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Degenerate quasilinear pseudoparabolic equations with memory terms and variational inequalities[☆]

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Abstract

The existence of solutions of degenerate quasilinear pseudoparabolic equations, where the term $\partial_t u$ is replace by $\partial_t b(u)$, with memory terms and quasilinear variational inequalities is shown. The existence of solutions of equations is proved under the assumption that the nonlinear function b is monotone and a gradient of a convex, continuously differentiable function. The uniqueness is proved for Lipschitz-continuous elliptic parts. The existence of solutions of quasilinear variational inequalities is proved under stronger assumptions, namely, the nonlinear function defining the elliptic part is assumed to be a gradient and the function b to be Lipschitz continuous.

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Keywords: Pseudoparabolic equations; Equations with memory terms; Degenerate parabolic equations

1. Introduction

The pseudoparabolic equations are used to model fluid flow in fissured porous media [3], two-phase flow in porous media with dynamical capillary pressure [7,11], and heat conduction in two-temperature systems [5]. Memory terms are obtained by modeling the flow in elastic media [6, 12,17]. Pseudoparabolic equations can be used also as regularization of ill-posed transport problems [2,19]. Pseudoparabolic variational inequalities appear in obstacle problems [20], and free boundary problems [9]. Degenerate evolution equations of the form $\frac{\mathrm{d}}{\mathrm{d}t}A(u) + B(u) \ni f$,

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where A and B are maximal monotone operators from a Hilbert space V to its dual V^*, are considered in [8]. Existence of a solution is proved by Yosida's approximation, if one of the operators is strongly monotone and the other is a subgradient, $A : V \to V^*$ is compact, one of the operators is coercive, $B : V \to V^*$ is bounded, and $A : V \to W^*$ is bounded, where W is a reflexive Banach space such that V is densely and compactly embedded in W. Corresponding nonlinear pseudoparabolic variational inequalities were studied in [9]. To prove the existence of a solution the penalty method is used. The existence is shown if $A : V \to V$ is compact perturbation of the identity and $B : V \to V$ is bounded. The uniqueness of the solution is known if the operator A is linear and self-adjoint and the operator B is strictly monotone. In [20] existence of a solution of a degenerate linear and in [15] of degenerate quasilinear pseudoparabolic variational inequality is proved by a regularization method combined with Galerkin's method.

In this article the existence of solutions of a system of degenerate quasilinear equations with memory terms is shown by Rothe–Galerkin's method. The crucial assumptions to guarantee existence are monotonicity and potentiality of the nonlinear function b, see [1]. The discretization of integral operators is used similarly to [13], i.e. we define an approximation as a function, piecewise constant on a partition of the time interval. The uniqueness follows by the strong monotonicity of the involved operators and can only be shown for the Lipschitz continuous elliptic part. To prove existence of solutions of inequalities a penalty method is used.

2. Quasilinear equations

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. In $Q_T = (0, T) \times \Omega$ the initial boundary value problem is given by

$$\begin{cases} \partial_t b^j(u) - \nabla \cdot (a(x)\nabla u_t)^j - \nabla \cdot d^j(t, x, u, \nabla u) + M^j(u) = f^j(u), \\ u^j = 0 \quad \text{on } (0, T) \times \partial\Omega, \\ b^j(u(0, x)) = b^j(u_0(x)) \quad \text{in } \Omega, \end{cases} \tag{1}$$

where the memory operator M is defined by

$$\langle M^j(t)(u), v^j \rangle = \int_\Omega \int_0^t K^j(t, s) g^j(s, x, \nabla u(s, x)) \mathrm{d}s \nabla v^j(t, x) \mathrm{d}x$$

for all functions $u, v \in L^p(0, T; H_0^{1,p}(\Omega)^l)$, for almost all $t \in (0, T)$.

The existence of a solution will be ensured by the following assumptions.

Assumption 1. A1. The vector field $b : \mathbb{R}^l \to \mathbb{R}^l$ is monotone nondecreasing and a continuous gradient, i.e. there exists a convex C^1 function $\Phi : \mathbb{R}^l \to \mathbb{R}$ such that $b = \nabla\Phi$, and $b(0) = 0$.

A2. The tensor field $a \in L^\infty(\Omega)^{N\times l\times N\times l}$, considered as a linear mapping on $L^\infty(\Omega)^{N\times l}$, is symmetric and elliptic, i.e. for $0 < a_0 \le a^0 < \infty$, a satisfies $a_0|\xi|^2 \le a(x)\xi\,\xi \le a^0|\xi|^2$ for $\xi \in \mathbb{R}^{N\times l}$ and for a.a. $x \in \Omega$.

A3. The diffusivity $d : (0, T) \times \Omega \times \mathbb{R}^l \times \mathbb{R}^{N\times l} \to \mathbb{R}^{N\times l}$ is continuous, elliptic, i.e. $d(t, x, \eta, \xi)\,\xi \ge d_0|\xi|^p$ for $\xi \in \mathbb{R}^{N\times l}$, for a.a. $(t, x) \in Q_T$, $d_0 > 0$, $p \ge 2$, strongly monotone, i.e. $(d(t, x, \eta, \xi_1) - d(t, x, \eta, \xi_2))\,(\xi_1 - \xi_2) \ge d_1|\xi_1 - \xi_2|^p$ for $\xi_1, \xi_2 \in \mathbb{R}^{N\times l}$, $\eta \in \mathbb{R}^l$, and for a.a. $(t, x) \in Q_T$, $d_1 > 0$, and satisfies the growth assumption $|d(t, x, \eta, \xi)| \le C(1 + |\eta|^{p-1} + |\xi|^{p-1})$ for $\eta \in \mathbb{R}^l$.

A4. The function $f : (0, T) \times \Omega \times \mathbb{R}^l \to \mathbb{R}^l$ is continuous and sublinear, i.e. $|f(t, x, \eta)| \leq C(1 + |\eta|)$ for $\eta \in \mathbb{R}^l$ and for a.a. $(t, x) \in Q_T$.

A5. The matrix field $g : (0, T) \times \Omega \times \mathbb{R}^{N\times l} \to \mathbb{R}^{N\times l}$ is continuous, satisfies the growth assumption $|g(t, x, \xi)| \leq C(1 + |\xi|^{p-1})$ for $\xi \in \mathbb{R}^{N\times l}$ and for a.a. $(t, x) \in Q_T$, and $|g(t, x, \xi_1) - g(t, x, \xi_2)| \leq C|\xi_1 - \xi_2|^{p-1}$ for $\xi_1, \xi_2 \in \mathbb{R}^{N\times l}$.

A6. The kernel $K : (0, T) \times (0, T) \to \mathbb{R}^l$ is weakly singular, i.e. for $0 \leq \gamma < 1/p$ and continuous $\omega : [0, T] \times [0, T] \to \mathbb{R}$ yields $|K(t, s)| \leq |t - s|^{-\gamma}\omega(t, s)$.

A7. The initial condition u_0 is in $H^1_0(\Omega)^l$, $b(u_0)$ is in $L^1(\Omega)^l$ and in $H^{-1}(\Omega)^l$.

The notion of a solution of the problem introduced above, will be given now.

Definition 2. A function $u : Q_T \to \mathbb{R}^l$ is called *a weak solution* if

1. $u \in L^p(0, T; H^{1,p}_0(\Omega)^l)$, $u \in L^\infty(0, T; H^1_0(\Omega)^l)$, $b(u) \in L^\infty(0, T; L^1(\Omega)^l)$, and $\partial_t(b(u) - \nabla \cdot (a(x)\nabla u)) \in L^q(0, T; H^{-1,q}(\Omega)^l)$,
2. u satisfies the equality

$$
\begin{aligned}
&-\int_0^T\int_\Omega (b(u)\, v_t + a(x)\nabla u\nabla v_t)\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega (b(u_0)\, v_t + a(x)\nabla u_0\nabla v_t)\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T\int_\Omega d(t, x, u, \nabla u)\nabla v\mathrm{d}x\mathrm{d}t + \int_0^T \langle M(u), v\rangle \mathrm{d}t = \int_0^T\int_\Omega f(t, x, u)\, v\mathrm{d}x\mathrm{d}t \qquad (2)
\end{aligned}
$$

for all $v \in L^p(0, T; H^{1,p}_0(\Omega)^l)$, $v_t \in L^2(0, T; H^1_0(\Omega)^l) \cap L^1(0, T; L^\infty(\Omega)^l)$ and $v(T) = 0$.

We define the function

$$
B(z) := b(z) \cdot z - \Phi(z) - \Phi(0) = \int_0^1 (b(z) - b(sz)) \cdot z\,\mathrm{d}s = \int_0^z (b(z) - b(s))\mathrm{d}s.
$$

The properties of the function B can be found in Lemma 14. The function B will be used in the integration by parts formula, see Lemma 15.

2.1. Existence

Theorem 3. *Suppose Assumption 1 is satisfied. Then there exists a weak solution of the problem (1).*

We approximate the differential equation by the time discretization, $h = T/n$, $t_i = ih$, $i = 0, \dots, n$, and obtain the discrete problem

$$
\begin{cases}
\frac{1}{h}(b(u_i) - b(u_{i-1})) - \frac{1}{h}\nabla \cdot (a(x)(\nabla u_i - \nabla u_{i-1})) \\
-\nabla \cdot d(t_i, x, u_{i-1}, \nabla u_i) + M(\hat{u}_{i-1}) - f(t_i, x, u_{i-1}) = 0 \quad \text{in } \Omega, \\
u_i(x) = 0 \quad \text{on } \partial\Omega,
\end{cases} \qquad (3)
$$

where the function $\hat{u}_{i-1}$ is defined by

$$
\hat{u}_{i-1} = \begin{cases} u_{j-1}, & t \in [t_{j-1}, t_j),\ j = 1, \dots, i - 1, \\ u_{i-1}, & t \in [t_{i-1}, T]. \end{cases}
$$

Thus, we obtain elliptic problems, which can be solved by Galerkin's procedure. Let $\{e_k\}_{k=1}^\infty$ be a basis of $H_0^{1,p}(\Omega)^l$ and $e_k \in L^\infty(\Omega)^l$. We are looking for functions $\{u_i^m\}_{i=1}^n$ in the subspace H_m, spanned by $\{e_1, \dots, e_m\}$,

$$u_i^m = \sum_{k=1}^m \alpha_{ik}^m \, e_k,$$

such that

$$\int_\Omega \frac{1}{h}(b(u_i^m) - b(u_{i-1}^m))\,\xi \,\mathrm{d}x + \int_\Omega \frac{1}{h} a(x)(\nabla u_i^m - \nabla u_{i-1}^m)\nabla \xi \,\mathrm{d}x \\ + \int_\Omega d(t_i, x, u_{i-1}^m, \nabla u_i^m)\nabla\xi \mathrm{d}x + \langle M(\hat{u}_{i-1}^m), \xi\rangle - \int_\Omega f(t_i, x, u_{i-1}^m)\,\xi \mathrm{d}x = 0 \tag{4}$$

holds for all $\xi \in H_m$. Here $u_0^m \in H_m$ is an approximation of u_0 in $H_0^1(\Omega)^l$.

Lemma 4. *There exists a solution u_i^m in H_m of the family of discretized Eq. (4).*

Proof. The existence will be shown by induction. Since u_0^m is given, u_{i-1}^m can be assumed to be known. The left-hand side of (4) defines a continuous mapping $J_{hm} : \mathbb{R}^m \to \mathbb{R}^m$ given by

$$J_{hm}^j(r) = \frac{1}{h}\int_\Omega \big(b(v)e_j + a(x)\nabla v \,\nabla e_j - b(u_{i-1}^m)e_j - a(x)\nabla u_{i-1}^m \nabla e_j\big)\,\mathrm{d}x \\ + \int_\Omega d(t_i, x, u_{i-1}^m, \nabla v)\nabla e_j \mathrm{d}x + \langle M(\hat{u}_{i-1}^m), e_j\rangle - \int_\Omega f(t_i, x, u_{i-1}^m) e_j \mathrm{d}x,$$

where $v = \sum_{j=1}^m r_j \, e_j$. This mapping satisfies the following estimates:

$$J_{hm}(r)r \geq (d_0 - c_1\delta)\int_\Omega |\nabla v|^p \mathrm{d}x + \frac{1}{h}\int_\Omega \Big(B(v) + \frac{a_0}{2}|\nabla v|^2\Big)\,\mathrm{d}x \\ - \frac{1}{h}\int_\Omega \left(B(u_{i-1}^m) + \frac{a^0}{2}|\nabla u_{i-1}^m|^2\right)\mathrm{d}x - c_2(\delta)c(\gamma)\sum_{k=1}^i h\int_\Omega |\nabla u_{k-1}^m|^p \mathrm{d}x \\ - c_3\int_\Omega |f(t_i, x, u_{i-1}^m)|^2\,\mathrm{d}x \geq c_4\int_\Omega |\nabla v|^2 \mathrm{d}x + c_5\int_\Omega |\nabla v|^p \mathrm{d}x - c_6,$$

as will be shown now. From the assumption on b and the definition of B it follows that

$$\frac{1}{h}\int_\Omega (b(v) - b(u_{i-1}^m))\,v\mathrm{d}x \geq \frac{1}{h}\int_\Omega B(v)\mathrm{d}x - \frac{1}{h}\int_\Omega B(u_{i-1}^m)\mathrm{d}x.$$

The assumptions on a and d imply

$$\int_\Omega d(t_i, x, u_{i-1}^m, \nabla v)\nabla v \mathrm{d}x \geq d_0 \int_\Omega |\nabla v|^p \mathrm{d}x,$$

$$\frac{1}{h}\int_\Omega a(x)(\nabla v - \nabla u_{i-1}^m)\nabla v \mathrm{d}x \geq \frac{a_0}{2h}\int_\Omega |\nabla v|^2 \mathrm{d}x - \frac{a^0}{2h}\int_\Omega |\nabla u_{i-1}^m|^2 \mathrm{d}x.$$

Applying Hölder's and Young's inequalities yields

$$\langle M(\hat{u}_{i-1}^m), v\rangle \leq \frac{c_1}{\delta}\int_\Omega \left(\int_0^{t_i} K(t_i, s)\, g(s, x, \nabla \hat{u}_{i-1}^m)\mathrm{d}s\right)^q \mathrm{d}x + c_2\delta\int_\Omega |\nabla v|^p \,\mathrm{d}x,$$

where $1/q + 1/p = 1$. The first integral can be estimated by using the assumptions on K and g, and the boundedness of ω,

$$\begin{aligned}\left|\int_0^{t_i} K(t_i,s)\, g(s,x,\nabla \hat{u}^m_{i-1})\mathrm{d}s\right| &\le \sum_{k=1}^{i}\int_{t_{k-1}}^{t_k} |K(t_i,s)||g(s,x,\nabla \hat{u}^m_{i-1})|\mathrm{d}s \\ &\le c_1 \sum_{k=1}^{i}(1+|\nabla u^m_{k-1}|^{p-1})\int_{t_{k-1}}^{t_k}(t_i-s)^{-\gamma}\,\mathrm{d}s \\ &\le \left(\sum_{k=1}^{i} h|\nabla u^m_{k-1}|^p\right)^{\frac{1}{q}}\left(\sum_{k=1}^{i} h(t_i-t_k)^{-\gamma p}\right)^{\frac{1}{p}} + c_2.\end{aligned}$$

Since $\gamma < 1/p$ we have $\sum_{k=1}^{i} h(t_i-t_k)^{-\gamma p} \le 1/(1-p\gamma) =: c(\gamma)$. Due to sublinearity of f and Poincaré's inequality,

$$\int_\Omega |f(t_i,x,u^m_{i-1})|^2\mathrm{d}x \le c_1\int_\Omega |\nabla u^m_{i-1}|^2\mathrm{d}x + c_2.$$

Hence, for $|r|$ big enough, $J(r)r \ge 0$ for all such r. The continuity of J implies the existence of a zero of $J(r)$, i.e. a solution of the discretized Equation (4), see [21, Proposition 2.1]. □

Now convergence of u^m_i to the solution u of the problem (1) for $n, m \to \infty$ is shown. For the proof a priori estimates, compactness arguments, and an integration by parts formula from [1], adapted for pseudoparabolic equations, see Lemma 15, are used. At first we obtain the estimates for u^m_i.

Lemma 5. *The estimates*

$$\begin{aligned}&\max_{1\le j\le n}\int_\Omega B(u^m_j)\mathrm{d}x \le C, \qquad \max_{1\le j\le n}\int_\Omega |\nabla u^m_j|^2\mathrm{d}x \le C,\\ &\sum_{i=1}^{n} h\int_\Omega |\nabla u^m_i|^p\mathrm{d}x \le C\end{aligned} \tag{5}$$

hold uniformly in m and n.

Proof. Choosing u^m_i as a test function in (4) and summing over i yield

$$\begin{aligned}&\sum_{i=1}^{j}\int_\Omega \frac{b(u^m_i)-b(u^m_{i-1})}{h}\,u^m_i\mathrm{d}x + \sum_{i=1}^{j}\int_\Omega a(x)\nabla\frac{u^m_i-u^m_{i-1}}{h}\nabla u^m_i\,\mathrm{d}x\\ &\quad+\sum_{i=1}^{j}\int_\Omega d(t_i,x,u^m_{i-1},\nabla u^m_i)\nabla u^m_i\mathrm{d}x + \sum_{i=1}^{j}\langle M(\hat{u}^m_{i-1}),u^m_i\rangle\\ &\quad=\sum_{i=1}^{j}\int_\Omega f(t_i,x,u^m_{i-1})\,u^m_i\mathrm{d}x.\end{aligned} \tag{6}$$

Each term will be dealt with separately. From the assumption on b and the definition of the function B it follows that

$$\sum_{i=1}^{j}\int_\Omega (b(u^m_i)-b(u^m_{i-1}))\,u^m_i\mathrm{d}x \ge \int_\Omega B(u^m_j)\mathrm{d}x - \int_\Omega B(u^m_0)\mathrm{d}x.$$

By Abel's summation formula we obtain

$$\sum_{i=1}^{j}\int_\Omega a(x)\nabla(u_i^m - u_{i-1}^m)\nabla u_i^m \mathrm{d}x \geq \frac{a_0}{2}\int_\Omega |\nabla u_j^m|^2 \mathrm{d}x - \frac{a^0}{2}\int_\Omega |\nabla u_0^m|^2 \mathrm{d}x.$$

The ellipticity assumption implies

$$\sum_{i=1}^{j}\int_\Omega d(t_i, x, u_{i-1}^m, \nabla u_i^m)\nabla u_i^m \mathrm{d}x \ \geq d_0 \sum_{i=1}^{j}\int_\Omega |\nabla u_i^m|^p \mathrm{d}x.$$

For the integral operator we have the estimate

$$\sum_{i=1}^{j}\langle M(\hat{u}_{i-1}^m), u_i^m\rangle \leq \frac{c_1}{\delta}\sum_{i=1}^{j}\int_\Omega \left(\int_0^{t_i} K(t_i, s)\, g(s, x, \nabla\hat{u}_{i-1}^m)\mathrm{d}s\right)^q \mathrm{d}x + c_2\delta\sum_{i=1}^{j}\int_\Omega |\nabla u_i^m|^p \mathrm{d}x.$$

Analogously to Lemma 4 by the assumptions on the function g and the kernel K we have

$$\left|\int_0^{t_i} K(t_i, s)\, g(s, x, \nabla\hat{u}_{i-1}^m)\mathrm{d}s\right| \leq c(\gamma)\left(\sum_{k=1}^{i} h|\nabla u_{k-1}^m|^p\right)^{1/q} + c_1.$$

The last integral in (6), due to sublinearity of f and Poincaré's inequality, is estimated by

$$\sum_{i=1}^{j}\int_\Omega f(x, t_i, u_{i-1}^m)\, u_i^m \mathrm{d}x \leq c_2\sum_{i=1}^{j}\int_\Omega |\nabla u_i^m|^2 \mathrm{d}x + c_3.$$

By the estimates above, from Eq. (6) we obtain the inequality

$$\begin{aligned}\int_\Omega B(u_j^m)\mathrm{d}x &+ \frac{a_0}{2}\int_\Omega |\nabla u_j^m|^2\mathrm{d}x + (d_0 - c_1\delta)\sum_{i=1}^{j} h\int_\Omega |\nabla u_i^m|^p \mathrm{d}x \\ &\leq \int_\Omega B(u_0^m)\mathrm{d}x + \frac{a^0}{2}\int_\Omega |\nabla u_0^m|^2\mathrm{d}x + c_2 c(\gamma)\sum_{i=1}^{j} h\sum_{k=1}^{i} h\int_\Omega |\nabla u_k^m|^p\mathrm{d}x \\ &\quad + c_3\sum_{i=1}^{j} h\int_\Omega |\nabla u_i^m|^2\mathrm{d}x + c_4.\end{aligned}$$

Using the discrete Gronwall's Lemma in the last inequality implies the estimates in Lemma 5. Gronwall's Lemma can be applied for all sufficiently small h and δ that satisfy $c_3 h < a_0/2$ and $c_2 c(\gamma)h < (d_0 - c_1\delta)$. □

To show the strong convergence of the approximation and equicontinuity of u in time with respect to $L^2(Q_T)$ the following lemma is needed.

Lemma 6. *The estimates*

$$\sum_{j=1}^{n-k} h\int_\Omega (b(u_{j+k}^m) - b(u_j^m))(u_{j+k}^m - u_j^m)\mathrm{d}x \leq Ckh,$$

$$\sum_{j=1}^{n-k} h \int_\Omega |\nabla u_{j+k}^m - \nabla u_j^m|^2 \mathrm{d}x \leq Ckh \tag{7}$$

hold uniformly with respect to m and n.

Proof. Summing up the Eq. (4) for $i = j+1, \dots, j+k$, then choosing $u_{j+k}^m - u_j^m$ as a test function, and finally summing up over $j = 1, \dots, n-k$ yields

$$\begin{aligned}
&\sum_{j=1}^{n-k} \int_\Omega \frac{1}{h}(b(u_{j+k}^m) - b(u_j^m))(u_{j+k}^m - u_j^m)\mathrm{d}x \\
&\quad + \sum_{j=1}^{n-k} \int_\Omega \frac{1}{h} a(x)\nabla(u_{j+k}^m - u_j^m)\nabla(u_{j+k}^m - u_j^m)\mathrm{d}x \\
&\quad + \sum_{j=1}^{n-k} \sum_{i=j+1}^{j+k} \left(\int_\Omega d(t_i, x, u_{i-1}^m, \nabla u_i^m)\nabla(u_{j+k}^m - u_j^m)\mathrm{d}x + \langle M(\hat{u}_{i-1}^m), u_{j+k}^m - u_j^m\rangle \right) \\
&\quad = \sum_{j=1}^{n-k} \sum_{i=j+1}^{j+k} \int_\Omega f(t_i, x, u_{i-1}^m)(u_{j+k}^m - u_j^m)\mathrm{d}x.
\end{aligned}$$

Due to the growth assumption on d we have

$$\sum_{i=1}^{n} \int_\Omega |d(t_i, x, u_{i-1}^m, \nabla u_i^m)|^q \mathrm{d}x \leq c_1 \sum_{i=1}^{n} \int_\Omega |\nabla u_i^m|^p \,\mathrm{d}x + c_2 \sum_{i=1}^{n} \int_\Omega |u_{i-1}^m|^p \,\mathrm{d}x + c_3.$$

The operator M and the function f can be estimated similarly to the last lemma. Then we obtain the following inequality

$$\begin{aligned}
&\sum_{j=1}^{n-k} \int_\Omega \frac{1}{h} \left((b(u_{j+k}^m) - b(u_j^m))(u_{j+k}^m - u_j^m) + a_0|\nabla u_{j+k}^m - \nabla u_j^m|^2 \right) \mathrm{d}x \\
&\quad \leq c_1 \sum_{i=1}^{n} \int_\Omega \left(|\nabla u_i^m|^2 + c_2(T)c(\gamma)|\nabla u_i^m|^p + |\nabla u_i^m|^p \right) \mathrm{d}x \\
&\qquad + k \sum_{j=1}^{n-k} \int_\Omega \left(|\nabla u_{j+k}^m|^2 + |\nabla u_{j+k}^m|^p + |u_{j+k}^m|^2 + |\nabla u_j^m|^2 + |\nabla u_j^m|^p + |u_j^m|^2 \right) \mathrm{d}x.
\end{aligned}$$

This, by using Lemma 5, implies the asserted estimates. □

Proof of Theorem 3. We define for $t \in (t_{i-1}, t_i]$ and $x \in \Omega$ the step functions by

$$\bar{u}_n^m(t, x) := u^m(t_i, x),$$

where the initial conditions are $\bar{u}_n^m(0, x) = u_0^m(x)$. From (5) we obtain

$$\begin{aligned}
&\sup_{0 \leq t \leq T} \int_\Omega B(\bar{u}_n^m(t))\mathrm{d}x \leq C, \qquad \sup_{0 \leq t \leq T} \int_\Omega |\nabla \bar{u}_n^m(t)|^2 \mathrm{d}x \leq C, \\
&\int_0^T \int_\Omega |\nabla \bar{u}_n^m|^p \mathrm{d}x\mathrm{d}t \leq C.
\end{aligned} \tag{8}$$

The growth assumptions on d, g, and f imply

$$\|d_n(t,x,\bar{u}^m_{n,h},\nabla\bar{u}^m_n)\|_{L^q(Q_T)^{N\times l}} \le C, \quad \|f_n(t,x,\bar{u}^m_{n,h})\|_{L^2(Q_T)^l} \le C,$$
$$\|M(\hat{u}^m_{n-1})\|_{L^q(0,T;H^{-1,q}(\Omega)^l)} \le C, \tag{9}$$

where $\bar{u}^m_{n,h}(t,x) := \bar{u}^m_n(t-h,x)$ for $t \in [h,T]$ and $\bar{u}^m_{n,h}(t,x) := u^m_0(x)$ for $t \in [0,h]$, $d_n(t,x,s,z) := d(t_i,x,s,z)$ for $t \in (t_{i-1},t_i]$, for $i = 1,\dots,n$, and $d_n(0,x,s,z) := d(0,x,s,z)$.

From (7) we have

$$\int_0^{T-\tau}\int_\Omega \left(b(\bar{u}^m_n(t+\tau)) - b(\bar{u}^m_n(t))\right)\left(\bar{u}^m_n(t+\tau) - \bar{u}^m_n(t)\right)\mathrm{d}x\mathrm{d}t \le C\,\tau,$$
$$\int_0^{T-\tau}\int_\Omega |\nabla\bar{u}^m_n(t+\tau) - \nabla\bar{u}^m_n(t)|^2\mathrm{d}x\,\mathrm{d}t \le C\,\tau, \tag{10}$$

where for $k \in \{0,\dots,n-1\}$, $kh \le \tau \le (k+1)h$.

The second estimate in (10) and Poincaré's inequality imply

$$\|\bar{u}^m_n - \bar{u}^m_{n,h}\|_{L^2(0,T;H^1_0(\Omega)^l)} \le \frac{C}{\sqrt{n}}. \tag{11}$$

From the Eq. (4) we obtain

$$\|\partial_h\left(b(\bar{u}^m_n) - \nabla\cdot(a(x)\nabla\bar{u}^m_n)\right)\|_{L^q(0,T;H^{-1,q}(\Omega)^l)} \le C. \tag{12}$$

Then the estimates in (8), (9) and (12) imply convergence of a subsequence of $\{\bar{u}^m_n\}$, again denoted by $\{\bar{u}^m_n\}$

$$\bar{u}^m_n \to u \quad \text{weakly in } L^p(0,T;H^{1,p}_0(\Omega)^l),$$
$$\bar{u}^m_n \to u \quad \text{weakly-}* \text{ in } L^\infty(0,T;H^1_0(\Omega)^l),$$
$$d(t,x,\bar{u}^m_{n,h},\nabla\bar{u}^m_n) \to \chi \quad \text{weakly in } L^q(Q_T)^{N\times l},$$
$$\partial_h\left(b(\bar{u}^m_n) - \nabla\cdot(a(x)\nabla\bar{u}^m_n)\right) \to \zeta \quad \text{weakly in } L^q(0,T;H^{-1,q}(\Omega)^l),$$
$$M(\hat{u}^m_{n-1}) \to \mu \quad \text{weakly in } L^q(0,T;H^{-1,q}(\Omega)^l)$$

as $m,n\to\infty$. The weak convergence of $\{\bar{u}^m_n\}$ in $L^p(0,T;H^{1,p}_0(\Omega)^l)$ and the second estimate in (10) imply, by Kolmogorov's Theorem [18], the strong convergence of $\{\bar{u}^m_n\}$ in $L^2(Q_T)^l$, and also the convergence a.e. in Q_T. Then we have also, due to (11), $\bar{u}^m_{n,h} \to u$ strongly in $L^2(Q_T)^l$ and a.e. in Q_T. Thus, since $|b(\bar{u}^m_n)| \le \delta B(\bar{u}^m_n) + \sup_{|\sigma|\le\frac{1}{\delta}}|b(\sigma)|$ [1] or [14], and sublinearity of f, due to the Dominated Convergence Theorem [10], and continuity of b in u and of f in t and u, we obtain the convergences $b(\bar{u}^m_n(t)) \to b(u)$ in $L^1(0,T;L^1(\Omega)^l)$, and $f_n(t,x,u^m_{n,h}) \to f(t,x,u)$ in $L^2(Q_T)^l$. From the continuity of B follows $B(\bar{u}^m_n) \to B(u)$ a.e. in Q_T. Since $\{B(\bar{u}^m_n)\}^m_n$ is bounded in $L^\infty(0,T;L^1(\Omega))$ and $B(\bar{u}^m_n)$ is non-negative we obtain, by Fatou's Lemma,

$$\frac{1}{\tau}\int_{t-\tau}^t\int_\Omega B(u)\mathrm{d}x\mathrm{d}t \le \liminf_{m,n\to\infty}\frac{1}{\tau}\int_{t-\tau}^t\int_\Omega B(\bar{u}^m_n)\mathrm{d}x\mathrm{d}t \le C$$

for all $t, t-\tau \in [0,T]$ and small τ, and, hereby, $B(u) \in L^\infty(0,T;L^1(\Omega))$. Due to the estimate $|b(u)| \le \delta B(u) + \sup_{|\sigma|\le\frac{1}{\delta}}|b(\sigma)|$ we have $b(u) \in L^\infty(0,T;L^1(\Omega)^l)$.

Passing to the limit for $m, n \to \infty$ in the Eq. (4) yields

$$\int_0^T \langle \zeta, v\rangle \mathrm{d}t + \int_0^T \int_\Omega \chi \nabla v \mathrm{d}x\mathrm{d}t + \int_0^T \langle \mu, v\rangle \mathrm{d}t = \int_0^T \int_\Omega f(t,x,u)\, v \mathrm{d}x\, \mathrm{d}t. \tag{13}$$

Since $b(\bar{u}_n^m(0)) = b(u_0^m)$ and $u_0^m \to u_0$ in $H_0^1(\Omega)^l$, we have that

$$\int_0^T \int_\Omega \partial_h b(\bar{u}_n^m)\, v \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega a(x)\partial_h \nabla \bar{u}_n^m \nabla v \mathrm{d}x\mathrm{d}t$$
$$= -\int_0^{T-h} \int_\Omega \left((b(\bar{u}_n^m) - b(u_0^m))\partial_{-h} v + a(x)\nabla(\bar{u}_n^m - u_0^m)\partial_{-h}\nabla v\right) \mathrm{d}x\mathrm{d}t$$
$$\to -\int_0^T \int_\Omega (b(u) - b(u_0))\, v_t \mathrm{d}x\mathrm{d}t - \int_0^T \int_\Omega a(x)\nabla(u - u_0)\nabla v_t \mathrm{d}x\mathrm{d}t$$

as $m, n \to \infty$, for $v \in L^p(0,T; H_0^{1,p}(\Omega)^l) \cap L^\infty(Q_T)^l$, $v_t \in L^2(0,T; H_0^1(\Omega)^l)$, $v_t \in L^\infty(0,T; L^\infty(\Omega)^l)$, and $v(T) = 0$. Since such v form a dense subspace of $L^p(0,T; H_0^{1,p}(\Omega)^l)$ and the boundedness (12), we obtain $\partial_t\, (b(u) - \nabla \cdot (a(x)\nabla u)) = \zeta$ as functions in $L^q(0,T; H^{-1,q}(\Omega)^l)$.

Now we prove $\bar{u}_n^m \to u$ strongly in $L^p(0,T; H_0^{1,p}(\Omega)^l)$. We choose in the discretized equation $\xi = \bar{u}_n^m - v_n^m$ and integrate over the interval $(0,\tau)$, where v_n^m in $L^p(0,T; H_m)$ is the approximation of u in $L^p(0,T; H_0^{1,p}(\Omega)^l)$, constant in each time interval $((k-1)h, kh)$.

$$\int_{Q_\tau} \partial_h b(\bar{u}_n^m)(\bar{u}_n^m - v_n^m)\mathrm{d}x\mathrm{d}t + \int_{Q_\tau} a(x)\partial_h \nabla \bar{u}_n^m (\nabla \bar{u}_n^m - \nabla v_n^m)\mathrm{d}x\mathrm{d}t$$
$$+ \int_{Q_\tau} d_n(t,x,\bar{u}_{n,h}^m, \nabla \bar{u}_n^m)\nabla(\bar{u}_n^m - v_n^m)\mathrm{d}x\mathrm{d}t + \int_0^\tau \langle M(\hat{u}_{n-1}^m), \bar{u}_n^m - v_n^m\rangle \mathrm{d}t$$
$$= \int_{Q_\tau} f_n(t,x,\bar{u}_{n,h}^m)(\bar{u}_n^m - v_n^m)\mathrm{d}x\mathrm{d}t. \tag{14}$$

By the convergences of $\{v_n^m\}$ and $\{\partial_h \left(b(\bar{u}_m^n) - \nabla \cdot (a(x)\nabla \bar{u}_m^n)\right)\}$ and the integration by parts formula, Lemma 15, yields

$$\int_0^\tau \int_\Omega \left(\partial_h b(\bar{u}_n^m)\, (\bar{u}_n^m - v_n^m) + a(x)\partial_h \nabla \bar{u}_n^m\, \nabla(\bar{u}_n^m - v_n^m)\right) \mathrm{d}x\mathrm{d}t$$
$$\geq \frac{1}{h}\int_{\tau-h}^\tau \int_\Omega B(\bar{u}_n^m)\mathrm{d}x\mathrm{d}t + \frac{1}{2h}\int_{\tau-h}^\tau \int_\Omega a(x)\nabla \bar{u}_n^m \nabla \bar{u}_n^m \mathrm{d}x\mathrm{d}t$$
$$- \int_\Omega B(u(\tau))\, \mathrm{d}x - \frac{1}{2}\int_\Omega a(x)\nabla u(\tau)\nabla u(\tau)\mathrm{d}x + c\varepsilon.$$

Fatou's Lemma implies

$$\liminf_{m,n\to\infty} \frac{1}{h}\int_{\tau-h}^\tau \int_\Omega \left(B(\bar{u}_n^m) + \frac{1}{2}a(x)\nabla \bar{u}_n^m \nabla \bar{u}_n^m\right) \mathrm{d}x\mathrm{d}t$$
$$\geq \int_\Omega B(u(\tau))\mathrm{d}x + \frac{1}{2}\int_\Omega a(x)\nabla u(\tau)\nabla u(\tau)\mathrm{d}x.$$

Thus, we obtain

$$\liminf_{m,n\to\infty} \int_0^\tau \int_\Omega \left(\partial_h b(\bar{u}_n^m)\, (\bar{u}_n^m - v_n^m) + a(x)\partial_h \nabla \bar{u}_n^m\, \nabla(\bar{u}_n^m - v_n^m)\right)\, \mathrm{d}x\mathrm{d}t \geq 0.$$

Strong convergence of $\{\bar{u}_n^m\}$ to u in $L^2(Q_T)^l$ and of $\{v_n^m\}$ to u in $L^p(0,T;H_0^{1,p}(\Omega))$, continuity of d, weak convergence of d in $L^q(Q_T)^{N\times l}$, and the Dominated Convergence Theorem, imply $d_n(t,x,\bar{u}_{n,h}^m,\nabla v_n^m) \to d(t,x,u,\nabla u)$ strongly in $L^q(Q_T)^{N\times l}$. Hence, strong monotonicity of d yields

$$\begin{aligned}
&\int_{Q_\tau} d_n(t,x,\bar{u}_{n,h}^m,\nabla\bar{u}_n^m)(\nabla\bar{u}_n^m-\nabla v_n^m)\mathrm{d}x\mathrm{d}t\\
&\quad=\int_{Q_\tau}(d_n(t,x,\bar{u}_{n,h}^m,\nabla\bar{u}_n^m)-d_n(t,x,\bar{u}_{n,h}^m,\nabla v_n^m))(\nabla\bar{u}_n^m-\nabla v_n^m)\mathrm{d}x\mathrm{d}t\\
&\qquad+\int_{Q_\tau} d_n(t,x,\bar{u}_{n,h}^m,\nabla v_n^m)\nabla(\bar{u}_n^m-v_n^m)\mathrm{d}x\,\mathrm{d}t \ge d_1\int_{Q_\tau}|\nabla(\bar{u}_n^m-v_n^m)|^p\mathrm{d}x\mathrm{d}t - c\,\varepsilon.
\end{aligned}$$

The integral operator satisfies the estimate

$$\begin{aligned}
&\int_0^\tau \langle M(\hat{u}_{n-1}^m),\bar{u}_n^m-v_n^m\rangle\,\mathrm{d}t\\
&\quad\le \frac{c_1}{\delta}\int_0^\tau\int_\Omega\left(\int_0^t K(t,s)(g(s,\nabla\bar{u}_{n-1}^m(s))-g(s,\nabla v_n^m(s)))\mathrm{d}s\right)^q\mathrm{d}x\mathrm{d}t\\
&\qquad+c_2\delta\|\bar{u}_n^m-v_n^m\|^p_{L^p(0,T;H_0^{1,p}(\Omega)^l)}.
\end{aligned}$$

Because of the weak singularity of the kernel K and the assumption on g we have

$$\begin{aligned}
&\int_0^t K(t,s)(g(s,\nabla\bar{u}_{n-1}^m(s))-g(s,\nabla v_n^m(s)))\mathrm{d}s\\
&\quad\le C\left(\int_0^t|K(t,s)|^p\mathrm{d}s\right)^{\frac{1}{p}}\left(\int_0^t|\nabla\bar{u}_{n-1}^m(s)-\nabla v_n^m(s)|^p\mathrm{d}s\right)^{\frac{1}{q}}.
\end{aligned}$$

Combining the last two estimates and the estimate (11) yields

$$\begin{aligned}
\int_0^\tau\langle M(\hat{u}_{n-1}^m),\bar{u}_n^m-v_n^m\rangle\mathrm{d}t &\le C_\delta\int_0^\tau\int_\Omega\int_0^t|\nabla\bar{u}_n^m-\nabla v_n^m|^p\mathrm{d}s\mathrm{d}x\mathrm{d}t\\
&\quad+C\delta\|\bar{u}_n^m-v_n^m\|^p_{L^p(0,T;H_0^{1,p}(\Omega)^l)}.
\end{aligned}$$

The strong convergences of $\{\bar{u}_n^m\}$ and $\{f_n(t,x,\bar{u}_{n,h}^m)\}$ imply

$$\int_0^\tau\int_\Omega f_n(t,x,\bar{u}_{n,h}^m)(\bar{u}_n^m-v_n^m)\mathrm{d}x\mathrm{d}t\le c\,\varepsilon.$$

The estimates of all terms in the Eq. (14) give

$$(d_1-C\delta)\int_0^\tau\int_\Omega|\nabla(\bar{u}_n^m-v_n^m)|^p\mathrm{d}x\mathrm{d}t\le C_1\int_0^\tau\int_\Omega\int_0^t|\nabla(\bar{u}_n^m-v_n^m)|^p\mathrm{d}s\mathrm{d}x\,\mathrm{d}t+C_2\varepsilon.$$

By Gronwall's Lemma

$$\int_0^\tau\int_\Omega|\nabla\bar{u}_n^m-\nabla v_n^m|^p\mathrm{d}x\,\mathrm{d}t\le C\,\varepsilon$$

holds. Thus, we have the strong convergence of $\bar{u}_n^m$ to u in $L^p(0,T;H_0^{1,p}(\Omega)^l)$. Continuity of d and g yield

$$d_n(t,x,\bar{u}_{n,h}^m,\nabla\bar{u}_n^m)\to d(t,x,u,\nabla u)\quad\text{a.e. in } Q_T$$

and

$$g_n(t,x,\nabla \bar{u}^m_{n-1}) \to g(t,x,\nabla u) \quad \text{a.e. in } Q_T.$$

The weak convergences of $d_n(t,x,\bar{u}^m_{n,h},\nabla \bar{u}^m_n)$ and $M(\hat{u}^m_{n-1})$ and the a.e. convergences imply $\chi = d(t,x,u,\nabla u)$ and $\mu = M(u)$. Thus, u is the solution of the problem (1). □

Remark 7. We can also consider the linear integral operator

$$\langle M(t)(u), v(t)\rangle = \int_\Omega \int_0^t K(t-s)\nabla u(s,x)\mathrm{d}s \nabla v(t,x)\mathrm{d}x,$$

for $u, v \in L^2(0,T; H^1_0(\Omega)^l)$ with positive-definite and weakly singular kernel $|K(t)| \le C|t|^{-\gamma}$, $0 \le \gamma < 1$. The kernel K is positive-definite iff

$$\int_0^T \int_0^t K(t-s)\beta(s)\mathrm{d}s\, \beta(t)\mathrm{d}t \ge 0.$$

This definiteness implies the monotonicity of the operator M. In this case we can weaken the assumption on d to show existence. Only monotonicity, but not strong monotonicity, is needed to apply the Minty–Browder theorem.

The existence can also be proved in the case of memory term operators of first order, i.e.

$$\langle M(t)(u), v(t)\rangle = \int_\Omega \int_0^t K(t-s)\cdot \nabla u(s,x)\mathrm{d}s\, v(t,x)\mathrm{d}x.$$

It is sufficient to assume monotonicity of d and weak singularity of the kernel K, i.e. $|K(t)| \le C|t|^{-\gamma}$, $0 \le \gamma < 1$. The convergence of $d_n(t,x,\bar{u}^m_{n,h},\nabla \bar{u}^m_n)$ to $d(t,x,u,\nabla u)$ follows from Minty–Browder theorem.

Though we considered the Dirichlet boundary conditions only, the results remain valid for other boundary conditions, that allow a Poincaré inequality. For more general boundary conditions we have to assume $B(u_0) \in L^1(\Omega)$, see Remark 16.

2.2. Uniqueness

Theorem 8. *Let Assumption 1, $p = 2$, and*

$$|d(t,x,\eta_1,\zeta_1) - d(t,x,\eta_2,\zeta_2)| \le C(|\eta_1-\eta_2| + |\zeta_1-\zeta_2|),$$
$$|f(t,x,\eta_1) - f(t,x,\eta_2)| \le C|\eta_1-\eta_2|$$

be satisfied for $(t,x) \in Q_T$, $\eta_1,\eta_2 \in \mathbb{R}^l$, and $\zeta_1,\zeta_2 \in \mathbb{R}^{N\times l}$. Then there exists at most one weak solution of the problem (1).

Proof. Suppose, there are two solutions $u^1, u^2 \in L^2(0,T; H^1_0(\Omega)^l)$. Then their difference satisfies

$$\begin{aligned} &-\int_0^T \int_\Omega \Big((b(u^1)-b(u^2))\, v_t + a(x)\nabla(u^1-u^2)\,\nabla v_t\Big)\mathrm{d}x\mathrm{d}t \\ &+\int_0^T \int_\Omega (d(t,x,u^1,\nabla u^1) - d(t,x,u^2,\nabla u^2))\,\nabla v \mathrm{d}x\mathrm{d}t \\ &+\int_0^T \langle M(u^1)-M(u^2), v\rangle \mathrm{d}t = \int_0^T \int_\Omega (f(t,x,u^1) - f(t,x,u^2))\, v\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{15}$$

because $b(u_0^1) = b(u_0^2)$ and $\nabla u_0^1 = \nabla u_0^2$. Since $\partial_t(b(u^i) - \nabla\cdot(a(x)\nabla u^i)) \in L^2(0,T,H^{-1}(\Omega)^l)$, we may assume $b(u^i) - \nabla\cdot(a(x)\nabla u^i) \in C(0,T;H^{-1}(\Omega)^l)$. Due to $u^i \in L^2(0,T;H_0^1(\Omega)^l)$ and $a \in L^\infty(\Omega)^l$ we obtain $\nabla\cdot(a(x)\nabla u^i) \in L^2(0,T,H^{-1}(\Omega)^l)$ and $b(u^i) \in L^2(0,T;H^{-1}(\Omega)^l)$. We choose for $s \leq T$

$$v_s(t) = \begin{cases} \int_t^s (u^1(\tau) - u^2(\tau))\mathrm{d}\tau, & t < s, \\ 0, & \text{otherwise} \end{cases}$$

and integrate by parts. Notice that $v_s(s) = 0$. Hereby we obtain

$$\begin{aligned} &\int_0^s \langle b(u^1) - b(u^2), u^1 - u^2\rangle \mathrm{d}t + a_0 \int_0^s \int_\Omega |\nabla u^1 - \nabla u^2|^2 \mathrm{d}x\mathrm{d}t \\ &\quad \leq \delta_0 \int_0^s \int_\Omega |\nabla u^1 - \nabla u^2|^2 \mathrm{d}x\mathrm{d}t + \frac{c_1}{\delta_0} \int_0^s \int_\Omega |\nabla v_s(t)|^2\, \mathrm{d}x\mathrm{d}t, \end{aligned} \tag{16}$$

where the last term satisfies the following estimate

$$\int_0^s \int_\Omega |\nabla v_s(t,x)|^2\, \mathrm{d}x\, \mathrm{d}t \leq c_2 \int_0^s \int_0^t \int_\Omega |\nabla(u^1(\tau,x) - u^2(\tau,x))|^2 \mathrm{d}x\mathrm{d}\tau\mathrm{d}t.$$

Using the monotonicity of the function b and Gronwall's Lemma for the inequality (16) yields

$$\int_0^s \int_\Omega |\nabla u^1 - \nabla u^2|^2 \mathrm{d}x\mathrm{d}t = 0$$

and $u^1 = u^2$ almost everywhere in Q_T. □

3. Variational inequalities

In this section we prove an existence theorem for the quasilinear pseudoparabolic inequality

$$\begin{aligned} &\int_{Q_T} [\partial_t b(u)(v-u) + a(x)\nabla u_t \nabla(v-u) + d(t,x,\nabla u)\nabla(v-u)]\, \mathrm{d}x\mathrm{d}t \\ &\quad \geq \int_{Q_T} f(t,x,u)(v-u)\mathrm{d}x\mathrm{d}t \end{aligned} \tag{17}$$

with initial condition

$$b(u(0,x)) = b(u_0(x)). \tag{18}$$

The initial value problem is completed by posing spatial boundary conditions. An intermediate subspace V, $H_0^{1,p}(\Omega)^l \subset V \subset H^{1,p}(\Omega)^l$, is chosen such that it is densely and continuously embedded in $L^2(\Omega)^l$, is densely and continuously embedded into a closed subspace $V_0 \subset H^1(\Omega)^l$. The spaces V and V_0 should satisfy Poincaré inequalities, i.e.

$$\|v\|_{L^p(\Omega)^l} \leq C\|\nabla v\|_{L^p(\Omega)^l} \quad \text{for } v \in V$$

and

$$\|v\|_{L^2(\Omega)^l} \leq C\|\nabla v\|_{L^2(\Omega)^l} \quad \text{for } v \in V_0.$$

A further constraint on u is given by the requirement $u \in K$, where K is chosen to be a closed and convex subset of V containing 0.

Assumption 9. A1. The vector field $b : \mathbb{R}^l \to \mathbb{R}^l$ is monotone nondecreasing, Lipschitz continuous, and a continuous gradient, i.e. there exists a convex C^1 function $\Phi : \mathbb{R}^l \to \mathbb{R}$ such that $b = \nabla \Phi$, and $b(0) = 0$.

A2. The tensor field $a \in L^\infty(\Omega)^{N\times l\times N\times l}$, considered as a linear mapping on $L^\infty(\Omega)^{N\times l}$, is symmetric and elliptic, i.e. for $0 < a_0 \le a^0 < \infty$, a satisfies $a_0|\xi|^2 \le a(x)\xi\,\xi \le a^0|\xi|^2$ for $\xi \in \mathbb{R}^{N\times l}$ and for a.a. $x \in \Omega$.

A3. The diffusivity $d : (0, T) \times \Omega \times \mathbb{R}^{N\times l} \to \mathbb{R}^{N\times l}$ is continuous, elliptic, i.e. $d_0 > 0$, $d(t, x, \xi)\,\xi \ge d_0|\xi|^p$, for $\xi \in \mathbb{R}^{N\times l}$ and for a.a. $(t, x) \in Q_T$, $p \ge 2$, and monotone, i.e. $(d(t, x, \xi_1) - d(t, x, \xi_2))\,(\xi_1 - \xi_2) \ge 0$ for all $\xi_1, \xi_2 \in \mathbb{R}^{N\times l}$, and satisfies the growth assumption $|d(t, x, \xi)| \le C(1 + |\xi|^{p-1})$ for $\xi \in \mathbb{R}^{N\times l}$, and is a gradient, i.e. there is a continuous function $D(t, x, \xi)$ s.t. $\nabla_\xi D = d$ and $|D(t, x, 0)| \le C$, $|\partial_t D(t, x, \xi)| \le C(1 + |\xi|^p)$ for $\xi \in \mathbb{R}^{N\times l}$, a.a. $(t, x) \in Q_T$.

A4. The function $f : (0, T) \times \Omega \times \mathbb{R}^l \to \mathbb{R}^l$ is continuous and sublinear, i.e. $|f(t, x, \eta)| \le C(1 + |\eta|)$ for $\eta \in \mathbb{R}^l$ and for a.a. $(t, x) \in Q_T$.

A5. The initial condition u_0 is in K.

Definition 10. A function $u : Q_T \to \mathbb{R}^l$ is called *a weak solution* if

1. $u \in L^\infty(0, T; V)$, $u_t \in L^2(0, T; V_0)$, $\partial_t b(u) \in L^2(Q_T)^l$, and $u(t) \in K$ for almost all $t \in (0, T)$,
2. u satisfies the inequality (17) for all test functions $v \in L^p(0, T; V)$ and $v(t) \in K$ for almost all $t \in (0, T)$,
3. u satisfies the initial condition (18), i.e. $b(u(t, x)) \to b(u_0(x))$ in $L^2(\Omega)^l$ as $t \to 0$.

Theorem 11. *Let Assumption 9 be satisfied. Then there exists a weak solution of the variational inequality (17) with the initial condition (18).*

For positive α we consider the penalized equation

$$\partial_t b(u) - \nabla\cdot(a(x)\nabla u_t) - \nabla\cdot d(t, x, \nabla u) + \alpha\mathcal{B}(u) = f(t, x, u),$$

where $\mathcal{B} : L^p(0, T; V) \to L^q(0, T; V^*)$ is the penalty operator, introduced in Definition 17 in Appendix A.

In the proof of this theorem some of the estimates of Section 2 are reused. By the Rothe–Galerkin approximation we obtain the family of functions $\{u^m_{\alpha,i}\}$, satisfying

$$\int_\Omega \frac{1}{h}(b(u^m_{\alpha,i}) - b(u^m_{\alpha,i-1}))\,\xi\,\mathrm{d}x + \int_\Omega a(x)\frac{1}{h}\nabla(u^m_{\alpha,i} - u^m_{\alpha,i-1})\nabla\xi\,\mathrm{d}x + \int_\Omega d(t_i, x, \nabla u^m_{\alpha,i})\nabla\xi\,\mathrm{d}x + \alpha\langle\mathcal{B}(u^m_{\alpha,i}), \xi\rangle = \int_\Omega f(t_i, x, u^m_{\alpha,i-1})\,\xi\,\mathrm{d}x \tag{19}$$

for $\xi \in H_m = \operatorname{span}\{e_1, \ldots, e_m\}$, where $\{e_j\}_{j=1}^\infty$ is a basis of V and $e_j \in L^\infty(\Omega)^l$, and $\{u^m_{\alpha,0}\}_m$ is an approximation of u_0 in V. Since the operator $\mathcal{B}$ is monotone, the existence of $u^m_{\alpha,i}$ can be proved in the same manner as in Section 2.

Similar to the proof of Lemma 5, using Assumption 9 and the monotonicity of $\mathcal{B}$ yields the estimates

$$\max_{0\le j\le n}\int_\Omega B(u^m_{\alpha,j})\mathrm{d}x \le C, \qquad \max_{0\le j\le n}\int_\Omega |\nabla u^m_{\alpha,j}|^2\mathrm{d}x \le C,$$

$$\sum_{i=1}^{j} h\int_\Omega |\nabla u^m_{\alpha,i}|^p\mathrm{d}x \le C, \qquad \alpha\sum_{i=1}^{j} h\langle\mathcal{B}(u^m_{\alpha,i}), u^m_{\alpha,i}\rangle \le C. \tag{20}$$

For the proof of the existence of a solution of the variational inequality we have to show that $\partial_t b(u) \in L^2(Q_T)^l$ and $u_t \in L^2(0, T; V_0)$. Since d is a gradient it is possible to prove the following.

Lemma 12. *The estimates*

$$\sum_{i=1}^{n} h \int_\Omega \frac{1}{h^2}(b(u^m_{\alpha,i}) - b(u^m_{\alpha,i-1}))(u^m_{\alpha,i} - u^m_{\alpha,i-1})\,\mathrm{d}x \le C,$$
$$\sum_{i=1}^{n} h \int_\Omega \left|\frac{\nabla u^m_{\alpha,i} - \nabla u^m_{\alpha,i-1}}{h}\right|^2 \mathrm{d}x \le C, \tag{21}$$
$$\max_{1\le j\le n} \int_\Omega |\nabla u^m_{\alpha,j}|^p \mathrm{d}x \le C$$

hold uniformly with respect to m, n, and α.

Proof. Choosing $\xi = (u^m_{\alpha,i} - u^m_{\alpha,i-1})$ as a test function in (19) and summing over i yields

$$\sum_{i=1}^{j} \int_\Omega \frac{1}{h}(b(u^m_{\alpha,i}) - b(u^m_{\alpha,i-1}))(u^m_{\alpha,i} - u^m_{\alpha,i-1})\,\mathrm{d}x$$
$$+ \sum_{i=1}^{j} \int_\Omega \frac{1}{h} a(x)\nabla(u^m_{\alpha,i} - u^m_{\alpha,i-1})\nabla(u^m_{\alpha,i} - u^m_{\alpha,i-1})\mathrm{d}x$$
$$+ \sum_{i=1}^{j} \int_\Omega d(t_i, x, \nabla u^m_{\alpha,i})\nabla(u^m_{\alpha,i} - u^m_{\alpha,i-1})\mathrm{d}x + \alpha \sum_{i=1}^{j} \langle \mathcal{B}(u^m_{\alpha,i}), u^m_{\alpha,i} - u^m_{\alpha,i-1}\rangle$$
$$= \sum_{i=1}^{j} \int_\Omega f(t_i, x, u^m_{\alpha,i-1})(u^m_{\alpha,i} - u^m_{\alpha,i-1})\mathrm{d}x. \tag{22}$$

Due to the assumption on d the third integral on the left can be rewritten in the form

$$I = \sum_{i=1}^{j} \int_\Omega d(t_i, x, \nabla u^m_{\alpha,i})\nabla(u^m_{\alpha,i} - u^m_{\alpha,i-1})\mathrm{d}x \ge \int_\Omega D(t_j, x, \nabla u^m_{\alpha,j})\mathrm{d}x$$
$$- \int_\Omega D(0, x, \nabla u^m_0)\mathrm{d}x - \sum_{i=1}^{j} \int_\Omega (D(t_i, x, \nabla u^m_{\alpha,i-1}) - D(t_{i-1}, x, \nabla u^m_{\alpha,i-1}))\mathrm{d}x.$$

Applying the assumed growth bounds of d to $|D(t, x, z)| \le \int_0^z |d(t, x, \xi)|\mathrm{d}\xi + |D(t, x, 0)|$ yields $|D(0, x, \nabla u^m_{\alpha,0})| \le C(1 + |\nabla u^m_{\alpha,0}|^p)$. The ellipticity assumption on d implies

$$D(t, x, \xi) - D(t, x, 0) = \int_0^1 \nabla D(t, x, s\xi)\,\xi\mathrm{d}s = \int_0^1 d(t, x, s\xi)\,s\,\xi\,s^{-1}\mathrm{d}s$$
$$\ge d_0|\xi|^p \int_0^1 s^{p-1}\mathrm{d}s = \frac{d_0}{p}|\xi|^p.$$

Since $|D(t, x, 0)| \le C$, we obtain $D(t_i, x, \nabla u^m_{\alpha,j}) \ge d_0/p|\nabla u^m_{\alpha,j}|^p - C$. Then, due to $|\partial_t D(t, x, z)| \le C(1 + |z|^p)$, we have

$$I \geq \frac{d_0}{p} \int_\Omega |\nabla u^m_{\alpha,j}|^p \mathrm{d}x - c_1 \int_\Omega |\nabla u^m_0|^p \mathrm{d}x - c_2 \sum_{i=1}^j h \int_\Omega |\nabla u^m_{\alpha,i-1}|^p \mathrm{d}x - c_3.$$

The penalty operator can be estimated by

$$\begin{aligned}
\sum_{i=1}^j \left\langle \mathcal{B}(u^m_{\alpha,i}), u^m_{\alpha,i} - u^m_{\alpha,i-1} \right\rangle &= \sum_{i=1}^j \left\langle J(u^m_{\alpha,i} - P_K u^m_{\alpha,i}), P_K u^m_{\alpha,i} - P_K u^m_{\alpha,i-1} \right\rangle \\
&\quad + \sum_{i=1}^j \left\langle J(u^m_{\alpha,i} - P_K u^m_{\alpha,i}), (u^m_{\alpha,i} - P_K u^m_{\alpha,i}) - (u^m_{\alpha,i-1} - P_K u^m_{\alpha,i-1}) \right\rangle \\
&\geq \frac{1}{p} \sum_{i=1}^j \left(\|u^m_{\alpha,i} - P_K u^m_{\alpha,i}\|^p_V - \|u^m_{\alpha,i-1} - P_K u^m_{\alpha,i-1}\|^p_V \right) \\
&= \frac{1}{p} \|(u^m_{\alpha,i} - P_K u^m_{\alpha,i})\|^p_V \geq 0.
\end{aligned}$$

Here $u_0 \in K$ and the property of P_K are used. Due to estimates for d and $\mathcal{B}$ we obtain from (22) the inequality

$$\begin{aligned}
&\sum_{i=1}^j \frac{1}{h} \int_\Omega (b(u^m_{\alpha,i}) - b(u^m_{\alpha,i-1}))(u^m_{\alpha,i} - u^m_{\alpha,i-1}) \mathrm{d}x \\
&\quad + \sum_{i=1}^j \frac{a_0 - \delta}{h} \int_\Omega |\nabla (u^m_{\alpha,i} - u^m_{\alpha,i-1})|^2 \mathrm{d}x + \frac{d_0}{p} \int_\Omega |\nabla u^m_{\alpha,j}|^p \mathrm{d}x \\
&\leq c_\delta \sum_{i=1}^j h \int_\Omega |f(t_i, x, u^m_{\alpha,i-1})|^2 \mathrm{d}x + \sum_{i=1}^j h \int_\Omega |\nabla u^m_{\alpha,i}|^p \mathrm{d}x + \int_\Omega |\nabla u^m_0|^p \mathrm{d}x.
\end{aligned}$$

Using the bounds (20) and sublinearity of f in the last inequality implies the assertion of the lemma. □

Since b is Lipschitz continuous we have

$$\sum_{i=1}^j h \int_\Omega \left| \partial_h b(u^m_{\alpha,i}) \right|^2 \mathrm{d}x \leq C. \tag{23}$$

Proof of Theorem 11. We define the Rothe functions piecewise for $t \in (t_{i-1}, t_i]$ and $x \in \Omega$ by

$$u^m_{\alpha,n}(t,x) = u^m_\alpha(t_{i-1}, x) + (t - t_{i-1}) \frac{u^m_\alpha(t_i, x) - u^m_\alpha(t_{i-1}, x)}{h}$$

and the step functions by

$$\bar{u}^m_{\alpha,n}(t,x) = u^m_\alpha(t_i, x),$$

where the initial conditions are $u^m_{\alpha,n}(0,x) = u^m_0(x)$ and $\bar{u}^m_{\alpha,n}(0,x) = u^m_0(x)$.

From (20), (21) and (23) we obtain

$$\begin{aligned}
&\sup_{0\leq t\leq T} \int_\Omega B(\bar{u}^m_{\alpha,n}) \mathrm{d}x \leq C, \qquad \sup_{0\leq t\leq T} \int_\Omega |\nabla \bar{u}^m_{\alpha,n}|^p \,\mathrm{d}x \leq C, \\
&\int_0^T \int_\Omega |\partial_h \nabla u^m_{\alpha,n}|^2 \mathrm{d}x \mathrm{d}t \leq C, \qquad \int_0^T \int_\Omega |\partial_h b(\bar{u}^m_{\alpha,n})|^2 \mathrm{d}x \,\mathrm{d}t \leq C.
\end{aligned} \tag{24}$$

The growth assumption on d implies

$$\|d_n(t,x,\nabla\bar{u}^m_{\alpha,n})\|_{L^q(Q_T)^{N\times l}} \leq C,$$

where $d_n(t,x,z) := d(t_i,x,z)$ for $t \in (t_{i-1},t_i]$ for $i = 1,\dots,n$, and $d_n(0,x,z) := d(0,x,z)$. The penalty operator is bounded, hence

$$\|\mathcal{B}(\bar{u}^m_{\alpha,n})\|_{L^q(0,T;V^*)} \leq C.$$

The third estimate in (24) and the Poincaré inequality imply

$$\|\bar{u}^m_{\alpha,n} - \bar{u}^m_{\alpha,n,h}\|_{L^2(0,T;V_0)} \leq \frac{C}{n}, \tag{25}$$

where $\bar{u}^m_{\alpha,n,h}(t,x) := \bar{u}^m_{\alpha,n}(t-h,x)$ for $t \in [h,T]$ and $\bar{u}^m_{\alpha,n,h}(t,x) := u^m_{\alpha,0}(x)$ for $t \in [0,h]$.

From (24) follows the existence of a subsequence of $\{\bar{u}^m_{\alpha,n}\}$ and of $\{u^m_{\alpha,n}\}$, resp., again denoted by $\{\bar{u}^m_{\alpha,n}\}$ and $\{u^m_{\alpha,n}\}$, resp., such that

$$\begin{aligned}
&\bar{u}^m_{\alpha,n} \to u_\alpha \quad \text{weakly} - * \text{ in } L^\infty(0,T;V),\\
&\partial_h u^m_{\alpha,n} \to \partial_t u_\alpha \quad \text{weakly in } L^2(0,T;V_0),\\
&\partial_h b(\bar{u}^m_{\alpha,n}) \to \eta_\alpha \quad \text{weakly in } L^2(Q_T)^l\\
&d_n(t,x,\nabla\bar{u}^m_{\alpha,n}) \to \chi_\alpha \quad \text{weakly in } L^q(Q_T)^{N\times l},\\
&\mathcal{B}(\bar{u}^m_{\alpha,n}) \to \theta \quad \text{weakly in } L^q(0,T;V^*),
\end{aligned}$$

as $m,n \to \infty$. The strong convergence of $\{u^m_{\alpha,n}\}$ in $L^2(Q_T)^l$ follows from the Compactness Lions–Aubin lemma [16]. This and the estimate (25) imply the strong convergence of $\{u^m_{\alpha,n,h}\}$ in $L^2(Q_T)^l$. The Lipschitz-continuity of b and $b(0) = 0$ imply $\|b(\bar{u}^m_{\alpha,n})\|_{L^2(Q_T)^l} \leq c\|\bar{u}^m_{\alpha,n}\|_{L^2(Q_T)^l} \leq C$. From the strong convergence of $\{u^m_{\alpha,n}\}$ in $L^2(Q_T)^l$, the continuity of b and f, and the sublinearity of f, due to the Dominated Convergence Theorem [10], follows $b(\bar{u}^m_{\alpha,n}) \to b(u_\alpha)$ in $L^2(Q_T)^l$ and $\eta_\alpha = \partial_t b(u_\alpha)$, and $f_n(t,x,\bar{u}^m_{\alpha,n,h}) \to f(t,x,u_\alpha)$ in $L^2(Q_T)^l$. From the continuity of B follows $B(\bar{u}^m_{\alpha,n}) \to B(u_\alpha)$ a.e. in Q_T. Since $\{B(\bar{u}^m_{\alpha,n})\}$ is bounded in $L^\infty(0,T;L^1(\Omega))$ and $B(\bar{u}^m_{\alpha,n})$ is non-negative we obtain, by Fatou's Lemma, that $B(u_\alpha) \in L^\infty(0,T;L^1(\Omega))$.

Using $u_\alpha \in L^p(0,T;V)$, $\partial_t u_\alpha \in L^2(0,T;V_0)$ and [10, Theorem 5.9.2], imply $u_\alpha \in C([0,T];V_0)$ and $u_\alpha(0) = u_0$. Due to the Lipschitz-continuity of b we obtain

$$\int_\Omega |b(u_\alpha(t)) - b(u_\alpha(s))|^2 \mathrm{d}x \leq c_1\|u_\alpha(t) - u_\alpha(s)\|^2_{V_0} \quad \text{for all } t,s \in [0,T].$$

This implies $b \in C([0,T];L^2(\Omega)^l)$ and $b(u_\alpha(0)) = b(u_0)$ in $L^2(\Omega)^l$.

Passing to the limit as $m,n \to \infty$ in the discretized Eq. (19) yields

$$\begin{aligned}
&\int_{Q_T} \partial_t b(u_\alpha)\, v\mathrm{d}x\mathrm{d}t + \int_{Q_T} a(x)\partial_t\nabla u_\alpha\nabla v\mathrm{d}x\mathrm{d}t + \int_{Q_T} \chi_\alpha\nabla v\mathrm{d}x\mathrm{d}t\\
&\quad + \alpha\int_0^T \langle\theta, v\rangle\,\mathrm{d}t = \int_{Q_T} f(t,x,u_\alpha)\, v\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{26}$$

Due to the monotonicity of d and $\mathcal{B}$ we will show

$$\int_{Q_T} \chi_\alpha\nabla v\mathrm{d}x\mathrm{d}t + \alpha\int_0^T \langle\theta, v\rangle\mathrm{d}t = \int_{Q_T} d(t,x,\nabla u_\alpha)\nabla v\mathrm{d}x\mathrm{d}t + \alpha\int_0^T \langle\mathcal{B}(u_\alpha), v\rangle\mathrm{d}t \tag{27}$$

for all functions $v \in L^p(0, T; V)$. Fatou's Lemma implies

$$\liminf_{m,n\to\infty} \int_{Q_\tau} a(x)\partial_t \nabla u^m_{\alpha,n} \nabla u^m_{\alpha,n} \mathrm{d}x\mathrm{d}t = \liminf_{m,n\to\infty} \frac{1}{2} \int_\Omega a(x) \nabla u^m_{\alpha,n} \nabla u^m_{\alpha,n} \mathrm{d}x$$
$$- \frac{1}{2} \int_\Omega a(x) \nabla u_0 \nabla u_0 \mathrm{d}x \geq \int_{Q_\tau} a(x) \partial_t \nabla u_\alpha \nabla u_\alpha \mathrm{d}x\,\mathrm{d}t.$$

Then from Eq. (19), convergence of $\{\bar{u}^m_{\alpha,n}\}$, and Eq. (26) we have

$$\limsup_{m,n\to\infty} \left(\int_{Q_T} d_n(t, x, \nabla \bar{u}^m_{\alpha,n}) \nabla \bar{u}^m_{\alpha,n} \mathrm{d}x\mathrm{d}t + \alpha \int_0^T \langle \mathcal{B}(\bar{u}^m_{\alpha,n}), \bar{u}^m_{\alpha,n} \rangle \mathrm{d}t \right)$$
$$\leq \int_{Q_T} f(t, x, u_\alpha)\, u_\alpha \mathrm{d}x\mathrm{d}t - \int_{Q_T} \partial_t b(u_\alpha)\, u_\alpha \mathrm{d}x\,\mathrm{d}t$$
$$- \liminf_{m,n\to\infty} \int_{Q_T} a(x) \partial_t \nabla u^m_{\alpha,n} \nabla u^m_{\alpha,n} \mathrm{d}x\mathrm{d}t \leq \int_0^T \langle \chi_\alpha, \nabla u_\alpha \rangle \mathrm{d}t + \alpha \int_0^T \langle \theta, u_\alpha \rangle \mathrm{d}t.$$

Since d and $\mathcal{B}$ are monotone, we have

$$\int_{Q_T} (d_n(\nabla u^m_{\alpha,n}) - d(\nabla w)) \nabla (u^m_{\alpha,n} - w) \mathrm{d}x\mathrm{d}t + \alpha \int_0^T \langle \mathcal{B}(u^m_{\alpha,n}) - \mathcal{B}(w), u^m_{\alpha,n} - w \rangle \mathrm{d}t \geq 0.$$

Passing to the limit as $m, n \to \infty$ yields

$$\int_{Q_T} (\chi_\alpha - d(t, x, \nabla w))\ (\nabla u_\alpha - \nabla w)\, \mathrm{d}x\mathrm{d}t + \alpha \int_0^T \langle \theta - \mathcal{B}(w), u_\alpha - w \rangle \mathrm{d}t \geq 0.$$

Choosing $w = u_\alpha - \lambda v$ for $v \in L^p(0, T; V)$ and $\lambda > 0$, continuity of d and hemicontinuity of $\mathcal{B}$ imply the equality (27) by Minty–Browder's argument.

Then for every α the function u_α satisfies the equation

$$\int_{Q_T} \partial_t b(u_\alpha)\, v \mathrm{d}x\,\mathrm{d}t + \int_{Q_T} a(x) \partial_t \nabla u_\alpha \nabla v \mathrm{d}x\,\mathrm{d}t + \int_{Q_T} d(t, x, \nabla u_\alpha) \nabla v \mathrm{d}x\mathrm{d}t$$
$$+ \alpha \int_0^T \langle \mathcal{B}(u_\alpha), v \rangle \mathrm{d}t = \int_{Q_T} f(t, x, u_\alpha)\, v \mathrm{d}x\mathrm{d}t. \tag{28}$$

Analogously as for $u^m_{\alpha,n}$, we obtain the estimates for u_α

$$\sup_{0\leq t\leq T} \int_\Omega B(u_\alpha) \mathrm{d}x \leq C, \qquad \sup_{0\leq t\leq T} \int_\Omega |\nabla u_\alpha|^p \mathrm{d}x \leq C,$$
$$\int_{Q_T} |\partial_t \nabla u_\alpha|^2 \mathrm{d}x\mathrm{d}t \leq C \qquad \int_{Q_T} |\partial_t b(u_\alpha)|^2 \mathrm{d}x\mathrm{d}t \leq C,$$
$$\int_{Q_T} |d(t, x, \nabla u_\alpha)|^q \mathrm{d}x\mathrm{d}t \leq C, \quad \alpha \int_0^T \langle \mathcal{B}(u_\alpha), u_\alpha \rangle \mathrm{d}t \leq C.$$

Then there exists a subsequence of $\{u_\alpha\}$, again denoted by $\{u_\alpha\}$, such that

$$u_\alpha \to u \quad \text{weakly}-* \text{ in } L^\infty(0, T; V),$$
$$\partial_t u_\alpha \to \partial_t u \quad \text{weakly in } L^2(0, T; V_0),$$
$$\partial_t b(u_\alpha) \to \eta \quad \text{weakly in } L^2(Q_T)^l,$$
$$d(t, x, \nabla u_\alpha) \to \chi \quad \text{weakly in } L^q(Q_T)^{N\times l}.$$

Due to the similar argumentation as for $u^m_{\alpha,n}$, we obtain the strong convergences $u_\alpha \to u$, $f(t,x,u_\alpha) \to f(t,x,u)$, $b(u_\alpha) \to b(u)$ in $L^2(Q_T)^l$ and $\eta = \partial_t b(u)$. Since $u \in L^p(0,T;V)$ and $u_t \in L^2(0,T;V_0)$ we have $u \in C([0,T];V_0)$ by [10, Theorem 5.9.2]. Then $u_\alpha(0) \to u(0)$ weakly in V_0 yields $u(0) = u_0$. Since b is Lipschitz continuous, we obtain $b \in C([0,T];L^2(\Omega)^l)$ and $b(u(0)) = b(u_0)$ in $L^2(\Omega)^l$, and thus the validity of the initial condition (18).

From Eq. (28) we obtain

$$\int_0^T \langle \mathcal{B}(u_\alpha), v\rangle \mathrm{d}t = \frac{1}{\alpha}\left(\int_{Q_T} f(t,x,u_\alpha)\, v \mathrm{d}x\mathrm{d}t - \int_{Q_T} \partial_t b(u_\alpha)\, v \mathrm{d}x\mathrm{d}t \right.$$
$$\left. - \int_{Q_T} a(x)\, \partial_t \nabla u_\alpha \nabla v \mathrm{d}x\mathrm{d}t - \int_{Q_T} d(t,x,\nabla u_\alpha)\nabla v \mathrm{d}x\mathrm{d}t\right)$$

for all $v \in L^p(0,T;V)$. Since all the terms on the right hand side are bounded in $L^q(0,T;V^*)$,

$$\mathcal{B}(u_\alpha) \to 0 \quad \text{in } L^q(0,T;V^*) \quad \text{as } \alpha \to \infty.$$

Applying the monotonicity of $\mathcal{B}$ to the sequence $\{u_\alpha\}$ yields

$$\int_0^T \langle \mathcal{B}(v), u_\alpha - v\rangle \mathrm{d}t \le \int_0^T \langle \mathcal{B}(u_\alpha), u_\alpha - v\rangle \mathrm{d}t.$$

Together with the estimate $\int_0^T \langle \mathcal{B}(u_\alpha), u_\alpha\rangle \mathrm{d}t \le C/\alpha$ and the convergence of $\mathcal{B}(u_\alpha) \to 0$ in $L^q(0,T;V^*)$ we obtain for $\alpha \to \infty$

$$\int_0^T \langle \mathcal{B}(v), u - v\rangle \mathrm{d}t \le 0.$$

We take $v = u - \lambda w$ for $\lambda > 0$ and $w \in L^p(0,T;V)$. Passing to the limit as $\lambda \to 0$ and using the hemicontinuity of $\mathcal{B}$ imply

$$\int_0^T \langle \mathcal{B}(u), w\rangle \mathrm{d}t \le 0 \quad \text{for all } w \in L^p(0,T;V).$$

Thus, $\mathcal{B}(u) = 0$ and $u \in K$ for almost all $t \in (0,T)$.

Now we show that u satisfies the inequality (17). We choose $u_\alpha - u$ as a test function in the Eq. (28) and obtain

$$\int_{Q_T} (\partial_t b(u_\alpha)(u_\alpha - u) + a(x)\partial_t \nabla u_\alpha \nabla(u_\alpha - u) + d(t,x,\nabla u_\alpha)\nabla(u_\alpha - u))\, \mathrm{d}x\mathrm{d}t$$
$$- \int_{Q_T} f(t,x,u_\alpha)\,(u_\alpha - u)\mathrm{d}x\mathrm{d}t = -\alpha \int_0^T \langle \mathcal{B}(u_\alpha) - \mathcal{B}(u), u_\alpha - u\rangle\, \mathrm{d}t \le 0,$$

since $\mathcal{B}(u) = 0$. Due to Fatou's Lemma and integration by parts

$$\liminf_{\alpha\to\infty} \int_{Q_T} a(x)\partial_t \nabla u_\alpha \nabla(u_\alpha - u)\mathrm{d}x\mathrm{d}t \ge 0.$$

Then, by using the convergence of u_α, we obtain

$$\limsup_{\alpha\to\infty} \int_{Q_T} d(t,x,\nabla u_\alpha)\nabla(u_\alpha - u)\mathrm{d}x\mathrm{d}t$$
$$\le \lim_{\alpha\to\infty} \int_{Q_T} f(t,x,u_\alpha)(u_\alpha - u)\mathrm{d}x\,\mathrm{d}t - \lim_{\alpha\to\infty} \int_{Q_T} \partial_t b(u_\alpha)(u_\alpha - u)\mathrm{d}x\,\mathrm{d}t = 0.$$

The monotonicity of d implies

$$\limsup_{\alpha\to\infty}\int_{Q_T} d(t,x,\nabla u_\alpha)\nabla(u_\alpha-u)\mathrm{d}x\mathrm{d}t \geq \lim_{\alpha\to\infty}\int_{Q_T} d(t,x,\nabla u)\nabla(u_\alpha-u)\mathrm{d}x\mathrm{d}t = 0.$$

Thus, we have

$$\lim_{\alpha\to\infty}\int_{Q_T} d(t,x,\nabla u_\alpha)\nabla(u_\alpha-u)\mathrm{d}x\mathrm{d}t = 0. \tag{29}$$

For the function $w=(1-\lambda)u+\lambda v$, where $v\in L^p(0,T;V)$ and $\lambda>0$, the monotonicity of d implies

$$\begin{aligned}0 &\leq \int_{Q_T}(d(t,x,\nabla u_\alpha)-d(t,x,\nabla w))\nabla(u_\alpha-w)\mathrm{d}x\mathrm{d}t\\ &= \int_{Q_T}(d(t,x,\nabla u_\alpha)-d(t,x,\nabla w))\nabla(u_\alpha-u)\mathrm{d}x\,\mathrm{d}t\\ &\quad+\lambda\int_{Q_T}(d(t,x,\nabla u_\alpha)-d(t,x,\nabla w))\nabla(u-v)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

The first integral on the right hand side converges to zero for $\alpha\to\infty$, due to the convergence of $\{u_\alpha\}$ and (29). Then we divide this inequality by λ, pass to the limits as $\alpha\to\infty$ and $\lambda\to 0$, and, due to continuity of d, obtain

$$\lim_{\alpha\to\infty}\int_{Q_T} d(t,x,\nabla u_\alpha)\nabla(u-v)\mathrm{d}x\mathrm{d}t \geq \int_{Q_T} d(t,x,\nabla u)\nabla(u-v)\mathrm{d}x\mathrm{d}t. \tag{30}$$

Now we choose $v-u_\alpha$ as a test function in (28), where $v\in L^p(0,T;V)$ and $v(t)\in K$ for a.a. $t\in(0,T)$, use the monotonicity of $\mathcal{B}$ and obtain

$$\begin{aligned}&\int_{Q_T}(\partial_t b(u_\alpha)(v-u_\alpha)+a(x)\partial_t\nabla u_\alpha\nabla(v-u_\alpha)+d(t,x,\nabla u_\alpha)\nabla(v-u_\alpha))\,\mathrm{d}x\mathrm{d}t\\ &\quad-\int_{Q_T} f(t,x,u_\alpha)(v-u_\alpha)\mathrm{d}x\,\mathrm{d}t = \alpha\int_0^T\langle\mathcal{B}(v)-\mathcal{B}(u_\alpha),v-u_\alpha\rangle\,\mathrm{d}t\geq 0,\end{aligned} \tag{31}$$

since $\mathcal{B}(v)=0$. By Fatou's Lemma we have

$$\liminf_{\alpha\to\infty}\int_{Q_T} a(x)\partial_t\nabla u_\alpha\nabla u_\alpha\mathrm{d}x\mathrm{d}t \geq \int_{Q_T} a(x)\partial_t\nabla u\nabla u\mathrm{d}x\,\mathrm{d}t.$$

The equality (29) and the inequality (30) yields

$$\begin{aligned}&\lim_{\alpha\to\infty}\int_{Q_T} d(t,x,\nabla u_\alpha)\nabla(v-u_\alpha)\mathrm{d}x\mathrm{d}t\\ &\quad= \lim_{\alpha\to\infty}\int_{Q_T}(d(t,x,\nabla u_\alpha)\nabla(v-u)+d(t,x,\nabla u_\alpha)\nabla(u-u_\alpha))\,\mathrm{d}x\mathrm{d}t\\ &\quad\leq \int_{Q_T} d(t,x,\nabla u)\nabla(v-u)\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Then taking the limit as $\alpha\to\infty$ in (31) and using the convergence of u_α imply that u satisfies the inequality (17). □

Remark 13. Assuming the strong monotonicity of d, i.e.

$$(d(t,x,\xi_1) - d(t,x,\xi_2))(\xi_1 - \xi_2) \geq d_1|\xi_1 - \xi_2|^p \quad \text{for } \mathrm{d}_1 > 0, \xi_1, \xi_2 \in \mathbb{R}^{N\times l}$$

ensures the strong convergence of $u_\alpha \to u$ in $L^p(0,T;V)$.

Appendix A. Integration by parts lemma

Lemma 14 (*[14]*). *The estimates*

$$\begin{aligned}
&(1) \quad B(z) = \int_0^1 (b(z) - b(\sigma z))z d\sigma \geq 0,\\
&(2) \quad B(z) - B(z_0) \geq (b(z) - b(z_0))z_0,\\
&(3) \quad b(z) - \Phi(z) + \Phi(0) = B(z) \leq b(z)z,\\
&(4) \quad |b(z)| \leq \delta B(z) + \sup_{|\sigma|\leq\frac{1}{\delta}} |b(\sigma)|,
\end{aligned}$$

hold true for all $z, z_0 \in \mathbb{R}^l$, and for positive δ.

Lemma 15. *Suppose $u \in L^p(0,T;H_0^{1,p}(\Omega)^l)$, $u \in L^\infty(0,T;H_0^1(\Omega)^l)$, $b(u) \in L^\infty(0,T;L^1(\Omega)^l)$, $B(u) \in L^\infty(0,T;L^1(\Omega))$, $\partial_t(b(u) - \nabla\cdot(a(x)\nabla u)) \in L^q(0,T;H^{-1,q}(\Omega)^l)$, $u_0 \in H_0^1(\Omega)$, $b(u_0) \in L^1(\Omega)$, and $b(u_0) \in H^{-1}(\Omega)$. Then for almost all t the following formula holds*

$$\begin{aligned}
\int_0^t \langle\partial_t(b(u) - \nabla\cdot(a(x)\nabla u)), u\rangle\, \mathrm{d}t &= \int_\Omega \left(B(u(t)) + \frac{1}{2}a(x)\nabla u(t)\nabla u(t)\right) \mathrm{d}x\\
&\quad - \int_\Omega \left(B(u_0) + \frac{1}{2}a(x)\nabla u_0\nabla u_0\right) \mathrm{d}x.
\end{aligned}$$

Proof. For almost all $(t,x) \in (h,T)\times\Omega$, where $u(t-h,x) = u_0(x)$ for $t \in (0,h)$ and $b(u(t-h,x)) = b(u_0(x))$ for $t \in (0,h)$, we have the inequalities

$$\begin{aligned}
&B(u(t)) - B(u(t-h)) + \frac{1}{2}a(x)\nabla u(t)\nabla u(t) - \frac{1}{2}a(x)\nabla u(t-h)\nabla u(t-h)\\
&\quad \leq (b(u(t)) - b(u(t-h)))u(t) + a(x)(\nabla u(t) - \nabla u(t-h))\nabla u(t)
\end{aligned}$$

and

$$\begin{aligned}
&B(u(t)) - B(u(t-h)) + \frac{1}{2}a(x)\nabla u(t)\nabla u(t) - \frac{1}{2}a(x)\nabla u(t-h)\nabla u(t-h)\\
&\quad \geq (b(u(t)) - b(u(t-h)))u(t-h) + a(x)(\nabla u(t) - \nabla u(t-h))\nabla u(t-h).
\end{aligned}$$

Now we multiply the first inequality by h^{-1} and integrate over $(0,\tau)\times\Omega$. Due to $u \in L^p(0,T;H_0^{1,p}(\Omega)^l)$ and $b(u) \in L^q(0,T;H^{-1,q}(\Omega)^l)$, we obtain

$$\begin{aligned}
&\frac{1}{h}\int_{\tau-h}^\tau \int_\Omega \left(B(u) + \frac{1}{2}a(x)\nabla u\nabla u\right) \mathrm{d}x\mathrm{d}t - \int_\Omega \left(B(u_0) + \frac{1}{2}a(x)\nabla u_0\nabla u_0\right) \mathrm{d}x\\
&\quad \leq \int_0^\tau \frac{1}{h}\langle b(u(t)) - b(u(t-h)), u(t)\rangle \mathrm{d}t\\
&\qquad + \int_0^\tau \int_\Omega \frac{1}{h}a(x)\nabla(u(t) - u(t-h))\nabla u(t)\mathrm{d}x\, \mathrm{d}t.
\end{aligned}$$

Multiplying the second inequality by h^{-1} and integrating over $(h, \tau) \times \Omega$ implies

$$\frac{1}{h}\int_{\tau-h}^{\tau}\int_{\Omega}\left(B(u)+\frac{1}{2}a(x)\nabla u\nabla u\right)\mathrm{d}x\mathrm{d}t-\frac{1}{h}\int_{0}^{h}\int_{\Omega}\left(B(u)+\frac{1}{2}a(x)\nabla u\nabla u\right)\mathrm{d}x\mathrm{d}t$$
$$\geq\int_{0}^{\tau-h}\frac{1}{h}\left(\langle b(u(t+h))-b(u(t)),u(t)\rangle+\int_{\Omega}a(x)\nabla(u(t+h)-u(t))\nabla u(t)\mathrm{d}x\right)\mathrm{d}t.$$

Since $\partial_h(b(u)-\nabla\cdot(a(x)\nabla u))=\frac{1}{h}\int_{\tau-h}^{\tau}\frac{\mathrm{d}}{\mathrm{d}t}(b(u)-\nabla(a(x)\nabla u))\mathrm{d}t$, we obtain

$$\int_{h}^{T}\|\partial_h(b(u)-\nabla\cdot(a(x)\nabla u))\|^q_{H^{-1,q}(\Omega)}\mathrm{d}t$$
$$\leq\int_{h}^{T}\frac{1}{h}\int_{0}^{h}\left\|\frac{\mathrm{d}}{\mathrm{d}z}(b(u(z+t-h))-\nabla\cdot(a(x)\nabla u(z+t-h)))\right\|^q_{H^{-1,q}(\Omega)}\mathrm{d}z\mathrm{d}t$$
$$\leq\frac{1}{h}\int_{0}^{h}\int_{0}^{T}\left\|\frac{\mathrm{d}}{\mathrm{d}t}(b(u)-\nabla\cdot(a(x)\nabla u))\right\|^q_{H^{-1,q}(\Omega)}\mathrm{d}t\mathrm{d}s\leq C.$$

Then $\partial_h(b(u)-\nabla\cdot(a(x)\nabla u))\rightarrow\chi$ in $L^q(0,T;H^{-1,q}(\Omega)^l)$. Due to $\partial_t(b(u)-\nabla\cdot(a(x)\nabla u))\in L^q(0,T;H^{-1,q}(\Omega)^l)$ and

$$\int_{Q_T}\partial_h(b(u)-\nabla\cdot(a(x)\nabla u))v\mathrm{d}x\mathrm{d}t=-\int_{Q_T}(b(u)-\nabla\cdot(a(x)\nabla u))\partial_{-h}v\,\mathrm{d}x\mathrm{d}t$$

for $v\in L^p(0,T;H^{1,p}(\Omega)^l)\cap L^\infty(Q_T)^l$, $v_t\in L^2(0,T;H^1_0(\Omega)^l)$ and $v(t,x)=0$ for $t\in(0,\delta)$ and $t\in(T-\delta,T)$, $0<\delta<T$, $x\in\Omega$, we have

$$\partial_h(b(u)-\nabla\cdot(a(x)\nabla u))\rightarrow\partial_t(b(u)-\nabla\cdot(a(x)\nabla u))\quad\text{in } L^q(0,T;H^{-1,q}(\Omega)^l).$$

Since $u\in L^p(0,T;H^{1,p}(\Omega)^l)$ we can take the limit as $h\rightarrow 0$ and obtain

$$\int_{\Omega}\left(B(u(\tau))+\frac{1}{2}a(x)\nabla u(\tau)\nabla u(\tau)\right)\mathrm{d}x-\int_{\Omega}\left(B(u_0)+\frac{1}{2}a(x)\nabla u_0\nabla u_0\right)\mathrm{d}x$$
$$\leq\int_{0}^{\tau}\langle\partial_t\left(b(u(t))-\nabla\cdot(a(x)\nabla u(t))\right),u(t)\rangle\mathrm{d}t,$$
$$\int_{\Omega}\left(B(u(\tau))+\frac{1}{2}a(x)\nabla u(\tau)\nabla u(\tau)\right)\mathrm{d}x-\int_{\Omega}\left(B(u_0)+\frac{1}{2}a(x)\nabla u_0\nabla u_0\right)\mathrm{d}x$$
$$\geq\int_{0}^{\tau}\langle\partial_t(b(u(t))-\nabla\cdot(a(x)\nabla u(t))),u(t)\rangle\mathrm{d}t.$$

These two inequalities imply the assertion of the lemma. By passing to the limit in the second inequality we used that

$$\liminf_{h\rightarrow 0}\frac{1}{h}\int_{0}^{h}\int_{\Omega}\left(B(u)+\frac{1}{2}a(x)\nabla u\nabla u\right)\mathrm{d}x\mathrm{d}t\geq\int_{\Omega}\left(B(u_0)+\frac{1}{2}a(x)\nabla u_0\nabla u_0\right)\mathrm{d}x.$$

Due to $\partial_t(b(u)-\nabla\cdot(a(x)\nabla u))\in L^q(0,T;H^{-1,q}(\Omega)^l)$ and the second estimate in Lemma 14, we obtain

$$\liminf_{h\rightarrow 0}\frac{1}{h}\int_{0}^{h}\int_{\Omega}\left(B(u)+\frac{1}{2}a(x)\nabla u\nabla u-B(u_0)-\frac{1}{2}a(x)\nabla u_0\nabla u_0\right)\mathrm{d}x\mathrm{d}t$$
$$\geq\liminf_{h\rightarrow 0}h\int_{0}^{1}\langle\partial_t(b(u)-\nabla\cdot(a(x)\nabla u)),u_0\rangle\mathrm{d}t=0.$$

Thus, we have the needed inequality. □

Remark 16. As was shown by Brezis and Browder [4], the assumptions $u_0 \in H_0^1(\Omega)$, $b(u_0) \in L^1(\Omega)$, $b(u_0) \in H^{-1}(\Omega)$, and $b(u_0)u_0 \geq 0$ yield $B(u_0) \in L^1(\Omega)$.

Definition 17 (*[16]*). Let V be a reflexive Banach space and K a closed convex subset in V. Then *a penalty operator* $\mathcal{B} : V \to V^*$ related to K is a monotone, bounded and hemicontinuous operator such that $\{v | v \in V, \mathcal{B}(v) = 0\} = K$.

Such an operator $\mathcal{B}$ is given by $\mathcal{B} = J(I - P_K)$, where $J : V \to V^*$ the dual mapping and $P_K : V \to K$ is the projection operator on K. In the case $V = H^{1,p}(\Omega)$, for some $p > 1$, the dual mapping J is defined by

$$\langle J(u), v\rangle = \int_\Omega (|u|^{p-2}uv + |\nabla u|^{p-2}\nabla u \nabla v)\mathrm{d}x$$

and the projection operator P_K satisfies

$$\langle J(u - P_K u), P_K u - v\rangle \geq 0 \quad \text{for } v \in K.$$

References

[1] H.W. Alt, S. Luckhaus, Quasilinear elliptic-parabolic differential equations, Mathematische Zeitschrift 183 (1983) 311–341.

[2] G.I. Barenblatt, M. Bertsch, R. Dal Passo, R.M. Ughii, A degenerate pseudoparabolic regularization of a nonlinear forward-backward heat equation arising in the theory of heat and mass exchange in stably stratified turbulent shear flow, SIAM Journal on Mathematical Analysis 24 (4–6) (1993) 1414–1439.

[3] G.I. Barenblatt, V.M. Entov, V.M. Ryzhik, Theory of Fluid Flow Through Natural Rocks, Kluwer, Dordrecht, 1990.

[4] H. Brezis, F. Browder, Sur une propriété des espaces de Sobolev, Comptes Rendus de l' Academie des Sciences 287 (1978) 113–115.

[5] P. Chen, M. Gurtin, On a theory of heat conduction involving two temperatures, Zeitschrift für angewandte Mathematik und Physik 19 (1968) 614–627.

[6] M.G. Crandall, S.-O. Londen, J.A. Nohel, An abstract nonlinear Volterra integrodifferential equation, Journal of Mathematical Analysis and Applications 64 (1978) 701–735.

[7] C. Cuesta, C.J. van Duijn, J. Hulshof, Infiltration in porous media with dynamic capillary pressure: travelling waves, Modelling, Analysis and Simulation (November 30) (1999) 1–19. Preprint CWI.

[8] E. DiBenedetto, R.E. Showalter, Implicit degenerate evolution equations and applications, SIAM Journal on Mathematical Analysis 12 (1981) 731–751.

[9] E. DiBenedetto, R.E. Showalter, A pseudo-parabolic variational inequality and Stefan problem, Nonlinear Analysis, Theory, Methods, Applications 6 (3) (1982) 279–291.

[10] L.C. Evans, Partial Differential Equations, American Mathematical Society, Providence, RI, 1998.

[11] W.G. Gray, S.M. Hassanizadeh, Thermodynamic basis of capillary pressure in porous media, Water Resources Research 29 (1993) 3389–3405.

[12] M. Heard, An abstract parabolic Volterra integrodifferential equation, SIAM Journal on Mathematical Analysis 13 (1) (1982) 81–105.

[13] J. Kacur, Solution to strongly nonlinear parabolic problems by a linear approximation schema, IMA Journal of Numerical Analysis 19 (1999) 119–145.

[14] J. Kacur, Method of Rothe in Evolution Equations, BSB Teubner, Leipzig, 1985.

[15] L. Kenneth Kuttler Jr., Degenerate variational inequalities of evolution, Nonlinear Analysis, Theory, Methods, Applications 8 (8) (1984) 837–850.

[16] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Paris, 1969.

[17] R.K. Miller, An Integrodifferential equation for rigid heat conductors with memory, Journal of Mathematical Analysis and Applications 66 (1978) 313–332.

[18] J. Necas, Les méthodes directes en théorie des équations elliptiques, Academie, Prague, 1967.

[19] A. Novick-Cohen, R.L. Pego, Stable patterns in a viscous diffusion equation, Transactions of the American Mathematical Society 324 (1) (1991) 331–351.
[20] F. Scarpini, Degenerate and pseudoparabolic variational inequalities, Numerical Functional Analysis and Optimization 9 (1987) 859–879.
[21] R.E. Showalter, Monotone operators in Banach space and nonlinear partial differential equations, Mathematical Surveys and Monographs (1996).