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# Pseudoparabolic equations with convection

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The existence of solutions of pseudoparabolic equations with convection by using discretization along characteristics is shown. The uniqueness of the solution of a pseudoparabolic equation is proved for a linear elliptic part and for a space dimension $N \leqslant 4$.

*Keywords*: pseudoparabolic equations; Rothe method; method of characteristics; convection–diffusion.

## 1. Introduction

The pseudoparabolic equations are used to model fluid flow in fissured porous media (Barenblatt *et al.*, 1990), two-phase flow in porous media with dynamical capillary pressure (Cuesta *et al.*, 1999; Gray & Hassanizadeh, 1993) and heat conduction in two-temperature systems (Chen & Gurtin, 1968). Pseudoparabolic equations can be used also as regularization of ill-posed transport problems (Barenblatt *et al.*, 1993).

To discretize a pseudoparabolic equation, Crank–Nicolson approximation in time combined with finite-element or finite-difference scheme is used (Ewing, 1975a,b, 1978; Ford & Ting, 1974; Wahlbin, 1975; Gilbert & Lundin, 1983). A predictor–corrector Galerkin approximation is considered in Ford (1976). The Euler–Galerkin method for quasi-linear pseudoparabolic equation is presented in Arnold *et al.* (1981). In special cases, when the differential operator acting upon the time derivative of the solution is invertible and dominates the elliptic operator, the pseudoparabolic equation is equivalent to a Banach-space-valued ordinary differential equation. In this manner, the strong convergence of a Galerkin approximation is proved in Gajewski & Zacharias (1973).

Pseudoparabolic equations with convection are obtained by modelling of two-phase flow in porous media with dynamical capillary pressure. The two phases in this model are water and air. For water in a homogeneous and isotropic porous medium, we have the momentum balance equation (Darcy's law)

$$q = -K(S)(\nabla p_{\mathrm{w}} + \rho g) \tag{1.1}$$

and the mass balance equation

$$\phi \partial_t (\rho S) + \nabla \cdot (\rho q) = 0. \tag{1.2}$$

Here, $q$ denotes the volumetric water flux, $S$ the water saturation, $K(S)$ the hydraulic conductivity, $p_{\mathrm{w}}$ the water pressure, $\rho$ the water density, $\phi$ the porosity and $g$ a gravity constant. To solve these equations, an additional relation between $p_{\mathrm{w}}$ and $S$ is needed. For this relation, it is assumed that the air pressure $p_{\mathrm{a}}$ is constant and the static condition holds:

$$p_{\mathrm{a}} - p_{\mathrm{w}} = p_{\mathrm{c}}(S),$$

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where $p_c$ denotes the capillary pressure. For the processes with slowly monotonically varying water saturation, this condition can be accepted. For the fast processes, e.g. capillary imbibition, the hysteresis and dynamical effect are important. To derive the dynamical relation between saturation $S$ and pressure difference $p_a - p_w$, Gray & Hassanizadeh (1993) gave a definition of the capillary pressure $p_c(S)$ as a thermodynamic parameter in terms of the free-energy functions of the phases, independent of $p_a - p_w$, and obtain the equation

$$p_a - p_w = p_c(S) - \phi L \partial_t S. \tag{1.3}$$

Now, from (1.1–1.3), a single equation for the water saturation can be obtained:

$$\phi \partial_t(\rho S) = \nabla \cdot \{\rho K(S)\rho g + \rho K(S)\nabla(-p_c(S) + \phi L \partial_t S)\}. \tag{1.4}$$

We consider a simplified version of (1.4) assuming linearity of the pseudoparabolic term

$$\partial_t u - \nabla \cdot (a(x)\partial_t \nabla u) + c(t,x,u) \cdot \nabla u - \nabla \cdot (d(t,x,u)\nabla u) = f(t,x,u).$$

The existence of solution of this pseudoparabolic equation with appropriate initial and boundary conditions can be shown using Rothe or Galerkin discretization method. In order to obtain a good numerical approximation for the convection term, a discretization along characteristics is used. Such type of discretization was used for parabolic equations in Douglas & Russell (1982), Dawson *et al.* (1994), Arbogast & Wheeler (1995), Bermejo (1995), Barrett & Knabner (1998), Kacur (2001) and Kacur & Keer (2001). An approximate solution is obtained as a solution to a discretized differential equation along the approximated characteristics. The change of the solution of the problem with convection along the characteristics is small compared to the change of the solution in time. Thus, the discretization along characteristics allows for large time steps in the time discretization.

In this article, the discretization along characteristics is applied to a pseudoparabolic equation with convection. The convergence of approximative solutions to the solution of the original problem is shown. The uniqueness is proved for a linear elliptic part, for a space dimension $N \leqslant 4$ and for Lipschitz continuous non-linear functions.

## 2. Pseudoparabolic equation with convection

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz continuous boundary. In $Q_T = (0,T) \times \Omega$, the initial boundary-value problem is given by

$$\begin{cases} \partial_t u - \nabla \cdot (a(x)\partial_t \nabla u) + c(t,x,u) \cdot \nabla u - \nabla \cdot (d(t,x,u)\nabla u) = f(t,x,u), \\ u(0,x) = u_0(x), \quad x \in \Omega, \\ u(t,x) = 0, \quad (t,x) \in (0,T) \times \partial\Omega. \end{cases} \tag{2.1}$$

The existence of a solution will be ensured by the following assumptions.

ASSUMPTION 2.1

- A1. The matrix field $a \in L^\infty(\Omega)^{N\times N}$ is symmetric and elliptic, i.e. for some $a_0$ and $a^0$, $0 < a_0 \leqslant a^0 < \infty$, $a$ satisfies $a_0|\xi|^2 \leqslant a(x)\xi\,\xi \leqslant a^0|\xi|^2$ for a.a. $x \in \Omega$ and for $\xi \in \mathbb{R}^N$.
- A2. The function $c$: $(0,T) \times \Omega \times \mathbb{R} \to \mathbb{R}^N$ is continuous and bounded $|c(t,x,z)| \leqslant c^0 < \infty$.
- A3. The matrix field $d$: $(0,T) \times \Omega \times \mathbb{R} \to \mathbb{R}^{N\times N}$ is continuous, elliptic, i.e. there exists some $d_0 > 0$ such that $d$ satisfies $d(t,x,z)\xi\xi \geqslant d_0|\xi|^2$ for $\xi \in \mathbb{R}^N$, and bounded, i.e. for some $d^0 < \infty$, $|d(t,x,z)| \leqslant d^0$ for almost all $(t,x) \in Q_T$ and $z \in \mathbb{R}$.

A4. The function $f\colon (0,T)\times\Omega\times\mathbb{R}\to\mathbb{R}$ is continuous and sublinear, i.e. $|f(t,x,z)|\leqslant C(1+|z|)$ for almost all $(t,x)\in Q_T$ and for $z\in\mathbb{R}$.

A5. The initial condition $u_0$ is in $H_0^1(\Omega)$.

DEFINITION 2.1 A function $u\colon Q_T\to\mathbb{R}$ is called a weak solution of (2.1) if $u\in H^1(0,T;H_0^1(\Omega))$, $u$ satisfies the initial condition, i.e. $u(t)\to u_0$ in $H_0^1(\Omega)$ as $t\to 0$, and $u$ satisfies the equality

$$\int_{Q_T} u_t v\,\mathrm{d}x\,\mathrm{d}t+\int_{Q_T} a(x)\nabla u_t\nabla v\,\mathrm{d}x\,\mathrm{d}t+\int_{Q_T} c(t,x,u)\nabla u v\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{Q_T} d(t,x,u)\nabla u\nabla v\,\mathrm{d}x\,\mathrm{d}t=\int_{Q_T} f(t,x,u)v\,\mathrm{d}x\,\mathrm{d}t \tag{2.2}$$

for all test functions $v\in L^2(0,T;H_0^1(\Omega))$.

The main theorem of this section contains the existence of such a solution.

THEOREM 2.1 (Existence). Under Assumption 2.1, there exists a solution of Problem (2.1).

At first, we explain the discretization method.

Equation (2.1) is of the form

$$\begin{cases}\partial_t u+v\cdot\nabla u-A(u)=f(t,x,u) & \text{in } Q_T,\\ u=0 & \text{on } (0,T)\times\partial\Omega,\\ u(0,x)=u_0 & \text{in } \Omega,\end{cases} \tag{2.3}$$

where $v(t,x)=c(t,x,u(t,x))$. Due to the characteristic method, the basic structure of the in-time discretized equation reads

$$\frac{u_i-u_{i-1}\circ\phi^i}{h}-A(u_i)=f(t_i,x,u_{i-1}),$$

where $\phi^i(x)=x-hv(t_i,x)$ is an approximation of $X(t_{i-1},t_i,x)$ for $h=T/n$, $t_i=ih$, $i=0,\dots,n$, and $X$ satisfies

$$\partial_t X(t,s,x)=v(t,X(t,s,x)),\qquad X(s,s,x)=x.$$

To make this idea work, there are some subtleties to be considered.

It is substantial that the characteristics $X$ do not intersect; otherwise, neither the backward transport $X(t_{i-1},t_i,x)$ nor $\phi^i(x)$ can be shown to exist. Provided

$$\|\nabla v(t)\|_{L^\infty(\Omega)}\leqslant c\quad \text{for all } t\in(0,T),$$

and therefore $\det(D\phi^i(x))\geqslant 1-hc>0$, the backward transport exists. However, this estimate may not be satisfied. To circumvent this problem, we consider for $\tau=h^\omega$, $0<\omega<1$, the smoothed version of $v_i(x):=v(t_i,x)$ by $v_i^\tau:=w_\tau * v_i$, where $w_\tau(x)=\frac{1}{\tau^N}w_1\left(\frac{x}{\tau}\right)$,

$$w_1(x)=\begin{cases}\kappa\exp\left(\frac{|x|^2}{|x|^2-1}\right), & \text{for } |x|\leqslant 1,\\ 0, & \text{otherwise,}\end{cases}\quad\text{and}\quad \int_{\mathbb{R}^N} w_1(x)\mathrm{d}x=1. \tag{2.4}$$

This concept will guarantee that $\|\nabla v_i^\tau\|_{L^\infty(\Omega)}$ will be uniformly bounded in $i = 1, \ldots, n$ for each fixed $\tau$.

Choose

$$\Omega_h = \{x \in \mathbb{R}^N, \operatorname{dist}(x, \Omega) < h\|v\|_{L^\infty(Q_T)}\}.$$

Then, $\overline{\Omega} = \cap_{h>0}\Omega_h$. Fix some $h^* > 0$ and $\Omega^* = \Omega_{h^*}$. Let $\Omega_i = \phi^i(\Omega)$. The boundedness of $v$ yields $\Omega_i \subset \Omega_h \subset \Omega^*$ for $h \leqslant h^*$. Since $h^* > 0$, there exists an extension $\tilde{u}_{i-1}$ of $u_{i-1}$ from $\Omega$ to $\Omega^*$, satisfying $\|\tilde{u}_{i-1}\|_{H^1(\Omega^*)} \leqslant c\|u_{i-1}\|_{H^1(\Omega)}$ uniformly in $u$. The function $u_{i-1}$ from $H_0^1(\Omega)$ can be extended by zero to a function $\tilde{u}_{i-1} \in H_0^1(\Omega^*)$ and $\|\tilde{u}_{i-1}\|_{H_0^1(\Omega^*)} \leqslant \|u_{i-1}\|_{H_0^1(\Omega)}$. This construction allows us to assume that $\tilde{u}_{i-1}$ is defined on all $\Omega_i$. Especially, $\tilde{u}_{i-1} \circ \phi^i$ is well defined.

We approximate the differential equation (2.1) by the time discretization, $h = T/n$, $t_i = ih$, $i = 0, \ldots, n$, and obtain

$$\frac{1}{h}(u_i - \tilde{u}_{i-1} \circ \phi^i) - \nabla \cdot \left(a(x)\frac{1}{h}\nabla(u_i - u_{i-1})\right) - \nabla \cdot (d(t_i, x, u_{i-1})\nabla u_i) = f(t_i, x, u_{i-1}),$$

$$u_i(x) = 0 \quad \text{on } \partial\Omega, \tag{2.5}$$

where $\phi^i(x) := x - hv_i^\tau(x)$ and $v_i(x) = c(t_i, x, u_{i-1})$. It is equivalent to

$$-\nabla \cdot \left(\left(a(x)\frac{1}{h} + d(t_i, x, u_{i-1})\right)\nabla u_i\right) + \frac{1}{h}u_i = f(t_i, x, u_{i-1}) + \frac{1}{h}\tilde{u}_{i-1} \circ \phi^i - \frac{1}{h}\nabla \cdot (a(x)\nabla u_{i-1}).$$

The existence and uniqueness of the solution $u_i$ of elliptic problems (2.5) follow from Lax–Milgram theorem (Evans, 1998).

In the proof of the *a priori* estimates, we use the following lemma.

LEMMA 2.1 (Kacur, 2001). There exists a $h_0 > 0$ such that $\phi^i$ is one to one and

$$\frac{1}{2}|x - y| \leqslant |\phi^i(x) - \phi^i(y)| \leqslant 2|x - y|, \quad \text{for all } x, y \in \Omega, \tag{2.6}$$

uniformly in $n$, $i = 1, \ldots, n$, and $h \leqslant h_0$.

*Proof.* Due to $\|v_i\|_{L^\infty(\Omega)} \leqslant C < \infty$, we have

$$\|v_i^\tau\|_{L^\infty(\Omega)} \leqslant C$$

and

$$\|\nabla v_i^\tau\|_{L^\infty(\Omega)} \leqslant C/\tau.$$

Since $\tau = h^\omega$ and $0 < \omega < 1$, we obtain for $\phi^i$,

$$(1 - h^{1-\omega}C)|x - y| \leqslant |\phi^i(x) - \phi^i(y)| \leqslant (1 + h^{1-\omega}C)|x - y|.$$

□

Now, we prove *a priori* estimates for $u_i$.

LEMMA 2.2 The estimates

$$\max_{1\leqslant j\leqslant n}\int_\Omega (|u_j|^2+|\nabla u_j|^2)\mathrm{d}x\leqslant C,$$
$$\sum_{i=1}^n h\int_\Omega |\nabla u_i|^2\,\mathrm{d}x\leqslant C \tag{2.7}$$

hold uniformly in $n$.

*Proof.* Testing (2.5) with $u_i$ and summing over $i$ yield

$$\sum_{i=1}^j \frac{1}{h}\int_\Omega (u_i-u_{i-1})u_i\,\mathrm{d}x+\sum_{i=1}^j \frac{1}{h}\int_\Omega (u_{i-1}-\tilde{u}_{i-1}\circ\phi^i)u_i\,\mathrm{d}x$$
$$+\sum_{i=1}^j \frac{1}{h}\int_\Omega a(x)\nabla(u_i-u_{i-1})\nabla u_i\,\mathrm{d}x$$
$$+\sum_{i=1}^j\int_\Omega d(t_i,x,u_{i-1})\nabla u_i\nabla u_i\,\mathrm{d}x=\sum_{i=1}^j\int_\Omega f(t_i,x,u_{i-1})u_i\,\mathrm{d}x.$$

Due to Assumption 2.1, Abel's summation formula and multiplication with $h$, we obtain

$$\int_\Omega |u_j|^2\,\mathrm{d}x+a_0\int_\Omega|\nabla u_j|^2\,\mathrm{d}x+d_0\sum_{i=1}^j h\int_\Omega|\nabla u_i|^2\,\mathrm{d}x$$
$$\leqslant\int_\Omega|u_0|^2\,\mathrm{d}x+a^0\int_\Omega|\nabla u_0|^2\,\mathrm{d}x+\sum_{i=1}^j\int_\Omega|(u_{i-1}-\tilde{u}_{i-1}\circ\phi^i)u_i|\mathrm{d}x+c_1\sum_{i=1}^j h\int_\Omega|u_i|^2\,\mathrm{d}x+c_2. \tag{2.8}$$

To estimate the third integral on the right-hand side, we use the equality

$$u_{i-1}-\tilde{u}_{i-1}\circ\phi^i=\int_0^1\nabla\tilde{u}_{i-1}(x+s(\phi^i(x)-x))\mathrm{d}s\,v_i^\tau(x)h.$$

Integration over $\Omega$ and boundedness of $v_i^\tau$ yield

$$\int_\Omega|u_{i-1}-\tilde{u}_{i-1}\circ\phi^i|^2\,\mathrm{d}x\leqslant C\int_0^1\int_\Omega|\nabla\tilde{u}_{i-1}(x+s(\phi^i(x)-x))|^2\,\mathrm{d}x\,\mathrm{d}s h^2.$$

Changing to the new variable $y=x+s(\phi^i(x)-x)$, using $y\in\Omega_i\subset\Omega^*$ and the monotonicity of the integral and applying the estimate $|\det D\phi(x)|\geqslant\frac{1}{2^N}$ yield

$$\int_\Omega|u_{i-1}-\tilde{u}_{i-1}\circ\phi^i|^2\,\mathrm{d}x\leqslant Ch^2\int_0^1\int_{\Omega^*}|\nabla\tilde{u}_{i-1}(y)|^2\,\mathrm{d}y\,\mathrm{d}s.$$

From the boundedness of the extension operator, it follows that

$$\|u_{i-1} - \tilde{u}_{i-1} \circ \phi^i\|_{L^2(\Omega)} \leqslant Ch\|\nabla \tilde{u}_{i-1}\|_{L^2(\Omega^*)} \leqslant C_1 h\|\nabla u_{i-1}\|_{L^2(\Omega)}.$$

Using this estimate yields

$$\sum_{i=1}^{j} \int_\Omega |(u_{i-1} - \tilde{u}_{i-1} \circ \phi^i)u_i| \mathrm{d}x \leqslant c_1 \sum_{i=1}^{j} h \int_\Omega |\nabla u_i|^2 \,\mathrm{d}x + c_2 \sum_{i=1}^{j} h \int_\Omega |u_i|^2 \,\mathrm{d}x.$$

Then, we obtain the inequality

$$\int_\Omega |u_j|^2 \,\mathrm{d}x + a_0 \int_\Omega |\nabla u_j|^2 \,\mathrm{d}x + d_0 \sum_{i=1}^{j} h \int_\Omega |\nabla u_i|^2 \,\mathrm{d}x \leqslant c_3 + c_4 \sum_{i=1}^{j} h \int_\Omega (|u_i|^2 + |\nabla u_i|^2) \mathrm{d}x.$$

Due to the discrete Gronwall lemma, we obtain the estimates (2.7). □

LEMMA 2.3 The estimate

$$\sum_{i=1}^{n} h \int_\Omega (|\partial_h u_i|^2 + |\partial_h \nabla u_i|^2) \mathrm{d}x \leqslant C \tag{2.9}$$

holds uniformly in $n$, where $\partial_h u_i := \frac{u_i - u_{i-1}}{h}$.

*Proof.* We test (2.5) with $u_i - u_{i-1}$, sum up over $i$ and obtain the equality

$$\sum_{i=1}^{j} h \int_\Omega \frac{u_i - \tilde{u}_{i-1} \circ \phi^i}{h} \partial_h u_i \,\mathrm{d}x + \sum_{i=1}^{j} h \int_\Omega a(x) \nabla \partial_h u_i \nabla \partial_h u_i \,\mathrm{d}x$$

$$+ \sum_{i=1}^{j} h \int_\Omega d(t_i, x, u_{i-1}) \nabla u_i \nabla \partial_h u_i \,\mathrm{d}x = \sum_{i=1}^{j} h \int_\Omega f(t_i, x, u_{i-1}) \partial_h u_i \,\mathrm{d}x.$$

By Assumption 2.1, we have the inequality

$$\sum_{i=1}^{j} h \int_\Omega |\partial_h u_i|^2 \,\mathrm{d}x + a_0 \sum_{i=1}^{j} h \int_\Omega |\nabla \partial_h u_i|^2 \,\mathrm{d}x$$

$$\leqslant c_1 \delta \sum_{i=1}^{j} h \int_\Omega |\nabla \partial_h u_i|^2 \,\mathrm{d}x + \frac{c_2 d^0}{\delta} \sum_{i=1}^{j} h \int_\Omega |\nabla u_i|^2 \,\mathrm{d}x + c_3 \delta \sum_{i=1}^{j} h \int_\Omega |\partial_h u_i|^2 \,\mathrm{d}x$$

$$+ \frac{c_4}{\delta} \sum_{i=1}^{j} h \int_\Omega |f(t_i, x, u_{i-1})|^2 \,\mathrm{d}x + \frac{c_5}{\delta} \sum_{i=1}^{j} h \int_\Omega \left| \frac{u_{i-1} - \tilde{u}_{i-1} \circ \phi^i}{h} \right|^2 \mathrm{d}x.$$

Similarly to Lemma 2.2, we obtain

$$\frac{u_{i-1}(x) - \tilde{u}_{i-1} \circ \phi^i(x)}{h} = \int_0^1 \nabla \tilde{u}_{i-1}(x + s(\phi^i(x) - x)) \mathrm{d}s v_i^\tau$$

and

$$\left\| \frac{u_{i-1} - \tilde{u}_{i-1} \circ \phi^i}{h} \right\|_{L^2(\Omega)} \leqslant C \|\nabla \tilde{u}_{i-1}\|_{L^2(\Omega^*)} \leqslant C \|\nabla u_{i-1}\|_{L^2(\Omega)}.$$

Then, we have the inequality

$$\sum_{i=1}^{j} h \int_\Omega |\partial_h u_i|^2 \,\mathrm{d}x + \sum_{i=1}^{j} h \int_\Omega |\nabla \partial_h u_i|^2 \,\mathrm{d}x \leqslant C_1 \sum_{i=1}^{j} h \int_\Omega |u_i|^2 \,\mathrm{d}x + C_2 \sum_{i=1}^{j} h \int_\Omega |\nabla u_i|^2 \,\mathrm{d}x.$$

Due to the estimates in Lemma 2.2, this inequality implies the estimate for the discrete time derivative $\partial_h u_i$. □

*Proof of Theorem 2.1.* By using the *a priori* estimates for $u_i$ and $\partial_h u_i$, we will show the convergence of an appropriate subsequence of the approximate solutions to a solution of the original problem (2.1).

Therefore, we define the Rothe functions piecewise for $t \in (t_{i-1}, t_i]$ and $x \in \Omega$ by

$$u^n(t, x) := u(t_{i-1}, x) + (t - t_{i-1}) \frac{u(t_i, x) - u(t_{i-1}, x)}{h}$$

and the step functions by

$$\bar{u}^n(t, x) := u(t_i, x),$$

where the initial conditions are $u^n(0, x) = u_0(x)$ and $\bar{u}^n(0, x) = u_0(x)$. From (2.7) and (2.9), we have the estimates

$$\sup_{0 \leqslant t \leqslant T} \int_\Omega (|\bar{u}^n|^2 + |\nabla \bar{u}^n|^2) \mathrm{d}x \leqslant C, \quad \int_{Q_T} |\nabla \bar{u}^n|^2 \,\mathrm{d}x \,\mathrm{d}t \leqslant C, \tag{2.10}$$

$$\int_{Q_T} (|\partial_h u^n|^2 + |\nabla \partial_h u^n|^2) \mathrm{d}x \,\mathrm{d}t \leqslant C, \quad \int_{Q_T} (|u^n - \bar{u}^n|^2 + |\nabla u^n - \nabla \bar{u}^n|^2) \mathrm{d}x \,\mathrm{d}t \leqslant \frac{C}{n^2}.$$

These estimates imply the existence of subsequences of $\{u^n\}$ and $\{\bar{u}^n\}$, respectively, again denoted by $\{u^n\}$ and $\{\bar{u}^n\}$, respectively, such that

$$\begin{aligned} \bar{u}^n &\to u \text{ weakly-}* \text{ in } L^\infty(0, T; H_0^1(\Omega)), \\ \bar{u}^n &\to u \text{ weakly in } L^2(0, T; H_0^1(\Omega)), \\ \partial_h u^n &\to \partial_t u \text{ weakly in } L^2(0, T; H_0^1(\Omega)), \end{aligned} \tag{2.11}$$

where $\partial_h u^n(t) := \frac{u^n(t) - u^n(t-h)}{h}$ and $u^n(t - h) = u_0$ for $t \in [0, h]$. Using the compactness Aubin–Lions lemma (see Lions, 1969) implies that $\bar{u}^n \to u$ strongly in $L^2(Q_T)$. Due to Evans (1998, Theorem 5.9.2) and $u \in H^1(0, T; H_0^1(\Omega))$, we obtain $u \in C([0, T]; H_0^1(\Omega))$ and $u(0) = u_0$.

Testing the discrete equation (2.5) with $v \in L^2(0, T; H_0^1(\Omega))$ yields

$$\begin{aligned} &\int_{Q_T} \partial_h u^n v \,\mathrm{d}x \,\mathrm{d}t + \int_{Q_T} \partial_h \nabla u^n \nabla v \,\mathrm{d}x \,\mathrm{d}t + \int_{Q_T} d_n(t, x, \bar{u}_h^n) \nabla \bar{u}^n \nabla v \,\mathrm{d}x \,\mathrm{d}t \\ &+ \int_{Q_T} \frac{1}{h} \left( \bar{u}_h^n - \tilde{\bar{u}}_h^n \circ \phi^n \right) v \,\mathrm{d}x \,\mathrm{d}t = \int_{Q_T} f_n(t, x, \bar{u}_h^n) v \,\mathrm{d}x \,\mathrm{d}t, \end{aligned} \tag{2.12}$$

where $\phi^n(t,x) = x - h\omega_\tau * c_n(t,x,\bar{u}_h^n)$, $\bar{u}_h^n(t,x) = \bar{u}^n(t-h,x)$, $c_n(t,x,z) = c(t_i,x,z)$, $d_n(t,x,z) = d(t_i,x,z)$ for $t \in (t_{i-1},t_i]$, for $i = 1,\ldots,n$, and $c_n(0,x,z) = c(0,x,z)$, $d_n(0,x,z) = d(0,x,z)$. The strong convergence of $\bar{u}^n$ and the last estimate in (2.10) imply that $\bar{u}_h^n \to u$ strongly in $L^2(Q_T)$ and $\bar{u}_h^n \to u$ a.e. in $Q_T$. The continuity of $d(t,x,z)$ in $t$ and $z$ and the convergence of $\bar{u}_h^n$ a.e. in $Q_T$ imply that $d_n(t,x,\bar{u}_h^n) \to d(t,x,u)$ a.e. in $Q_T$. Due to $\bar{u}^n \to u$ weakly in $L^2(0,T;H_0^1(\Omega))$ and boundedness of $d_n(t,x,\bar{u}_h^n)$ and $d(t,x,u)$, we obtain strong convergence of $d_n(t,x,\bar{u}_h^n)$ to $d(t,x,u)$ in $L^2(Q_T)$ and weak convergence of $d_n(t,x,\bar{u}_h^n)\nabla\bar{u}^n$ in $L^2(Q_T)$ to $d(t,x,u)\nabla u$ since

$$\int_{Q_T} d_n(t,x,\bar{u}_h^n)\nabla\bar{u}^n\varphi\,\mathrm{d}x\,\mathrm{d}t \to \int_{Q_T} d(t,x,u)\nabla u\varphi\,\mathrm{d}x\,\mathrm{d}t$$

for any smooth $\varphi$. The convergence of $f_n(t,x,\bar{u}_h^n) \to f(t,x,u)$ a.e. in $Q_T$ follows from the continuity of $f$ and the a.e. convergence of $\bar{u}_h^n$ in $Q_T$. Due to the sublinearity of $f$ and the dominated convergence theorem (Evans, 1998), we obtain $f_n(t,x,\bar{u}_h^n) \to f(t,x,u)$ in $L^2(Q_T)$. The continuity of $c$ implies that $c_n(t,x,\bar{u}_h^n) \to c(t,x,u)$ a.e. in $Q_T$. From the boundedness of $c_n(t,x,\bar{u}_h^n)$ and $c(t,x,u)$ in $L^\infty(Q_T)$ follows $c_n(t,x,\bar{u}_h^n) \to c(t,x,u)$ strongly in $L^2(Q_T)$ and $c_n(t,x,\bar{u}_h^n)$ converges weakly-$*$ in $L^\infty(Q_T)$.

Now, we have to prove that

$$\int_{Q_T} \frac{1}{h}\left(\bar{u}_h^n - \widetilde{\bar{u}}_h^n \circ \phi^n\right) v\,\mathrm{d}x\,\mathrm{d}t \to \int_{Q_T} c(t,x,u)\nabla u v\,\mathrm{d}x\,\mathrm{d}t$$

for $n \to \infty$, where $h = \frac{T}{n}$. The equality

$$\int_{Q_T} \frac{1}{h}\left(\bar{u}_h^n - \widetilde{\bar{u}}_h^n \circ \phi^n\right) v\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T}\int_0^1 \nabla\widetilde{\bar{u}}_h^n(x + s(\phi^n(t,x)-x))\mathrm{d}s\,\omega_\tau * c_n(t,x,\bar{u}_h^n) v\,\mathrm{d}x\,\mathrm{d}t$$

holds. Since $c_n(t,x,\bar{u}_h^n) \to c(t,x,u)$ a.e. in $Q_T$, we have $\omega_\tau * c_n(t,x,\bar{u}_h^n) \to c(t,x,u)$ a.e. in $Q_T$ as $n \to \infty$.

The assumed boundedness of $c$ yields

$$\|\omega_\tau * c_n(t,x,\bar{u}_h^n)\|_{L^\infty(Q_T)} \leqslant c^0.$$

We need to show that $\nabla z_n \to \nabla u$ weakly in $L^2(Q_T)$, where

$$\nabla z_n(t,x) := \int_0^1 \nabla\widetilde{\bar{u}}_h^n(t, x + s(\phi^n(t,x)-x))\mathrm{d}s.$$

Due to

$$\int_{Q_T} |\nabla z_n|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C_1,$$

there exists a $\chi \in L^2(Q_T)$ such that $\nabla z_n \to \chi$ weakly in $L^2(Q_T)$. Now, we show that $z_n \to u$ in $L^2(Q_T)$. Integrating the difference

$$\begin{aligned} z_n(t,x) - \bar{u}_h^n(t,x) &= \int_0^1 \left(\widetilde{\bar{u}}_h^n(t, x + s(\phi^n(t,x)-x)) - \bar{u}_h^n(t,x)\right)\mathrm{d}s \\ &= \int_0^1\int_0^1 \nabla\widetilde{\bar{u}}_h^n(t, x + sr(\phi^n(t,x)-x))\mathrm{d}s\,\mathrm{d}r\,\omega_\tau * c_n(t,x,\bar{u}_h^n)h \end{aligned}$$

over $Q_T$ and using the boundedness of $c_n$ imply that

$$\int_{Q_T} |z_n(t,x) - \bar{u}_h^n(t,x)|^2 \, \mathrm{d}x \, \mathrm{d}t \leqslant c^0 h^2 \int_0^1 \int_0^1 \int_{Q_T} \left| \nabla \widetilde{\bar{u}}_h^n(t, x + sr(\phi^n(t,x) - x)) \right|^2 \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}s \, \mathrm{d}r.$$

From the boundedness of the extension operator and the *a priori* estimates for $\bar{u}_h^n$, it follows that

$$\left\| \nabla \widetilde{\bar{u}}_h^n \right\|_{L^2((0,T)\times\Omega^*)} \leqslant C_2 \| \nabla \bar{u}_n^h \|_{L^2(Q_T)} \leqslant C_3.$$

Then, we have

$$\int_0^T \int_\Omega |z_n(t,x) - \bar{u}_h^n(t,x)|^2 \, \mathrm{d}x \, \mathrm{d}t \leqslant C h^2.$$

Due to the fact that $\bar{u}_h^n \to u$ in $L^2(Q_T)$, we obtain $z_n \to u$ in $L^2(Q_T)$. Then, $\nabla z_n \to \chi$ weakly in $L^2(Q_T)$ implies that $\chi = \nabla u$. Passing (2.12) to the limit as $n \to \infty$, it follows that the function $u$ is a solution of Problem (2.1). □

THEOREM 2.2 (Uniqueness). Let Assumption 2.1 be satisfied, where $d$ depends only on time and space. Let $N \leqslant 4$ and

$$|f(t,x,z^1) - f(t,x,z^2)| \leqslant C|z^1 - z^2|, \quad |c(t,x,z^1) - c(t,x,z^2)| \leqslant C|z^1 - z^2|$$

for $z^1, z^2 \in \mathbb{R}$ and $(t,x) \in Q_T$. Then, there exists at most one weak solution of (2.1).

*Proof.* Suppose that $u_1$ and $u_2$ solve Problem (2.1). Then, the difference $u = u_1 - u_2$ satisfies the equality

$$\int_{Q_\tau} u_t v \, \mathrm{d}x \, \mathrm{d}t + \int_{Q_\tau} a(x) \nabla u_t \nabla v \, \mathrm{d}x \, \mathrm{d}t + \int_{Q_\tau} (c(t,x,u_1)\nabla u_1 - c(t,x,u_2)\nabla u_2) v \, \mathrm{d}x \, \mathrm{d}t$$
$$+ \int_{Q_\tau} d(t,x) \nabla u \nabla v \, \mathrm{d}x \, \mathrm{d}t = \int_{Q_\tau} (f(t,x,u_1) - f(t,x,u_2)) v \, \mathrm{d}x \, \mathrm{d}t. \tag{2.13}$$

We choose the test function $v = u$. The third integral in the last equality is estimated by

$$\int_{Q_\tau} (c(t,x,u_1)\nabla u_1 - c(t,x,u_2)\nabla u_2) u \, \mathrm{d}x \, \mathrm{d}t$$
$$= \int_{Q_\tau} c(t,x,u_1) \nabla u \, u \, \mathrm{d}x \, \mathrm{d}t + \int_{Q_\tau} (c(t,x,u_1) - c(t,x,u_2)) \nabla u_2 u \, \mathrm{d}x \, \mathrm{d}t$$
$$\leqslant c_1 \int_{Q_\tau} |u|^2 \, \mathrm{d}x \, \mathrm{d}t + c_2 \int_{Q_\tau} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t + c_3 \left( \int_{Q_\tau} |u|^4 \, \mathrm{d}x \, \mathrm{d}t \right)^{\frac{1}{2}} \left( \int_{Q_\tau} |\nabla u_2|^2 \, \mathrm{d}x \, \mathrm{d}t \right)^{\frac{1}{2}}.$$

Sobolev's embedding theorem yields

$$\left( \int_{Q_\tau} |u|^4 \, \mathrm{d}x \, \mathrm{d}t \right)^{\frac{1}{2}} \leqslant c_4 \int_{Q_\tau} |u|^2 \, \mathrm{d}x \, \mathrm{d}t + c_5 \int_{Q_\tau} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t$$

since $u \in L^\infty(0,T;H_0^1(\Omega))$ and $N \leqslant 4$. Applying these estimates, ellipticity of $a$ and $d$ and the Lipschitz continuity of $f$ to (2.13) implies that

$$\int_\Omega (|u(\tau)|^2 + |\nabla u(\tau)|^2)\mathrm{d}x \leqslant C \int_{Q_\tau} (|u|^2 + |\nabla u|^2)\mathrm{d}x\,\mathrm{d}t.$$

Due to Gronwall's lemma, we obtain

$$\int_\Omega (|u(\tau)|^2 + |\nabla u(\tau)|^2)\mathrm{d}x \leqslant 0$$

and $u_1 = u_2$ almost everywhere in $Q_T$. □

REMARK 2.1 Here, the zero Dirichlet boundary conditions were considered. This restriction is not essential and the results can be obtained also for other boundary conditions.

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