\
&\partial_t(\partial_t \mathbf{u}_f) - \mu \operatorname{div}(\mathbf{e}(\partial_t \mathbf{u}_f) - p_f I) = 0 && \text{in } G_f
\end{aligned}
$$

Plant tissues biomechanics: Random geometry



- Ω_f measurable set, $\mathcal{P}(\Omega_f) > 0$, $\mathcal{P}(\Omega \setminus \Omega_f) > 0$
- $\Omega_e = \Omega \setminus \Omega_f$
- $\Omega_\Gamma \subset \Omega$, with $\mathcal{P}(\Omega_\Gamma) > 0$ and $\mathcal{P}(\Omega_\Gamma \cap \Omega_j) > 0$ for $j = e, f$

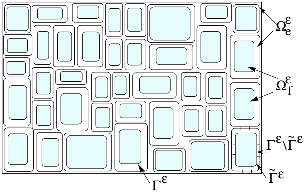
- For $\mathcal{P}$-a.a. $\omega \in \Omega$ define a random system of subdomains in $\mathbb{R}^3$

$$G_j(\omega) = \{x \in \mathbb{R}^3 : \ \mathcal{T}(x)\omega \in \Omega_j\}, \quad j = e, f$$
$$G_\Gamma(\omega) = \{x \in \mathbb{R}^3 : \ \mathcal{T}(x)\omega \in \Omega_\Gamma\}$$

$$\Gamma(\omega) = \partial G_f(\omega), \qquad \widetilde{\Gamma}(\omega) = \Gamma(\omega) \cap G_\Gamma(\omega)$$

1. $G_f(\omega)$ countable number of disjoined Lipschitz domains for $\mathcal{P}$-a.a. $\omega \in \Omega$
2. The distance between two connected components of $G_f(\omega)$ and diameter of $G_f(\omega)$ are uniformly bounded from above and below.
3. The surface $\widetilde{\Gamma}(\omega) \subset \Gamma(\omega)$ is open on $\Gamma(\omega)$ and Lipschitz continuous

Plant tissues biomechanics: Random geometry



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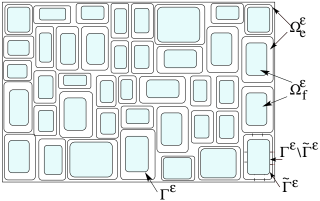
Random geometry



$$G_f^\varepsilon = \{x \in \mathbb{R}^3 : \ \mathcal{T}(x/\varepsilon)\omega \in \Omega_f\} \cap G$$

cell inside

$$G_e^\varepsilon = G \setminus G_f^\varepsilon$$

cell wall + middle lamella

$$G_\Gamma^\varepsilon = \{x \in \mathbb{R}^3 : \ \mathcal{T}(x/\varepsilon)\omega \in \Omega_\Gamma\} \cap G$$

$\Gamma^\varepsilon = \partial G_f^\varepsilon$ cell membrane

$\widetilde{\Gamma}^\varepsilon = \Gamma^\varepsilon \cap G_\Gamma^\varepsilon$ part of the cell membrane impermeable to calcium ions

Statistically homogeneous (stationary) random fields

$$\mathbb{E}(x, \omega, \xi) = \widetilde{\mathbb{E}}(\mathcal{T}(x)\omega, \xi), \qquad K_p(x, \omega) = \widetilde{K}_p(\mathcal{T}(x)\omega)$$

$\widetilde{\mathbb{E}}(\cdot, \xi) : \Omega \to \mathbb{R}^{3^4}$, $\widetilde{K}_p(\cdot) : \Omega \to \mathbb{R}^{3\times 3}$ measurable functions, $\xi \in \mathbb{R}$.

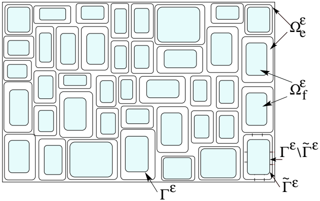
$\mathbb{E}^\varepsilon(x, \xi) = \mathbb{E}\big(x/\varepsilon, \omega, \xi\big)$, $K_p^\varepsilon(x) = K_p\big(x/\varepsilon, \omega\big)$ for $\omega \in \Omega$, $x \in \mathbb{R}^3$, $\xi \in \mathbb{R}$

Random geometry

$$G_f^\varepsilon = \{x \in \mathbb{R}^3 : \mathcal{T}(x/\varepsilon)\omega \in \Omega_f\} \cap G$$

cell inside

$$G_e^\varepsilon = G \setminus G_f^\varepsilon$$ cell wall + middle lamella



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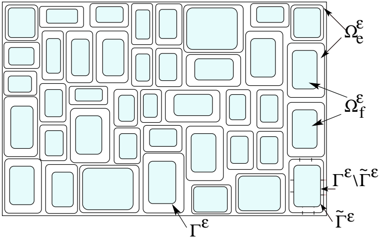
Plant tissue biomechanics: Poroelasticity



$$\partial_t^2 \mathbf{u}_e^\varepsilon - \operatorname{div}(\mathbb{E}^\varepsilon(b_{e,3}^\varepsilon)\mathbf{e}(\mathbf{u}_e^\varepsilon)) + \nabla p_e^\varepsilon = 0$$
$$\partial_t p_e^\varepsilon - \operatorname{div}(K_p^\varepsilon \nabla p_e^\varepsilon - \partial_t \mathbf{u}_e^\varepsilon) = 0$$
$$\partial_t^2 \mathbf{u}_f^\varepsilon - \varepsilon^2 \mu \operatorname{div}(\mathbf{e}(\partial_t \mathbf{u}_f^\varepsilon)) + \nabla p_f^\varepsilon = 0$$
$$\operatorname{div} \partial_t \mathbf{u}_f^\varepsilon = 0$$

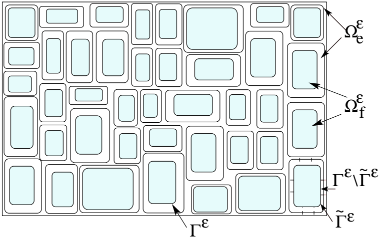
$$(\mathbb{E}^\varepsilon(b_{e,3}^\varepsilon)\,\mathbf{e}(\mathbf{u}_e^\varepsilon) - p_e^\varepsilon I)\,\nu = (\varepsilon^2 \mu\, \mathrm{e}(\partial_t \mathbf{u}_f^\varepsilon) - p_f^\varepsilon I)\,\nu \quad \text{on } \Gamma^\varepsilon$$
$$\Pi_\tau \partial_t \mathbf{u}_e^\varepsilon = \Pi_\tau \partial_t \mathbf{u}_f^\varepsilon \quad \text{on } \Gamma^\varepsilon$$
$$n \cdot (\varepsilon^2 \mu\, \mathrm{e}(\partial_t \mathbf{u}_f^\varepsilon) - p_f^\varepsilon)\,\nu = -p_e^\varepsilon \quad \text{on } \Gamma^\varepsilon$$
$$(-K_p^\varepsilon \nabla p_e^\varepsilon + \partial_t \mathbf{u}_e^\varepsilon) \cdot \nu = \partial_t \mathbf{u}_f^\varepsilon \cdot \nu \quad \text{on } \Gamma^\varepsilon$$

$\Pi_\tau w$ - tangential components

On the external boundaries :

$$\mathbb{E}^\varepsilon(b_{e,3}^\varepsilon)\mathrm{e}(u_e^\varepsilon)\,\nu = F_u, \qquad (K_p^\varepsilon \nabla p_e^\varepsilon - \partial_t u_e^\varepsilon) \cdot \nu = F_p \quad \text{on } \partial\Omega$$

Plant tissue biomechanics: Poroelasticity



$$\partial_t^2 \mathbf{u}_e^\varepsilon - \mathrm{div}(\mathbb{E}^\varepsilon(b_{e,3}^\varepsilon)\mathbf{e}(\mathbf{u}_e^\varepsilon)) + \nabla p_e^\varepsilon = 0$$

$$\partial_t p_e^\varepsilon - \mathrm{div}(K_p^\varepsilon \nabla p_e^\varepsilon - \partial_t \mathbf{u}_e^\varepsilon) = 0$$

$$\partial_t^2 \mathbf{u}_f^\varepsilon - \varepsilon^2 \mu\, \mathrm{div}(\mathbf{e}(\partial_t \mathbf{u}_f^\varepsilon)) + \nabla p_f^\varepsilon = 0$$

$$\mathrm{div}\, \partial_t \mathbf{u}_f^\varepsilon = 0$$

$$(\mathbb{E}^\varepsilon(b_{e,3}^\varepsilon)\, \mathbf{e}(\mathbf{u}_e^\varepsilon) - p_e^\varepsilon I)\, \nu = (\varepsilon^2 \mu\, \mathrm{e}(\partial_t \mathbf{u}_f^\varepsilon) - p_f^\varepsilon I)\, \nu \qquad \text{on } \Gamma^\varepsilon$$

$$\Pi_\tau \partial_t \mathbf{u}_e^\varepsilon = \Pi_\tau \partial_t \mathbf{u}_f^\varepsilon \qquad \text{on } \Gamma^\varepsilon$$

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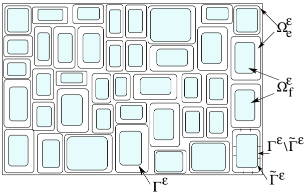
Plant tissue biomechanics: Chemistry



$$\partial_t b_e^\varepsilon = \mathrm{div}(D_b \nabla b_e^\varepsilon) + g_b(c_e^\varepsilon, b_e^\varepsilon, \mathbf{e}(\mathbf{u}_e^\varepsilon)) \quad \text{in } \Omega_e^\varepsilon$$

$$\partial_t c_e^\varepsilon = \mathrm{div}(D_e \nabla c_e^\varepsilon) + g_e(c_e^\varepsilon, b_e^\varepsilon, \mathbf{e}(\mathbf{u}_e^\varepsilon)) \quad \text{in } \Omega_e^\varepsilon$$

$$\partial_t c_f^\varepsilon = \mathrm{div}(D_f \nabla c_f^\varepsilon - \mathcal{G}(\partial_t \mathbf{u}_f^\varepsilon) c_f^\varepsilon) + g_f(c_f^\varepsilon) \quad \text{in } \Omega_f^\varepsilon$$

$$D_b \nabla b_e^\varepsilon \cdot \nu = \varepsilon\, R(b_e^\varepsilon) \quad \text{on } \Gamma^\varepsilon$$

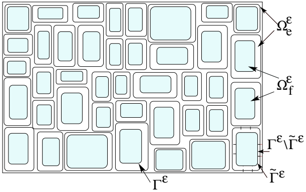
$$c_e^\varepsilon = c_f^\varepsilon, \qquad D_e \nabla c_e^\varepsilon \cdot \nu = (D_f \nabla c_f^\varepsilon - \mathcal{G}(\partial_t \mathbf{u}_f^\varepsilon) c_f^\varepsilon) \cdot \nu \quad \text{on } \Gamma^\varepsilon \setminus \widetilde{\Gamma}^\varepsilon$$

$$D_e \nabla c_e^\varepsilon \cdot \nu = 0, \qquad (D_f \nabla c_f^\varepsilon - \mathcal{G}(\partial_t \mathbf{u}_f^\varepsilon) c_f^\varepsilon) \cdot \nu = 0 \quad \text{on } \widetilde{\Gamma}^\varepsilon$$

On the external boundaries :

$$D_b \nabla b_e^\varepsilon \cdot \nu = F_b(b_e^\varepsilon), \qquad D_e \nabla c_e^\varepsilon \cdot \nu = F_c(c_e^\varepsilon) \qquad \text{on } \partial\Omega$$

Plant tissue biomechanics: Chemistry



$$\partial_t b_e^\varepsilon = \mathrm{div}(D_b \nabla b_e^\varepsilon) + g_b(c_e^\varepsilon, b_e^\varepsilon, \mathbf{e}(\mathbf{u}_e^\varepsilon)) \qquad \text{in } \Omega_e^\varepsilon$$

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$$D_e \nabla c_e^\varepsilon \cdot \nu = 0, \qquad (D_f \nabla c_f^\varepsilon - \mathcal{G}(\partial_t \mathbf{u}_f^\varepsilon) c_f^\varepsilon) \cdot \nu = 0 \qquad \text{on } \widetilde{\Gamma}^\varepsilon$$

On the external boundaries :

$$D_b \nabla b_e^\varepsilon \cdot \nu = F_b(b_e^\varepsilon), \qquad D_e \nabla c_e^\varepsilon \cdot \nu = F_c(c_e^\varepsilon) \qquad \text{on } \partial\Omega$$

Macroscopic equations

$$\vartheta_e \partial_t^2 \mathbf{u}_e - \mathrm{div}(\mathbb{E}^{\mathrm{hom}}(b_{e,3})\mathbf{e}(\mathbf{u}_e)) + \nabla p_e + \int_\Omega \partial_t^2 \mathbf{u}_f \, \chi_{\Omega_f} d\mathcal{P}(\omega) = 0 \qquad \text{in } G_\tau$$

$$\vartheta_e \partial_t p_e - \mathrm{div}(K_p^{\mathrm{hom}} \nabla p_e - K_u \partial_t \mathbf{u}_e - Q(\partial_t \mathbf{u}_f)) = 0 \qquad \text{in } G_\tau$$

and

$$\int_\Omega \Big[\partial_t^2 \mathbf{u}_f \, \varphi + \mu \, \mathbf{e}_\omega(\partial_t \mathbf{u}_f) \mathbf{e}_\omega(\varphi) + \nabla p_e \, \varphi\Big] \chi_{\Omega_f} d\mathcal{P}(\omega) - \int_\Omega P_e^1 \, \chi_{\Omega_e} \, \varphi \, d\mathcal{P}(\omega) = 0$$

$$\mathrm{div}_\omega \partial_t \mathbf{u}_f = 0 \quad \text{in } G_T \times \Omega, \qquad \partial_t \mathbf{u}_f(0) = \mathbf{u}_{f0}^1 \quad \text{in } G \times \Omega$$

$$\Pi_\tau \partial_t \mathbf{u}_f(t, x, \mathcal{T}(\widetilde{x})\omega) = \Pi_\tau \partial_t \mathbf{u}_e(t, x) \quad \text{for } (t, x) \in G_T, \ \widetilde{x} \in \Gamma(\omega), \ \mathcal{P}\text{-a.s. in } \Omega$$

$$P_e^1(t, x, \omega) = \sum_{k=1}^{3} \partial_{x_k} p_e(t, x) W_p^k(\omega) + \partial_t u_e^k(t, x) W_u^k(\omega) + Q_f(\omega, \partial_t u_f)$$

$$\forall \, \varphi \in L^2(G_T; H^1(\Omega))^3, \quad \mathrm{div}_\omega \varphi = 0 \text{ in } G_T \times \Omega, \ \Pi_\tau \varphi(t, x, \mathcal{T}(\widetilde{x})\omega) = 0$$
$$\text{for } (t, x) \in G_T, \ \widetilde{x} \in \Gamma(\omega) \text{ and } \mathcal{P}\text{-a.s. in } \Omega$$

$$\mathbf{e}_\omega(\psi)_{jl} = \tfrac{1}{2}\big(\partial_\omega^j \phi_l + \partial_\omega^l \psi_j\big)$$

Macroscopic equations for b and c

$$\vartheta_e \partial_t b - \mathrm{div}(D_{b,\mathrm{eff}} \nabla b) = \int_\Omega g_b(c, b, \mathbb{U}(b,\omega)\mathbf{e}(u_e))\, \chi_{\Omega_e}\, d\mathcal{P}(\omega) + \int_\Omega R(b_e)\, d\boldsymbol{\mu}(\omega) \quad \text{in } G_T$$

$$\partial_t c - \mathrm{div}(D_{\mathrm{eff}} \nabla c - u_{\mathrm{eff}}\, c) = \vartheta_f g_f(c) + \int_\Omega g_e(c, b, \mathbb{U}(b,\omega)\mathbf{e}(u_e))\, \chi_{\Omega_e}\, d\mathcal{P}(\omega) \quad \text{in } G_T$$

where $\vartheta_j = \int_\Omega \chi_{\Omega_j}(\omega)\, d\mathcal{P}(\omega)$, for $j = e, f$, and

$$\mathbb{U}(b,\omega) = \{\mathbb{U}_{klij}(b,\omega)\}_{k,l,i,j=1,2,3} = \left\{ b_{kl}^{ij} + W_{e,\mathrm{sym},kl}^{ij} \right\}_{k,l,i,j=1,2,3}$$

W_e^{ij} solutions of the cell problems, $\mathrm{b}_{kl} = (b_{kl}^{ij})_{i,j=1,2,3}$, $\mathrm{b}_{kl} = e_k \otimes e_l$
$\boldsymbol{\mu}(\omega)$ is the Palm measure of the random measure of the surfaces $\Gamma(\omega)$

Macroscopic tensors

- $\mathbb{E}^{\text{hom}}_{ijkl}(b) = \int_\Omega \big[\widetilde{\mathbb{E}}_{ijkl}(\omega,b) + \big(\widetilde{\mathbb{E}}(\omega,b) W^{kl}_{e,\text{sym}}\big)_{ij}\big]\, \chi_{\Omega_e}\, d\mathcal{P}(\omega)$,
- $K^{\text{hom}}_{ij} = \int_\Omega \big[\widetilde{K}_{ij}(\omega) + \big(\widetilde{K}(\omega) W^{j}_{p}\big)_{i}\big]\, \chi_{\Omega_e}\, d\mathcal{P}(\omega)$,
- $K_{u,ij} = \int_\Omega \big[\delta_{ij} - \big(\widetilde{K}(\omega) W^{j}_{u}\big)_{i}\big]\, \chi_{\Omega_e}\, d\mathcal{P}(\omega)$,
- $Q(\partial_t u_f) = \int_\Omega \partial_t u_f\, \chi_{\Omega_f}\, d\mathcal{P}(\omega) - \int_\Omega \widetilde{K}(\omega) Q_f(\omega, \partial_t u_f)\, \chi_{\Omega_e}\, d\mathcal{P}(\omega)$

- $\int_\Omega \widetilde{\mathbb{E}}(\omega,b)\big(W^{kl}_{e,\text{sym}} + \mathrm{b}_{kl}\big)\, \Phi\, \chi_{\Omega_e}\, d\mathcal{P}(\omega) = 0$ for all $\Phi \in L^2_{\text{pot}}(\Omega)^3$,
- $\int_\Omega \widetilde{K}(\omega)\big(W^{k}_{p} + e_k\big)\, \zeta\, \chi_{\Omega_e}\, d\mathcal{P}(\omega) = 0$ for all $\zeta \in L^2_{\text{pot}}(\Omega)$,
- $\int_\Omega \big(\widetilde{K}(\omega) W^{k}_{u} - e_k\big)\, \zeta\, \chi_{\Omega_e}\, d\mathcal{P}(\omega) = 0$ for all $\zeta \in L^2_{\text{pot}}(\Omega)$,
- $\int_\Omega \big(\widetilde{K}(\omega)\, Q_f\, \chi_{\Omega_e} + \partial_t u_f\, \chi_{\Omega_f}\big)\, \zeta\, d\mathcal{P}(\omega) = 0$ for $\zeta \in L^2_{\text{pot}}(\Omega)$,

with $\mathrm{b}_{kl} = \frac{1}{2}(e_k \otimes e_l + e_l \otimes e_k)$ and $\{e_j\}_{j=1}^3$ – canonical basis of $\mathbb{R}^3$.

Effective diffusion coefficients and velocity

- Macroscopic diffusion coefficients

$$D^{ij}_{b,\mathrm{eff}} = \int_\Omega \left[D^{ij}_b + (D_b\, w^j_b)_i\right] \chi_{\Omega_e}\, d\mathcal{P}(\omega)$$

$$D^{ij}_{\mathrm{eff}} = \int_\Omega \left[D^{ij}(\omega) + \left(D(\omega)\, w^j\right)_i\right] d\mathcal{P}(\omega)$$

where $D(\omega) = D_e \chi_{\Omega_e}(\omega) + D_f \chi_{\Omega_f}(\omega)$ for $\omega \in \Omega$

- $w^j_b \in L^2_{\mathrm{pot}}(\Omega)$, $w^j \in L^2_{\mathrm{pot},\Gamma}(\Omega)$ solutions of the cell problems

$$\int_\Omega D_b(w^j_b + e_j)\,\zeta\,\chi_{\Omega_e}\, d\mathcal{P}(\omega) = 0 \quad \text{for all } \zeta \in L^2_{\mathrm{pot}}(\Omega),$$

$$\int_\Omega D(\omega)(w^j + e_j)\,\eta\, d\mathcal{P}(\omega) = 0 \quad \text{for all } \eta \in L^2_{\mathrm{pot},\Gamma}(\Omega).$$

- Effective velocity

$$u_{\mathrm{eff}}(t,x) = \int_\Omega D_f\, Z(t,x,\omega)\,\chi_{\Omega_f}\, d\mathcal{P}(\omega)$$

- $Z \in L^\infty(G_T; L^2_{\mathrm{pot}}(\Omega))$ satisfies

$$\int_\Omega (D_f Z - \mathcal{G}(\partial_t u_f))\,\zeta\,\chi_{\Omega_f}\, dP(\omega) = 0 \quad \text{for all } \zeta \in L^2_{\mathrm{pot}}(\Omega), \text{ a.e. } (t,x) \in G_T$$

Convergence on boundaries of random microstr.

Lemma For $u \in H^1(\Omega, \mathcal{P})$ we have $u \in L^2(\Omega, \boldsymbol{\mu})$ and the embedding is continuous
$\boldsymbol{\mu}$ is the Palm measure of the random stationary measure μ_ω of surfaces $\Gamma(\omega)$ for realisations $\omega \in \Omega$

Lemma Let μ_ω random measure of $\Gamma(\omega)$ and $d\mu^\varepsilon_\omega(x) = \varepsilon^n d\mu_\omega(x/\varepsilon)$

- For $\boxed{\|b^\varepsilon\|_{L^p(G_e^\varepsilon)} + \|\nabla b^\varepsilon\|_{L^p(G_e^\varepsilon)} \leq C}$ and $b^\varepsilon \to b$ stochastic two-scale, $b \in L^p(0, T; W^{1,p}(G))$, with $p \in (1, \infty)$, then

$$\lim_{\varepsilon \to 0} \int_{G_T} b^\varepsilon(t,x)\, \phi(t,x) \psi(\mathcal{T}(x/\varepsilon)\omega) d\mu^\varepsilon_\omega(x) dt = \int_{G_T} \int_\Omega b(t,x) \phi(t,x) \psi(\omega) d\boldsymbol{\mu}(\omega)\, dxdt \tag{2}$$

  for any $\phi \in C^\infty(0, T; C_0^\infty(\mathbb{R}^d))$ and $\psi \in C(\Omega)$ and

$$\int_{G_T} \int_\Omega |b|^p d\boldsymbol{\mu}(\omega) dxdt \leq C \int_{G_T} \int_\Omega |b|^p d\mathcal{P} dxdt$$

- $\boxed{\|b^\varepsilon\|_{L^p(G_e^\varepsilon)} + \varepsilon \|\nabla b^\varepsilon\|_{L^p(G_e^\varepsilon)} \leq C}$ and $b^\varepsilon \to b$ stochastic two-scale, $b \in L^p(G_T, W^{1,p}(\Omega, d\mathcal{P}))$, with $p \in (1, \infty)$, then we have (2) and

$$\int_{G_T} \int_\Omega |b|^p d\boldsymbol{\mu}(\omega) dxdt \leq C$$

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The ergodic setting

Definition Let $(\Omega, \mathcal{F})$ be a measurable space and $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$
$\tilde{\mu} : \Omega \times \mathcal{B}(\mathbb{R}^d) \to \mathbb{R}_+ \cup \{\infty\}$ is a random measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ if
$\mu_\omega(A) = \tilde{\mu}(\omega, A)$ is

- $\mathcal{F}$-measurable in $\omega \in \Omega$ for each $A \in \mathcal{B}(\mathbb{R}^d)$ and
- a measure in $A \in \mathcal{B}(\mathbb{R}^d)$ for each $\omega \in \Omega$.

Definition The random measure μ_ω is stationary if for $\phi \in C_0^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \phi(y-x) d\mu_\omega(y) = \int_{\mathbb{R}^d} \phi(y) d\mu_{\mathcal{T}(x)\omega}(y)$$

i.e. random function

$$F_\phi(x, \omega) = \int_{\mathbb{R}^d} \phi(y-x) d\mu_\omega(y)$$

is stationary and measurable.

The ergodic theorem for random fields

Definition The *Palm measure* of the random measure μ_ω is a measure $\boldsymbol{\mu}$ on $(\Omega, \mathcal{F})$ defined by:

$$\boldsymbol{\mu}(F) = \int_\Omega \int_{\mathbb{R}^d} \mathbb{I}_{[0,1)^d}(x)\, \mathbb{I}_F(\mathcal{T}(x)\omega)\, d\mu_\omega(x) d\mathcal{P}(\omega), \quad F \in \mathcal{F}$$

- $d\mu_\omega(x) = \rho(\mathcal{T}(x)\omega)dx$ on $\mathbb{R}^d$: $d\boldsymbol{\mu}(\omega) = \rho(\omega)d\mathcal{P}(\omega)$ on Ω

Theorem (Ergodic theorem (see Zhikov & Piatnitsky 2006))

Let $\{\mathcal{T}(x)\}_{x\in\mathbb{R}^n}$ be ergodic and the stationary random measure μ_ω has finite intensity $m(\mu_\omega) > 0$. Then

$$\lim_{t\to\infty} \frac{1}{t^d\,|A|} \int_{tA} g(\mathcal{T}(x)\omega) d\mu_\omega(x) = \int_\Omega g(\omega) d\boldsymbol{\mu}(\omega) \qquad \mathcal{P}\text{-a.s.}$$

for all bounded Borel sets A with $|A| > 0$, and all $g \in L^1(\Omega, \boldsymbol{\mu})$

- For $\boldsymbol{\mu} = \mathcal{P}$ - classical ergodic theorem of Birkhoff

$$\lim_{t\to\infty} \frac{1}{t^d|A|} \int_{tA} g(\mathcal{T}(x)\,\omega) dx = \int_\Omega g(\omega) d\mathcal{P} \quad \mathcal{P}\text{-a.s.}$$

for all bounded Borel sets A with $|A| > 0$, and all $g \in L^1(\Omega, \boldsymbol{\mu})$

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