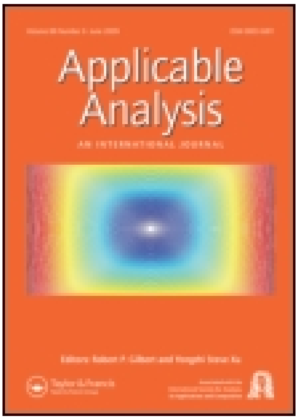

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Unfolding-based corrector estimates for a reaction-diffusion system predicting concrete corrosion

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We use the periodic unfolding technique to derive corrector estimates for a reaction–diffusion system describing concrete corrosion penetration in the sewer pipes. The system, defined in a periodically perforated domain, is semi-linear, partially dissipative and coupled to a nonlinear ordinary differential equation posed on the solid–water interface at the pore level. After discussing the solvability of the pore scale model, we apply the periodic unfolding techniques (adapted to treat the presence of perforations) not only to derive macroscopic (upscaled) model equations, but also to prepare a proper framework for obtaining a convergence rate (corrector estimates) of the averaging procedure.

Keywords: corrector estimates; periodic unfolding; homogenization; sulphate corrosion of concrete; reaction–diffusion systems

AMS Subject Classifications: 35B27; 47Q10; 74Q15; 35K57; 35K60

1. Introduction

Concrete corrosion is a slow natural process that leads to the deterioration of concrete structures (buildings, bridges, highways, etc.) leading yearly to huge financial losses everywhere in the world. In this article, we focus on one of the many mechanisms of chemical corrosion, namely the sulphation of concrete, and aim to describe it macroscopically by a system of averaged reaction–diffusion equations whose effective coefficients depend on the particular shape of the microstructure. The final aim of our research is to become capable to predict quantitatively the durability of a (well-understood) cement-based material under a controlled experimental setup (well-defined boundary conditions). The striking thing is that in spite of the fact that the basic physical-chemistry of this relatively easy material is known [1], we have no control on how the microstructure changes (in time and space) and to

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which extent these spatio-temporal changes affect the observable macroscopic behaviour of the material. The research reported here goes along the line open in [2], where a formal asymptotic expansion ansatz was used to derive macroscopic equations for a corrosion model, posed in a domain with locally periodic microstructure (see [3] for a rigorous averaging approach of a reduced model defined in a domain with locally periodic microstructures). A two-scale convergence approach for periodic microstructures was studied in [4], while preliminary multiscale simulations are reported in [5]. Within this article we consider a partially dissipative reaction–diffusion system defined in a domain with periodically distributed microstructure. This system was originally proposed in [6] as a free-boundary problem. The model equations describe the corrosion of sewer pipes made of concrete when sulphate ions penetrate the non-saturated porous matrix of the concrete viewed as a 'composite'. The typical concrete microstructure includes solid, water and air parts, see Figure 1. One could argue that the microstructure of a concrete is neither uniformly periodic nor locally periodic, and the randomness of the pores and of their distributions should be taken into account. Based on our experience, periodic representations of concrete microstructures often provide good qualitative descriptions. For what the macroscopic corrosion process is concerned, the derivation of corrector estimates (for the periodic case) is crucial for the identification of convergence rates of microscopic solutions. The stochastic geometry of the concrete will be studied as future work with the hope to shed some light on eventual connections between the role played by a locally periodic distributed microstructure versus stationary random(-distributed) pores. In this spirit, we think that there is much to be learnt from [7].

The main novelty of this article is twofold: on the one hand, we obtain corrector estimates under optimal regularity assumptions on solutions of the microscopic model and obtain the desired convergence rate (hence, we now have a confidence measure of our averaging results); on the other hand, we apply for the first time an unfolding technique to derive corrector estimates in perforated media. The main ideas of the methodology were presented in [8,9] and applied to linear elliptic equations with oscillating coefficients, posed in a fixed domain. Our approach strongly relies on these results. However, novel aspects of the method, related to the presence of perforations in the considered microscopic domain, are treated here for the first time; see Section 3. The main advantage of using the unfolding technique to prove corrector estimates is that only H^1-regularity of solutions of microscopic equations and of unit cell problems is required, compared to standard methods (mostly based on energy-type estimates) used in the derivation of corrector estimates. As a natural consequence of this fact, the set of choices of microstructures is now much larger.

This article is structured as follows: after introducing model equations and the assumed microscopic geometry of the concrete material, the Section 2 goes on with the main assumptions and basic estimates ensuring both the solvability of the microscopic problem and the convergence of microscopic solutions to a solution of the macroscopic equations, as $\varepsilon \to 0$. In Section 3, we state and prove the corrector estimates for the concrete corrosion model, Theorem 3.6, determining the range of validity of the upscaled model.

Note that the technique developed in this article can be applied in a straightforward way to derive convergence rates for solutions of other classes of partial differential equations, posed in domains with periodically distributed microstructures.

2. Problem description

2.1. *Geometry*

We assume that concrete piece consists of a system of pores periodically distributed inside the three-dimensional cube $\Omega = [a, b]^3$ with $a, b \in \mathbb{R}$ and $b > a$. Since usually the concrete in sewer pipes is not completely dry, we consider a partially saturated porous material. We assume that every pore has three distinct non-overlapping parts: a solid part, the water film which surrounds the solid part and an air layer bounding the water film and filling the space of Y as shown in Figure 1. Note that the dark (black) parts indicate the water-filled parts in the material where most of our model equations are defined. The reference pore, $Y = [0, 1]^3$, has three pairwise disjoint domains Y_0, Y_1 and Y_2 with smooth boundaries Γ_1 and Γ_2 as shown in the figure. Moreover, $Y = \bar{Y}_0 \cup \bar{Y}_1 \cup \bar{Y}_2$.

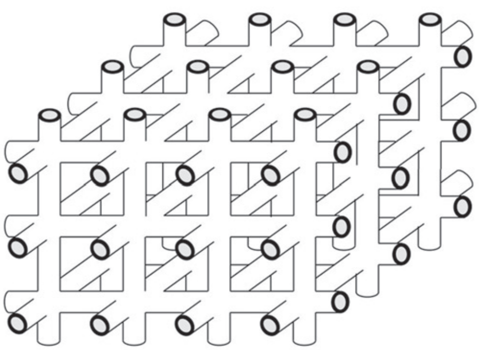
Let ε be a small factor denoting the ratio between the characteristic length of the pore Y and the characteristic length of the domain Ω. Let χ_1 and χ_2 be the characteristic functions of the sets Y_1 and Y_2, respectively. The shifted set Y_1^k is defined by $Y_1^k := Y_1 + \Sigma_{j=0}^{3} k_j e_j$ for $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$, where e_j is the jth unit vector. The union of all Y_1^k multiplied by ε that are contained within Ω defines the perforated domain Ω_1^ε, namely $\Omega_1^\varepsilon := \cup_{k \in \mathbb{Z}^3} \{\varepsilon Y_1^k \mid \varepsilon Y_1^k \subset \Omega\}$.

Similarly, Ω_2^ε, Γ_1^ε and Γ_2^ε denote the union of εY_2^k, $\varepsilon \Gamma_1^k$ and $\varepsilon \Gamma_2^k$, contained in Ω.

2.2. *Microscopic equations*

We consider the microscopic model

$$\begin{cases} \partial_t u^\varepsilon - \nabla \cdot (D_u^\varepsilon \nabla u^\varepsilon) = -f(u^\varepsilon, v^\varepsilon) & \text{in } (0, T) \times \Omega_1^\varepsilon, \\ \partial_t v^\varepsilon - \nabla \cdot (D_v^\varepsilon \nabla v^\varepsilon) = f(u^\varepsilon, v^\varepsilon) & \text{in } (0, T) \times \Omega_1^\varepsilon, \\ \partial_t w^\varepsilon - \nabla \cdot (D_w^\varepsilon \nabla w^\varepsilon) = 0 & \text{in } (0, T) \times \Omega_2^\varepsilon, \\ \partial_t r^\varepsilon = \eta(u^\varepsilon, r^\varepsilon) & \text{on } (0, T) \times \Gamma_1^\varepsilon, \end{cases} \tag{1}$$



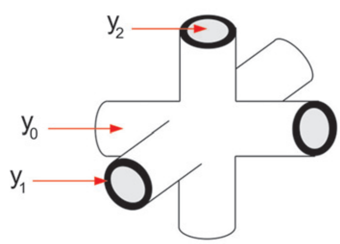


Figure 1. Left: Periodic approximation of the concrete piece. Right: Our choice of the microstructure. Y_0, Y_1 and Y_2 are the solid, water and air phases of Y, respectively.

with the initial conditions

$$\begin{cases} u^\varepsilon(0,x) = u_0(x), \quad v^\varepsilon(0,x) = v_0(x) & \text{in } \Omega_1^\varepsilon, \\ w^\varepsilon(0,x) = w_0(x) \quad \text{in } \Omega_2^\varepsilon, \quad r^\varepsilon(0,x) = r_0(x) & \text{on } \Gamma_1^\varepsilon \end{cases} \tag{2}$$

and the boundary conditions

$$u^\varepsilon = 0, v^\varepsilon = 0 \quad \text{on } (0,T) \times \partial\Omega \cap \partial\Omega_1^\varepsilon, \quad w^\varepsilon = 0 \quad \text{on } (0,T) \times \partial\Omega \cap \partial\Omega_2^\varepsilon, \tag{3}$$

together with

$$\begin{cases} D_u^\varepsilon \nabla u^\varepsilon \cdot \nu = -\varepsilon\eta(u^\varepsilon, r^\varepsilon) & \text{on } (0,T) \times \Gamma_1^\varepsilon, \\ D_v^\varepsilon \nabla v^\varepsilon \cdot \nu = 0 & \text{on } (0,T) \times \Gamma_1^\varepsilon, \\ D_u^\varepsilon \nabla u^\varepsilon \cdot \nu = 0 & \text{on } (0,T) \times \Gamma_2^\varepsilon, \\ D_v^\varepsilon \nabla v^\varepsilon \cdot \nu = \varepsilon(a^\varepsilon(x) w^\varepsilon - b^\varepsilon(x) v^\varepsilon) & \text{on } (0,T) \times \Gamma_2^\varepsilon, \\ D_w^\varepsilon \nabla w^\varepsilon \cdot \nu = -\varepsilon(a^\varepsilon(x) w^\varepsilon - b^\varepsilon(x) v^\varepsilon) & \text{on } (0,T) \times \Gamma_2^\varepsilon. \end{cases} \tag{4}$$

Concrete corrosion is modelled by diffusion and reaction of three microscopically active chemical species: sulphuric acid u^ε, hydrogen sulphide v^ε in water phase, and hydrogen sulphide gas w^ε in air layers. The transfer of hydrogen sulphide from air into a water film is modelled by interface reaction given by Henry's law [6] where a^ε and b^ε are mass-transfer coefficients. The catalysis of hydrogen sulphide into sulphuric acid is defined via a nonlinear reaction. The mass concentration of a chemical compound bound on surface (gypsum), produced through reaction of sulphuric acid with solid matrix, is represented by r^ε. The evolution of r^ε typically define criteria on how important the corrosion process is [5]. We refer the reader to [2,4] for the mathematical modelling of the corrosion mechanism as well as for details on the structure of the bulk and surface production terms by reaction $f(\cdot)$ and $\eta(\cdot)$.

Assumption 2.1

(A1) $D_i, \partial_t D_i \in L^\infty(0,T; L^\infty_{\text{per}}(Y))^{3\times 3}$, $i \in \{u,v,w\}$, $(D_i(t,x)\xi, \xi) \geq D_i^0|\xi|^2$ for $D_i^0 > 0$, *for every* $\xi \in \mathbb{R}^3$ *and a.a.* $(t,x) \in (0,T) \times Y$.

(A2) $k \in L^\infty_{\text{per}}(\Gamma_1)$ *is nonnegative and* $\eta(\alpha,\beta) = k(y)R(\alpha)Q(\beta)$, *where* $R\colon \mathbb{R} \to \mathbb{R}_+$, $Q\colon \mathbb{R} \to \mathbb{R}_+$ *are sublinear and locally Lipschitz continuous. Furthermore,* $R(\alpha) = 0$ *for* $\alpha < 0$ *and* $Q(\beta) = 0$ *for* $\beta \geq \beta_{\max}$, *with some* $\beta_{\max} > 0$.

(A3) $f \in C^1(\mathbb{R}^2)$ *is sublinear and globally Lipschitz continuous in both variables, i.e.* $|f(\alpha,\beta)| \leq C_f(1 + |\alpha| + |\beta|)$, $|f(\alpha_1,\beta_1) - f(\alpha_2,\beta_2)| \leq C_L(|\alpha_1 - \alpha_2| + |\beta_1 - \beta_2|)$ *and* $f(\alpha,\beta) = 0$ *for* $\alpha < 0$ *or* $\beta < 0$.

(A4) $a, b \in L^\infty_{\text{per}}(\Gamma_2)$, $a(y)$ *and* $b(y)$ *are positive for a.a.* $y \in \Gamma_2$ *and there exist* M_v, M_w *such that* $b(y)M_v = a(y)\, M_w$ *for a.a.* $y \in \Gamma_2$.

(A5) *Initial data* $(u_0, v_0, w_0, r_0) \in [H^2(\Omega) \cap H_0^1(\Omega) \cap L^\infty(\Omega)]^3 \times L^\infty_{\text{per}}(\Gamma_1)$ *are nonnegative and* $v_0(x) \leq M_v$, $w_0(x) \leq M_w$ *a.e. in* Ω.

We define the oscillating coefficients $D_i^\varepsilon(t,x) := D_i\left(t, \frac{x}{\varepsilon}\right)$, where $i \in \{u,v,w\}$, $a^\varepsilon(x) := a\left(\frac{x}{\varepsilon}\right)$, $b^\varepsilon(x) := b\left(\frac{x}{\varepsilon}\right)$, $k^\varepsilon(x) := k\left(\frac{x}{\varepsilon}\right)$, as well as the space $H^1_{\partial\Omega}(\Omega_i^\varepsilon) := \{u \in H^1(\Omega_i^\varepsilon) : u = 0 \text{ on } \partial\Omega \cap \partial\Omega_i^\varepsilon\}$, $i = 1,2$.

Definition 2.2 We call $(u^\varepsilon, v^\varepsilon, w^\varepsilon, r^\varepsilon)$ a weak solution of (1)–(4) if $u^\varepsilon, v^\varepsilon \in L^2(0,T; H^1_{\partial\Omega}(\Omega_1^\varepsilon)) \cap H^1(0,T; L^2(\Omega_1^\varepsilon))$, $w^\varepsilon \in L^2(0,T; H^1_{\partial\Omega}(\Omega_2^\varepsilon)) \cap H^1(0,T; L^2(\Omega_2^\varepsilon))$, $r^\varepsilon \in H^1(0,T;$

$L^2(\Gamma_1^\varepsilon)$) and satisfies the following equations:

$$\int_0^T \int_{\Omega_1^\varepsilon} \big(\partial_t u^\varepsilon \phi + D_u^\varepsilon \nabla u^\varepsilon \nabla \phi + f(u^\varepsilon, v^\varepsilon)\phi\big) \mathrm{d}x\, \mathrm{d}t = -\varepsilon \int_0^T \int_{\Gamma_1^\varepsilon} \eta(u^\varepsilon, r^\varepsilon)\phi\, \mathrm{d}\gamma\, \mathrm{d}t, \tag{5}$$

$$\int_0^T \int_{\Omega_1^\varepsilon} \big(\partial_t v^\varepsilon \phi + D_v^\varepsilon \nabla v^\varepsilon \nabla \phi - f(u^\varepsilon, v^\varepsilon)\phi\big) \mathrm{d}x\, \mathrm{d}t = \varepsilon \int_0^T \int_{\Gamma_2^\varepsilon} \big(a^\varepsilon\, w^\varepsilon - b^\varepsilon\, v^\varepsilon\big)\phi\, \mathrm{d}\gamma\, \mathrm{d}t, \tag{6}$$

$$\int_0^T \int_{\Omega_2^\varepsilon} \big(\partial_t w^\varepsilon \varphi + D_w^\varepsilon \nabla w^\varepsilon \nabla \varphi\big) \mathrm{d}x\, \mathrm{d}t = -\varepsilon \int_0^T \int_{\Gamma_2^\varepsilon} \big(a^\varepsilon\, w^\varepsilon - b^\varepsilon\, v^\varepsilon\big)\varphi\, \mathrm{d}\gamma\, \mathrm{d}t, \tag{7}$$

$$\varepsilon \int_0^T \int_{\Gamma_1^\varepsilon} \partial_t r^\varepsilon \psi\, \mathrm{d}\gamma\, \mathrm{d}t = \varepsilon \int_0^T \int_{\Gamma_1^\varepsilon} \eta(u^\varepsilon, r^\varepsilon)\psi\, \mathrm{d}\gamma\, \mathrm{d}t \tag{8}$$

for all $\phi \in L^2(0, T; H^1_{\partial\Omega}(\Omega_1^\varepsilon))$, $\varphi \in L^2(0, T; H^1_{\partial\Omega}(\Omega_2^\varepsilon))$, $\psi \in L^2((0, T) \times \Gamma_1^\varepsilon)$ and $u^\varepsilon(t) \to u_0$, $v^\varepsilon(t) \to v_0$ in $L^2(\Omega_1^\varepsilon)$, $w^\varepsilon(t) \to w_0$ in $L^2(\Omega_2^\varepsilon)$, $r^\varepsilon(t) \to r_0$ in $L^2(\Gamma_1^\varepsilon)$ as $t \to 0$.

LEMMA 2.3 *Under the Assumption* 2.1, *solutions of the problem* (1)–(4) *satisfy the following* a priori *estimates*:

$$\begin{aligned}
\|u^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_1^\varepsilon))} + \|\nabla u^\varepsilon\|_{L^2((0,T)\times\Omega_1^\varepsilon)} &\le C,\\
\|v^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_1^\varepsilon))} + \|\nabla v^\varepsilon\|_{L^2((0,T)\times\Omega_1^\varepsilon)} &\le C,\\
\|w^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_2^\varepsilon))} + \|\nabla w^\varepsilon\|_{L^2((0,T)\times\Omega_2^\varepsilon)} &\le C,\\
\varepsilon^{1/2}\|r^\varepsilon\|_{L^\infty(0,T;L^2(\Gamma_1^\varepsilon))} + \varepsilon^{1/2}\|\partial_t r^\varepsilon\|_{L^2((0,T)\times\Gamma_1^\varepsilon)} &\le C,
\end{aligned} \tag{9}$$

where the constant C is independent of ε.

Proof First, we consider as test functions $\phi = u^\varepsilon$ in (5), $\phi = v^\varepsilon$ in (6), $\psi = w^\varepsilon$ in (7) and use Assumption 2.1, Young's inequality and the trace inequality, i.e.

$$\begin{aligned}
&\varepsilon \int_0^t \int_{\Gamma_2^\varepsilon} w^\varepsilon v^\varepsilon \mathrm{d}\gamma\, \mathrm{d}\tau\\
&\quad \le C \int_0^t \int_{\Omega_2^\varepsilon} (|w^\varepsilon|^2 + \varepsilon^2 |\nabla w^\varepsilon|^2) \mathrm{d}\gamma\, \mathrm{d}\tau + C \int_0^t \int_{\Omega_1^\varepsilon} (|v^\varepsilon|^2 + \varepsilon^2 |\nabla v^\varepsilon|^2) \mathrm{d}\gamma\, \mathrm{d}\tau.
\end{aligned}$$

Then, adding the obtained inequalities, choosing ε conveniently and applying Gronwall's inequality imply the first three estimates in the the lemma.

Taking $\psi = r^\varepsilon$ as a test function in (8) and using (A2) from Assumption 2.1 and the estimates for u^ε, yield the estimate for r^ε. The test function $\psi = \partial_t r^\varepsilon$ in (8), the sublinearity of R, the boundedness of Q and the estimates for u^ε imply the boundedness of $\varepsilon^{1/2}\|\partial_t r^\varepsilon\|_{L^2((0,T)\times\Gamma_1^\varepsilon)}$. ■

LEMMA 2.4 (Positivity and boundedness) *Let Assumption* 2.1 *be fulfilled. Then the following estimates hold*:

(i) $u^\varepsilon(t), v^\varepsilon(t) \ge 0$ *a.e. in* Ω_1^ε, $w^\varepsilon(t) \ge 0$ *a.e. in* Ω_2^ε *and* $u^\varepsilon(t), r^\varepsilon(t) \ge 0$ *a.e. on* Γ_1^ε, *for a.a.* $t \in (0, T)$.

(ii) $u^\varepsilon(t) \le M_u e^{A_u t}$, $v^\varepsilon(t) \le M_v e^{A_v t}$ *a.e. in* Ω_1^ε , $w^\varepsilon(t) \le M_w e^{A_w t}$ *a.e. in* Ω_2^ε, *and* $u^\varepsilon(t) \le M_u e^{A_u t}$, $r^\varepsilon(t) \le M_r e^{A_r t}$ *a.e. on* Γ_1^ε, *for a.a.* $t \in (0, T)$, *with some positive numbers* A_j, M_j, *where* $j = u, v, w, r$.

Proof (i) To show the positivity of a weak solution we consider $u^{\varepsilon-}$ as test function in (5), $v^{\varepsilon-}$ in (6), $w^{\varepsilon-}$ in (7) and $r^{\varepsilon-}$ in (8), where $\phi^- = \min\{0, \phi\}$ with $\phi^+ \phi^- = 0$. The integrals involving $f(u^\varepsilon, v^\varepsilon)u^{\varepsilon-}$, $f(u^\varepsilon, v^\varepsilon)v^{\varepsilon-}$ and $\eta(u^\varepsilon, r^\varepsilon)u^{\varepsilon-}$ are zero, since by Assumption 2.1 $f(u, v)$ is zero for negative u or v and $\eta(u, r)$ is zero for negative u. In the integrals over Γ_2^ε we use the positivity of a and b and the estimate $v^\varepsilon w^{\varepsilon-} = (v^{\varepsilon+} + v^{\varepsilon-})w^{\varepsilon-} \le v^{\varepsilon-}w^{\varepsilon-}$. Due to the positivity of η, the right-hand side in the equation for r^ε, with the test function $\psi = r^{\varepsilon-}$, is nonpositive. Adding the obtained inequalities, applying both Young's and the trace inequalities, considering ε sufficiently small, we obtain, due to positivity of the initial data and using Gronwall's inequality, that

$$\|u^{\varepsilon-}(t)\|_{L^2(\Omega_1^\varepsilon)} + \|v^{\varepsilon-}(t)\|_{L^2(\Omega_1^\varepsilon)} + \|w^{\varepsilon-}(t)\|_{L^2(\Omega_2^\varepsilon)} + \|r^{\varepsilon-}(t)\|_{L^2(\Gamma_1^\varepsilon)} \le 0,$$

for a.a. $t \in (0, T)$. Thus, negative parts of the involved concentrations are equal to zero a.e. in $(0, T) \times \Omega_i^\varepsilon$, $i = 1, 2$, or in $(0, T) \times \Gamma_1^\varepsilon$, respectively.

(ii) To show the boundedness of solutions, we consider $U_M^\varepsilon = (u^\varepsilon - e^{A_u t}M_u)^+$ as a test function in (5), $V_M^\varepsilon = (v^\varepsilon - e^{A_v t}M_v)^+$ in (6) $W_M^\varepsilon = (w^\varepsilon - e^{A_w t}M_w)^+$ in (7), where $(\phi - M)^+ = \max\{0, \phi - M\}$ and M_i, $i = u, v, w$, are positive numbers, such that $u_0(x) \le M_u$, $v_0(x) \le M_v$, $w_0(x) \le M_w$ a.e in Ω, also $A_v = A_w$ and M_v, M_w are given by (A4) in Assumption 2.1. Note $U_M^\varepsilon, V_M^\varepsilon \in H^1_{\partial\Omega}(\Omega_1^\varepsilon)$, $W_M^\varepsilon \in H^1_{\partial\Omega}(\Omega_2^\varepsilon)$ [10]. Adding the obtained equations and using Assumption 2.1 yield

$$\begin{aligned}
&\int_0^\tau \left(\int_{\Omega_1^\varepsilon} \partial_t(|U_M^\varepsilon|^2 + |V_M^\varepsilon|^2) + |\nabla U_M^\varepsilon|^2 + |\nabla V_M^\varepsilon|^2 \mathrm{d}x + \int_{\Omega_2^\varepsilon} \partial_t |W_M^\varepsilon|^2 + |\nabla W_M^\varepsilon|^2 \mathrm{d}x \right) \mathrm{d}t \\
&\quad \le C \int_0^\tau \Bigg[\int_{\Omega_1^\varepsilon} \Big(\big(e^{A_u t}M_u(C_f - A_u) + C_f e^{A_v t}M_v\big) U_M^\varepsilon + |U_M^\varepsilon|^2 + |V_M^\varepsilon|^2 + \varepsilon^2 |\nabla V_M^\varepsilon|^2 \\
&\qquad + \big(C_f e^{A_u t}M_u + e^{A_v t}M_v(C_f - M_v)\big) V_M^\varepsilon \Big) \mathrm{d}x + \int_{\Omega_2^\varepsilon} \Big(|W_M^\varepsilon|^2 + \varepsilon^2 |\nabla W_M^\varepsilon|^2 \Big) \mathrm{d}x \Bigg] \mathrm{d}t.
\end{aligned}$$

Choosing A_u, M_u such that $C_f e^{A_u t}M_u + C_f e^{A_v t}M_v - A_u e^{A_u t}M_u \le 0$ and $C_f e^{A_u t}M_u + C_f e^{A_v t}M_v - A_v e^{A_v t}M_v \le 0$ for a.a. $t \in (0, T)$, and ε sufficiently small, Gronwall's inequality implies the estimates for u^ε, v^ε, w^ε, stated in the lemma.

Lemma A.1 in the appendix and H^1-estimates for u^ε in Lemma 2.3 imply $u^\varepsilon(t) \ge 0$ and $u^\varepsilon(t) \le e^{A_u t}M_u$ a.e on Γ_1^ε for a.a. $t \in (0, T)$. The assumption on η and Equation (8) with the test function $(r^\varepsilon - e^{A_r t}M_r)^+$, where $r_0(x) \le M_r$ a.e. on Γ_1, yield

$$\begin{aligned}
&\varepsilon \int_0^\tau \int_{\Gamma_1^\varepsilon} \left(\frac{1}{2} \partial_t |(r^\varepsilon - e^{A_r t}M_r)^+|^2 + A_r e^{A_r t}M_r (r^\varepsilon - e^{A_r t}M_r)^+ \right) \mathrm{d}\gamma\, \mathrm{d}t \\
&\quad = \varepsilon \int_0^\tau \int_{\Gamma_1^\varepsilon} \eta(u^\varepsilon, r^\varepsilon)(r^\varepsilon - e^{A_r t}M_r)^+ \mathrm{d}\gamma\, \mathrm{d}t \le C_\eta(A_u, M_u)\varepsilon \int_0^\tau \int_{\Gamma_1^\varepsilon} (r^\varepsilon - e^{A_r t}M_r)^+ \mathrm{d}\gamma\, \mathrm{d}t.
\end{aligned}$$

This, for A_r and M_r, such that $C_\eta \le A_r M_r e^{A_r T}$, implies the boundedness of r^ε on Γ_1^ε for a.e. $t \in (0, T)$. ■

LEMMA 2.5 *Under Assumption* 2.1, *we obtain the estimates*, *independent of* ε:

$$\|\partial_t u^\varepsilon\|_{L^2((0,T)\times\Omega_1^\varepsilon)} + \|\partial_t v^\varepsilon\|_{L^2(0,T;H^1(\Omega_1^\varepsilon))} + \|\partial_t w^\varepsilon\|_{L^2(0,T;H^1(\Omega_2^\varepsilon))} \leq C.$$

Proof We test (5) with $\phi = \partial_t u^\varepsilon$, and using the structure of η, the regularity assumptions on R and Q and the boundedness of u^ε and r^ε on Γ_1^ε, we estimate the boundary integral by

$$\begin{aligned}
&\varepsilon \int_0^\tau \int_{\Gamma_1^\varepsilon} \eta(u^\varepsilon, r^\varepsilon)\partial_t u^\varepsilon \mathrm{d}\gamma\, \mathrm{d}t \\
&\quad = \varepsilon \int_0^\tau \int_{\Gamma_1^\varepsilon} k^\varepsilon \big(\partial_t\big(\mathcal{R}(u^\varepsilon)Q(r^\varepsilon)\big) - \mathcal{R}(u^\varepsilon)Q'(r^\varepsilon)\partial_t r^\varepsilon\big)\mathrm{d}\gamma\, \mathrm{d}t \\
&\quad \leq C\int_{\Omega_1^\varepsilon} \big(|u^\varepsilon|^2 + \varepsilon^2|\nabla u^\varepsilon|^2 + |u_0|^2 + \varepsilon^2|\nabla u_0|^2\big)\mathrm{d}x + C\varepsilon \int_0^\tau \int_{\Gamma_1^\varepsilon} \big(1 + |\partial_t r^\varepsilon|^2\big)\mathrm{d}\gamma\, \mathrm{d}t,
\end{aligned}$$

where $\mathcal{R}(\alpha) = \int_0^\alpha R(\xi)\mathrm{d}\xi$. Then, Assumption 2.1, estimates in Lemma 2.3 and the fact that $D_u^0/2 - \varepsilon^2 \geq 0$ for appropriate ε imply the estimate for $\partial_t u^\varepsilon$.

In order to estimate $\partial_t v^\varepsilon$ and $\partial_t w^\varepsilon$, we differentiate the corresponding equations with respect to the time variable and then test the result with $\partial_t v^\varepsilon$ and $\partial_t w^\varepsilon$, respectively. Due to assumptions on f and using the trace inequality, we obtain

$$\begin{aligned}
&\int_{\Omega_1^\varepsilon} |\partial_t v^\varepsilon|^2 \mathrm{d}x + C\int_0^\tau \int_{\Omega_1^\varepsilon} |\nabla \partial_t v^\varepsilon|^2 \mathrm{d}x\, \mathrm{d}t \\
&\quad \leq C\int_0^\tau \int_{\Omega_2^\varepsilon} \big(|\partial_t w^\varepsilon|^2 + \varepsilon^2|\nabla\partial_t w^\varepsilon|^2\big)\mathrm{d}x\, \mathrm{d}t \\
&\qquad + C\int_0^\tau \int_{\Omega_1^\varepsilon} \big(|\partial_t u^\varepsilon|^2 + |\partial_t v^\varepsilon|^2 + |\nabla v^\varepsilon|^2\big)\mathrm{d}x\, \mathrm{d}t + \int_{\Omega_1^\varepsilon} |\partial_t v^\varepsilon(0)|^2 \mathrm{d}x,
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
&\int_{\Omega_2^\varepsilon} |\partial_t w^\varepsilon|^2 \mathrm{d}x + C\int_0^\tau \int_{\Omega_2^\varepsilon} |\nabla\partial_t w^\varepsilon|^2 \mathrm{d}x\, \mathrm{d}t \\
&\quad \leq C\int_0^\tau \int_{\Omega_2^\varepsilon} \big(|\partial_t w^\varepsilon|^2 + |\nabla w^\varepsilon|^2\big)\mathrm{d}x\, \mathrm{d}t \\
&\qquad + \int_{\Omega_2^\varepsilon} |\partial_t w^\varepsilon(0)|^2 \mathrm{d}x + C\int_0^\tau \int_{\Omega_1^\varepsilon} \big(|\partial_t v^\varepsilon|^2 + \varepsilon^2|\nabla\partial_t v^\varepsilon|^2\big)\mathrm{d}x\, \mathrm{d}t.
\end{aligned} \tag{11}$$

The regularity assumptions imply that $\|\partial_t v^\varepsilon(0)\|_{L^2(\Omega_1^\varepsilon)}$ and $\|\partial_t w^\varepsilon(0)\|_{L^2(\Omega_2^\varepsilon)}$ can be estimated by the H^2-norm of v_0 and w_0. Adding (10) and (11), making use of estimates for $\partial_t u^\varepsilon$, ∇v^ε and ∇w^ε and applying Gronwall's lemma, give the desired estimates. ■

LEMMA 2.6 (Existence and uniqueness) *Let Assumption* 2.1 *be fulfilled. Then there exists a unique global-in-time weak solution in the sense of Definition* 2.2.

Proof The Lipschitz continuity of f, local Lipschitz continuity of η and the boundedness of u^ε and r^ε on Γ_1^ε ensure the uniqueness result. The existence of weak solutions follows by a standard Galerkin approach [11] using the a priori estimates in Lemmas 2.3, 2.4 and 2.5. ■

2.3. *Unfolded limit equations*

Definition 2.7 [12–15] (1) For any function ϕ Lebesgue-measurable on perforated domain Ω_i^ε, the unfolding operator $\mathcal{T}^\varepsilon_{Y_i}\colon \Omega_i^\varepsilon \to \Omega\times Y_i$, $i=1,2$, is defined

$$\mathcal{T}^\varepsilon_{Y_i}(\phi)(x,y) = \begin{cases} \phi\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{a.e. for } y\in Y_i,\ x\in\tilde{\Omega}^\varepsilon_{\text{int}}, \\ 0 & \text{a.e. for } y\in Y_i,\ x\in\Omega\setminus\tilde{\Omega}^\varepsilon_{\text{int}}, \end{cases}$$

where $k := [\frac{x}{\varepsilon}]$ is the unique integer combination, such that $x-[\frac{x}{\varepsilon}]$ belongs to Y_i, and $\tilde{\Omega}^\varepsilon_{\text{int}} = \text{Int}(\cup_{k\in\mathbb{Z}^3}\{\varepsilon\overline{Y^k}, \varepsilon Y^k\subset\Omega\})$.

We note that for $w\in H^1(\Omega)$ it holds that $\mathcal{T}^\varepsilon_{Y_i}(w|_{\Omega_i^\varepsilon}) = \mathcal{T}^\varepsilon_{Y}(w)|_{\Omega\times Y_i}$.

(2) For any function ϕ Lebesgue-measurable on oscillating boundary Γ_i^ε, the boundary unfolding operator $\mathcal{T}^\varepsilon_{\Gamma_i}\colon \Gamma_i^\varepsilon\to\Omega\times\Gamma_i$, $i=1,2$ is defined by

$$\mathcal{T}^\varepsilon_{\Gamma_i}(\phi)(x,y) = \begin{cases} \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{a.e. for } y\in \Gamma_i,\ x\in\tilde{\Omega}^\varepsilon_{\text{int}}, \\ 0 & \text{a.e. for } y\in \Gamma_i,\ x\in\Omega\setminus\tilde{\Omega}^\varepsilon_{\text{int}}. \end{cases}$$

LEMMA 2.8 *Under Assumption 2.1, there exist* $u,v,w\in L^2(0,T;H^1_0(\Omega))\cap H^1(0,T;L^2(\Omega))$, $\tilde{u}$, $\tilde{v}\in L^2((0,T)\times\Omega;H^1_{\text{per}}(Y_1))$, $\tilde{w}\in L^2((0,T)\times\Omega;H^1_{\text{per}}(Y_2))$, *and* $r\in H^1(0,T,L^2(\Omega\times\Gamma_1))$ *such that (up to a subsequence) for* $\varepsilon\to 0$

$$\begin{aligned}
&\mathcal{T}^\varepsilon_{Y_1}(u^\varepsilon)\to u,\quad \mathcal{T}^\varepsilon_{Y_1}(v^\varepsilon)\to v && \textit{in } L^2((0,T)\times\Omega;H^1(Y_1)),\\
&\partial_t\mathcal{T}^\varepsilon_{Y_1}(u^\varepsilon)\rightharpoonup\partial_t u,\quad \partial_t\mathcal{T}^\varepsilon_{Y_1}(v^\varepsilon)\rightharpoonup\partial_t v && \textit{in } L^2((0,T)\times\Omega\times Y_1),\\
&\mathcal{T}^\varepsilon_{Y_2}(w^\varepsilon)\to w,\quad \partial_t\mathcal{T}^\varepsilon_{Y_2}(w^\varepsilon)\rightharpoonup\partial_t w && \textit{in } L^2((0,T)\times\Omega;H^1(Y_2)),\\
&\mathcal{T}^\varepsilon_{Y_1}(\nabla u^\varepsilon)\rightharpoonup\nabla u+\nabla_y\tilde{u} && \textit{in } L^2((0,T)\times\Omega\times Y_1),\\
&\mathcal{T}^\varepsilon_{Y_1}(\nabla v^\varepsilon)\rightharpoonup\nabla v+\nabla_y\tilde{v} && \textit{in } L^2((0,T)\times\Omega\times Y_1),\\
&\mathcal{T}^\varepsilon_{Y_2}(\nabla w^\varepsilon)\rightharpoonup\nabla w+\nabla_y\tilde{w} && \textit{in } L^2((0,T)\times\Omega\times Y_2),
\end{aligned}\tag{12}$$

and

$$\begin{aligned}
&\mathcal{T}^\varepsilon_{\Gamma_1}(r^\varepsilon)\to r,\quad \partial_t\mathcal{T}^\varepsilon_{\Gamma_1}(r^\varepsilon)\rightharpoonup\partial_t r,\quad \mathcal{T}^\varepsilon_{\Gamma_1}(u^\varepsilon)\to u && \textit{in } L^2((0,T)\times\Omega\times\Gamma_1),\\
&\mathcal{T}^\varepsilon_{\Gamma_2}(v^\varepsilon)\to v,\quad \mathcal{T}^\varepsilon_{\Gamma_2}(w^\varepsilon)\to w && \textit{in } L^2((0,T)\times\Omega\times\Gamma_2).
\end{aligned}\tag{13}$$

Proof Applying estimates in Lemmas 2.3, 2.5 and convergence theorem [13,16] see Theorem A.3 in the appendix, implies the convergences for u^ε, v^ε, w^ε in (12). The strong convergence of r^ε is achieved by showing that $\mathcal{T}^\varepsilon_{\Gamma_1}(r^\varepsilon)$ is a Cauchy sequence in $L^2((0,T)\times\Omega\times\Gamma_1)$, for the proof see [4,17]. *A priori* estimate for $\partial_t r^\varepsilon$ and the convergence properties of $\mathcal{T}^\varepsilon_{\Gamma_1}$ [13] imply the convergences of $\partial_t\mathcal{T}^\varepsilon_{\Gamma_1}(r^\varepsilon)$. To show the other convergences in (13), we make use of the trace theorem [10], and of the strong convergence of $\mathcal{T}^\varepsilon_{Y_1}(u^\varepsilon)$, i.e. $\|\mathcal{T}^\varepsilon_{\Gamma_1}(u^\varepsilon)-u\|_{L^2((0,T)\times\Omega\times\Gamma_1)}\le C\|\mathcal{T}^\varepsilon_{Y_1}(u^\varepsilon)-u\|_{L^2((0,T)\times\Omega;H^1(Y_1))}\to 0$ as $\varepsilon\to 0$. ■

THEOREM 2.9 *Under Assumption* 2.1, *the sequences of weak solutions of the problem* (1)–(4) *converges as* $\varepsilon \to 0$ *to a weak solution* (u, v, w, r) *of a macroscopic model*, *i.e.* $u, v, w \in L^2(0,T; H_0^1(\Omega)) \cap H^1(0,T; L^2(\Omega))$, $r \in H^1(0,T; L^2(\Omega \times \Gamma_1))$ *and* u, v, w, r *satisfy the macroscopic equations*

$$
\begin{aligned}
&\int_0^T \int_{\Omega\times Y_1} \partial_t u\phi_1 + D_u(t,y)\left(\nabla u + \sum_{j=1}^n \frac{\partial u}{\partial x_j}\nabla_y \omega_u^j\right)(\nabla\phi_1 + \nabla_y\tilde{\phi}_1) + f(u,v)\phi_1\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = -\int_0^T \int_{\Omega\times\Gamma_1} \eta(u,r)\phi_1 \mathrm{d}\gamma_y\,\mathrm{d}x\,\mathrm{d}t, \\
&\int_0^T \int_{\Omega\times Y_1} \partial_t v\phi_1 + D_v(t,y)\left(\nabla v + \sum_{j=1}^n \frac{\partial v}{\partial x_j}\nabla_y \omega_v^j\right)(\nabla\phi_1 + \nabla_y\tilde{\phi}_1) - f(u,v)\phi_1 \mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^T \int_{\Omega\times\Gamma_2} (a(y)w - b(y)v)\phi_1\,\mathrm{d}\gamma_y\,\mathrm{d}x\,\mathrm{d}t, \\
&\int_0^T \int_{\Omega\times Y_2} \partial_t w\phi_2 + D_w(t,y)\left(\nabla w + \sum_{j=1}^n \frac{\partial w}{\partial x_j}\nabla_y \omega_w^j\right)(\nabla\phi_2 + \nabla_y\tilde{\phi}_2)\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = -\int_0^T \int_{\Omega\times\Gamma_2} (a(y)w - b(y)v)\phi_2\,\mathrm{d}\gamma_y\,\mathrm{d}x\,\mathrm{d}t, \\
&\int_0^T \int_{\Omega\times\Gamma_1} \partial_t r\psi\,\mathrm{d}\gamma_y\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\Omega\times\Gamma_1} \eta(u,r)\psi\,\mathrm{d}\gamma_y\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
\tag{14}
$$

for $\phi_1, \phi_2 \in L^2(0,T; H_0^1(\Omega))$, $\tilde{\phi}_1 \in L^2((0,T)\times\Omega; H^1_{\mathrm{per}}(Y_1))$, $\tilde{\phi}_2 \in L^2((0,T)\times\Omega; H^1_{\mathrm{per}}(Y_2))$ *and* $\psi \in L^2((0,T)\times\Omega\times\Gamma_1)$, *where* ω_u^j, ω_v^j *and* ω_w^j, *for* $j = 1, \ldots, n$, *are solutions of the correspondent unit cell problems*

$$
\begin{aligned}
-\nabla_y(D_\zeta(t,y)\nabla_y\omega_\zeta^j) &= \sum_{k=1}^n \partial_{y_k} D_\zeta^{kj}(t,y) \quad \text{in } Y_1,\ \zeta = u, v, \\
-D_\zeta(t,y)\nabla\omega_\zeta^j \cdot \nu &= \sum_{k=1}^n D_\zeta^{kj}(t,y)\nu_k \quad \text{on } \Gamma_1 \cup \Gamma_2, \\
\omega_\zeta^j &\quad \text{is } Y\text{-periodic}, \int_{Y_1} \omega_\zeta^j(y)\mathrm{d}y = 0,
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
-\nabla_y(D_w(t,y)\nabla_y\omega_w^j) &= \sum_{k=1}^n \partial_{y_k} D_w^{kj}(t,y) \quad \text{in } Y_2, \\
-D_w(t,y)\nabla\omega_w^j \cdot \nu &= \sum_{k=1}^n D_w^{kj}(t,y)\nu_k \quad \text{on } \Gamma_2, \quad \omega_w^j \text{ is } Y\text{-periodic}, \quad \int_{Y_2} \omega_w^j(y)\mathrm{d}y = 0.
\end{aligned}
\tag{16}
$$

Proof Due to considered geometry of Ω_1^ε and Ω_2^ε we have

$$
\int_0^T \int_{\Omega_i^\varepsilon} u^\varepsilon\phi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\Omega\times Y_i} \mathcal{T}_{Y_i}^\varepsilon(u^\varepsilon)\mathcal{T}_{Y_i}^\varepsilon(\phi)\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t, \quad i = 1, 2.
$$

Applying the unfolding operator to (5)–(8), using $\mathcal{T}^{\varepsilon}_{Y_1} D_i(t, \frac{x}{\varepsilon}) = D_i(t, y), i \in \{u, v\}$ and $\mathcal{T}^{\varepsilon}_{Y_2} D_w(t, \frac{x}{\varepsilon}) = D_w(t, y)$, considering the limit as $\varepsilon \to 0$ and the convergences stated in Theorem 2.8, we obtain the unfolded limit problem. Similarly as for microscopic problem, using local Lipschitz continuity of η and f and boundedness of macroscopic solutions, which follows directly from the boundedness of microscopic solutions, we can show the uniqueness of a solution of the macroscopic model. Thus, the whole sequence of microscopic solutions converge to a solution of the limit problem. The functions $\tilde{u}, \tilde{v}, \tilde{w}$ are defined in terms of u, v, w and solutions $\omega_u^j, \omega_v^j, \omega_w^j$ of unit cell problems (15) and (16), see [4,17].

3. Corrector estimates

First of all, we introduce the definition of local average and averaging operators. After that, we show some technical estimates needed in the following. We define $(\mathbb{R}^n)^i = \mathbb{R}^n \cap \{\varepsilon(Y_i + \xi), \xi \in \mathbb{Z}^n\}$, $\tilde{\Omega}_i^{\varepsilon,l} = \{x \in (\mathbb{R}^n)^i : \operatorname{dist}(x, \Omega_i^{\varepsilon}) < l\sqrt{n}\varepsilon\}$, $\tilde{\Omega}^{\varepsilon,l} = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \Omega) < l\sqrt{n}\varepsilon\}$, for $l = 1, 2$, and $\tilde{\Gamma}^{\varepsilon}_{i,\mathrm{int}} = \cup_{k \in \mathbb{Z}^3}\{\varepsilon\Gamma_i^k, \varepsilon Y^k \subset \Omega\}$, where $i = 1, 2$.

Definition 3.1 [8,14] (1) For any $\phi \in L^p(\Omega_i^{\varepsilon})$, $p \in [1, \infty]$ and $i = 1, 2$, we define the local average operator ('mean in the cells') $\mathcal{M}^{\varepsilon}_{Y_i}: L^p(\Omega_i^{\varepsilon}) \to L^p(\Omega)$

$$\mathcal{M}^{\varepsilon}_{Y_i}(\phi)(x) = \frac{1}{|Y_i|}\int_{Y_i} \mathcal{T}^{\varepsilon}_{Y_i}(\phi)(x, y)\mathrm{d}y = \frac{1}{\varepsilon^n |Y_i|}\int_{\varepsilon[\frac{x}{\varepsilon}] + \varepsilon Y_i} \phi(y)\mathrm{d}y, \quad x \in \Omega.$$

(2) The operator $Q^{\varepsilon}_{Y_i}: L^p(\tilde{\Omega}_i^{\varepsilon,2}) \to W^{1,\infty}(\Omega)$, for $p \in [1, \infty]$ and $i = 1, 2$, is defined as Q_1-interpolation of $M^{\varepsilon}_{Y_i}(\phi)$, i.e. $Q^{\varepsilon}_{Y_i}(\phi)(\varepsilon\xi) = \mathcal{M}^{\varepsilon}_{Y_i}(\phi)(\varepsilon\xi)$ for $\xi \in \mathbb{Z}^n$ and

$$Q^{\varepsilon}_{Y_i}(\phi)(x) = \sum_{k \in \{0,1\}^n} Q^{\varepsilon}_{Y_i}(\phi)(\varepsilon\xi + \varepsilon k)\bar{x}_1^{k_1} \ldots \bar{x}_n^{k_n} \quad \text{for } x \in \varepsilon(Y + \xi),\ \xi \in \mathbb{Z}^n,$$

where for $x \in \varepsilon(Y + \xi)$ and $k = (k_1, \ldots, k_n) \in \{0, 1\}^n$ points $\bar{x}_l^{k_l}$ are given by

$$\bar{x}_l^{k_l} = \begin{cases} \dfrac{x_l - \varepsilon\xi_l}{\varepsilon}, & \text{if } k_l = 1, \\ 1 - \dfrac{x_l - \varepsilon\xi_l}{\varepsilon}, & \text{if } k_l = 0. \end{cases}$$

(3) The operator $Q^{\varepsilon}_{Y_i}: W^{1,p}(\Omega_i^{\varepsilon}) \to W^{1,\infty}(\Omega)$, $p \in [1, \infty]$ and $i = 1, 2$, is defined by $Q^{\varepsilon}_{Y_i}(\phi) = Q^{\varepsilon}_{Y_i}(\mathcal{P}(\phi))|_{\Omega_i^{\varepsilon}}$, where $Q^{\varepsilon}_{Y_i}$ is given in (2). and $\mathcal{P}: W^{1,p}(\Omega_i^{\varepsilon}) \to W^{1,p}((\mathbb{R}^n)^i)$ is an extension operator, in case there exists $\mathcal{P}$, $\|\mathcal{P}(\phi)\|_{W^{1,p}((\mathbb{R}^n)^i)} \leq C\|\phi\|_{W^{1,p}(\Omega_i^{\varepsilon})}$.

Note $\mathcal{T}^{\varepsilon}_{Y_i} \circ \mathcal{M}^{\varepsilon}_{Y_i}(\phi) = \mathcal{M}^{\varepsilon}_{Y_i}(\phi)$ for $\phi \in L^p(\Omega_i^{\varepsilon})$ and $\mathcal{M}^{\varepsilon}_{Y_i}(\phi)(x) = \mathcal{M}_{Y_i}(\mathcal{T}^{\varepsilon}_{Y_i}(\phi))(x)$, where $\mathcal{M}_{Y_i}$ is the mean value over Y_i, additionally $\sum_{k \in \{0,1\}^n} \bar{x}_1^{k_1} \ldots \bar{x}_n^{k_n} = 1$.

Definition 3.2 [13,16] (1) For $p \in [1, +\infty]$ and $i = 1, 2$, the averaging operator $\mathcal{U}^{\varepsilon}_{Y_i}: L^p(\Omega \times Y_i) \to L^p(\Omega_i^{\varepsilon})$ is defined as

$$\mathcal{U}^{\varepsilon}_{Y_i}(\Phi)(x) = \begin{cases} \dfrac{1}{|Y|}\displaystyle\int_Y \Phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z, \left\{\frac{x}{\varepsilon}\right\}_Y\right)\mathrm{d}z & \text{for a.a. } x \in \tilde{\Omega}^{\varepsilon}_{i,\mathrm{int}}, \\ 0 & \text{for a.a. } x \in \Omega_i^{\varepsilon} \setminus \tilde{\Omega}^{\varepsilon}_{i,\mathrm{int}}. \end{cases}$$

(2) $\mathcal{U}^{\varepsilon}_{\Gamma_i}: L^p(\Omega \times \Gamma_i) \to L^p(\Gamma^{\varepsilon}_i)$ is defined as

$$\mathcal{U}^{\varepsilon}_{\Gamma_i}(\Phi)(x) = \begin{cases} \dfrac{1}{|Y|}\displaystyle\int_Y \Phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z, \left\{\frac{x}{\varepsilon}\right\}_Y\right)\mathrm{d}z & \text{for a.a. } x \in \tilde{\Gamma}^{\varepsilon}_{i,\mathrm{int}}, \\ 0 & \text{for a.a. } x \in \Gamma^{\varepsilon}_i \setminus \tilde{\Gamma}^{\varepsilon}_{i,\mathrm{int}}. \end{cases}$$

For $\omega^i \in H^1_{\mathrm{per}}(Y_i)$, due to $\nabla_y \omega^i(y) = \nabla_y \mathcal{T}^{\varepsilon}_{Y_i}\left(\omega^i\left(\frac{x}{\varepsilon}\right)\right) = \varepsilon \mathcal{T}^{\varepsilon}_{Y_i}\left(\nabla_x \omega^i\left(\frac{x}{\varepsilon}\right)\right)$ and $\mathcal{U}^{\varepsilon}_{Y_i}(\nabla_y \omega^i(y)) = \varepsilon \mathcal{U}^{\varepsilon}_{Y_i}\left(\mathcal{T}^{\varepsilon}_{Y_i}\left(\nabla_x \omega^i\left(\frac{x}{\varepsilon}\right)\right)\right) = \varepsilon \nabla_x \omega^i(\frac{x}{\varepsilon}) = \nabla_y \omega^i\left(\frac{x}{\varepsilon}\right)$, we have that $\mathcal{U}^{\varepsilon}_{Y_i}(\nabla_y \omega^i(y)) = \nabla_y \omega^i\left(\frac{x}{\varepsilon}\right)$.

3.1. *Basic estimates*

In this section, we prove some technical estimates, used in the derivation of corrector estimates.

Proposition 3.3 *For $\phi_1 \in L^2(0, T; H^1(\Omega))$ and $\phi_2 \in L^2(0, T; H^1(\Omega^{\varepsilon}_i))$ we have*

$$\begin{aligned} \|\phi_1 - \mathcal{M}^{\varepsilon}_{Y_i}(\phi_1)\|_{L^2((0,T)\times\Omega)} &\le \varepsilon C \|\nabla \phi_1\|_{L^2((0,T)\times\Omega)}, \\ \|\phi_2 - \mathcal{M}^{\varepsilon}_{Y_i}(\phi_2)\|_{L^2((0,T)\times\Omega^{\varepsilon}_i)} &\le \varepsilon C \|\nabla \phi_2\|_{L^2((0,T)\times\Omega^{\varepsilon}_i)}. \end{aligned} \tag{17}$$

Proof This proof is similar to [8]. For $\phi_1 \in L^2(0, T; H^1(\Omega))$ we can write

$$x \to \phi_1|_{\varepsilon(\xi+Y)}(x) - \mathcal{M}^{\varepsilon}_{Y_i}(\phi_1)(\varepsilon\xi) \in L^2(0, T; H^1(\varepsilon\xi + \varepsilon Y)) \quad \text{with } \varepsilon(\xi + Y) \subset \Omega.$$

Using $Y_i \subset Y$ and applying Poincaré's inequality, we obtain

$$\begin{aligned} &\int_0^T \int_{\varepsilon(\xi+Y)} |\phi_1 - \mathcal{M}^{\varepsilon}_{Y_i}(\phi_1)(\varepsilon\xi)|^2 \mathrm{d}x\, \mathrm{d}t \\ &= \int_0^T \int_{\xi+Y} \left|\phi_1(\varepsilon y) - \frac{1}{|Y_i|}\int_{\xi+Y_i} \phi_1(\varepsilon z)\mathrm{d}z\right|^2 \varepsilon^n \mathrm{d}y\, \mathrm{d}t \\ &\le C\varepsilon^n \int_0^T \int_{\xi+Y} |\nabla_y \phi_1(\varepsilon y)|^2 \mathrm{d}y\, \mathrm{d}t = C\varepsilon^2 \int_0^T \int_{\varepsilon(\xi+Y)} |\nabla_x \phi_1(x)|^2 \mathrm{d}x\, \mathrm{d}t. \end{aligned}$$

Then, we add all inequalities for $\xi \in \mathbb{Z}^n$, such that $\varepsilon(\xi + Y) \subset \Omega$, and obtain the first estimate in (17). The second estimate follows from the decomposition of Ω^{ε}_i into $\cup_{\xi \in \mathbb{Z}^n} \varepsilon(\xi + Y_i)$ and Poincaré's inequality as in the previous estimate. ■

Lemma 3.4 *For $\phi \in L^2(0, T; H^2(\tilde{\Omega}^{\varepsilon,2}))$, $\phi_2 \in L^2(0, T; H^1(\tilde{\Omega}^{\varepsilon,2}_i))$, $\omega \in H^1_{\mathrm{per}}(Y_i)$, with $i = 1, 2$, we have the following estimates*

$$\begin{aligned} &\|\nabla\phi - \mathcal{M}^{\varepsilon}_{Y_i}(\nabla\phi)\|_{L^2((0,T)\times\Omega)} \le \varepsilon C \|\phi\|_{L^2(0,T;H^2(\Omega))}, \\ &\|(\mathcal{M}^{\varepsilon}_{Y_i}(\partial_{x_i}\phi) - \mathcal{Q}^{\varepsilon}_{Y_i}(\partial_{x_i}\phi))\nabla_y\omega\|_{L^2((0,T)\times\Omega^{\varepsilon}_i)} \le \varepsilon C \|\phi\|_{L^2(0,T;H^2(\tilde{\Omega}^{\varepsilon,2}))} \|\nabla\omega\|_{L^2(Y_i)}, \\ &\|\mathcal{Q}^{\varepsilon}_{Y_i}(\phi_2) - \mathcal{M}^{\varepsilon}_{Y_i}(\phi_2)\|_{L^2((0,T)\times\Omega)} \le \varepsilon C \|\nabla\phi_2\|_{L^2((0,T)\times\tilde{\Omega}^{\varepsilon,2}_i)}, \\ &\|\mathcal{Q}^{\varepsilon}_{Y_i}(\phi) - \phi\|_{L^2((0,T)\times\Omega)} \le \varepsilon C \|\nabla\phi\|_{L^2((0,T)\times\tilde{\Omega}^{\varepsilon,2})}, \end{aligned}$$

$$
\begin{aligned}
&\|\mathcal{Q}^{\varepsilon}_{Y_i}(\phi_2)-\phi_2\|_{L^2((0,T)\times\Omega_i^\varepsilon)} \le \varepsilon C\|\nabla\phi_2\|_{L^2((0,T)\times\tilde\Omega_i^{\varepsilon,2})},\\
&\|\phi-\mathcal{T}^{\varepsilon}_{\Gamma_i}(\phi)\|_{L^2((0,T)\times\Omega\times\Gamma_i)} \le \varepsilon C\|\nabla\phi\|_{L^2((0,T)\times\Omega)}+\varepsilon C\|\nabla\phi\|_{L^2((0,T)\times\Omega_i^\varepsilon)},\\
&\|\nabla\mathcal{Q}^{\varepsilon}_{Y_i}(\phi_2)\|_{L^2((0,T)\times\Omega)} \le C\|\nabla\phi_2\|_{L^2((0,T)\times\tilde\Omega_i^{\varepsilon,2})},\\
&\|\mathcal{M}_{Y_i}(\omega)-\omega\|_{L^2(Y_i)} \le C\|\nabla_y\omega\|_{L^2(Y_i)},\\
&\|\mathcal{T}^{\varepsilon}_{Y_i}(\mathcal{Q}^{\varepsilon}_{Y_i}(\phi_2))-\mathcal{Q}^{\varepsilon}_{Y_i}(\phi_2)\|_{L^2((0,T)\times\Omega\times Y_i)} \le \varepsilon C\|\nabla\phi_2\|_{L^2((0,T)\times\tilde\Omega_i^{\varepsilon,2})}.
\end{aligned}
\tag{18}
$$

Proof The first inequality follows directly from the first estimate in (17) applied to $\nabla\phi$. To show the second estimate, we use the definition of the operator $\mathcal{Q}^{\varepsilon}_{Y_i}$, the equality $\sum_{k\in\{0,1\}^n}\bar{x}_1^{k_1}\dots\bar{x}_n^{k_n}=1$, and obtain

$$
\mathcal{Q}^{\varepsilon}_{Y_i}(\phi)(x)-\mathcal{M}^{\varepsilon}_{Y_i}(\phi)(x)=\sum_{k\in\{0,1\}^n}\big(\mathcal{Q}^{\varepsilon}_{Y_i}(\phi)(\varepsilon\xi+\varepsilon k)-\mathcal{M}^{\varepsilon}_{Y_i}(\phi)(\varepsilon\xi)\big)\bar{x}_1^{k_1}\dots\bar{x}_n^{k_n}.
$$

Then, it follows

$$
\begin{aligned}
&\int_{\varepsilon(\xi+Y_i)}\big|\mathcal{Q}^{\varepsilon}_{Y_i}(\phi)(x)-\mathcal{M}^{\varepsilon}_{Y_i}(\phi)(x)\big|^2\left|\nabla_y\omega\left(\frac{x}{\varepsilon}\right)\right|^2\mathrm{d}x\\
&\quad\le 2^n\sum_{k\in\{0,1\}^n}\big|\mathcal{Q}^{\varepsilon}_{Y_i}(\phi)(\varepsilon\xi+\varepsilon k)-\mathcal{Q}^{\varepsilon}_{Y_i}(\phi)(\varepsilon\xi)\big|^2\varepsilon^n\int_{Y_i}|\nabla_y\omega(y)|^2\,\mathrm{d}y.
\end{aligned}
$$

For any $\phi\in W^{1,p}(\mathrm{Int}(\overline{Y_i\cup(Y_i+e_j)}))$, the following estimate holds:

$$
\begin{aligned}
|\mathcal{M}_{Y_i+e_j}(\phi)-\mathcal{M}_{Y_i}(\phi)| &= |\mathcal{M}_{Y_i}(\phi(\cdot+e_j)-\phi(\cdot))|\\
&\le \|\phi(\cdot+e_j)-\phi(\cdot)\|_{L^p(Y_i)}\le C\|\nabla\phi\|_{L^p(Y_i\cup(Y_i+e_j))}.
\end{aligned}
$$

Thus, by the definition of $\mathcal{Q}^{\varepsilon}_{Y_i}(\phi)$ and by a scaling argument this implies

$$
|\mathcal{Q}^{\varepsilon}_{Y_i}(\phi)(\varepsilon\xi+\varepsilon k)-\mathcal{Q}^{\varepsilon}_{Y_i}(\phi)(\varepsilon\xi)|\le\varepsilon C\|\nabla\phi\|_{L^2(\varepsilon(\xi+Y_i)\cup\varepsilon(\xi+k+Y_i))}.
\tag{19}
$$

We sum over $\xi\in\mathbb{Z}^n$ with $\varepsilon(\xi+Y_i)\subset\tilde\Omega_i^\varepsilon$ and obtain the desired estimate. Using (19) we also obtain that

$$
\begin{aligned}
&\int_\Omega|\mathcal{Q}^{\varepsilon}_{Y_i}(\phi_2)-\mathcal{M}^{\varepsilon}_{Y_i}(\phi_2)|^2\mathrm{d}x\\
&\quad\le\varepsilon^2C\sum_{\varepsilon(\xi+Y_i)\subset\tilde\Omega_i^{\varepsilon,1}}\varepsilon^n\sum_{k\in\{0,1\}^n}\|\nabla\phi_2\|^2_{L^2(\varepsilon(\xi+Y_i)\cup\varepsilon(\xi+k+Y_i))}\\
&\quad\le\varepsilon^2C\int_{\tilde\Omega_i^{\varepsilon,2}}|\nabla\phi_2|^2\mathrm{d}x.
\end{aligned}
$$

In the same way, using the estimates stated in Proposition 3.3, the fourth and fifth estimates in (18) follows from:

$$
\begin{aligned}
&\|\mathcal{Q}^{\varepsilon}_{Y_i}(\phi_2)-\phi_2\|_{L^2((0,T)\times\Omega_i^\varepsilon)}\\
&\quad\le\|\mathcal{Q}^{\varepsilon}_{Y_i}(\phi_2)-\mathcal{M}^{\varepsilon}_{Y_i}(\phi_2)\|_{L^2((0,T)\times\Omega)}+\|\mathcal{M}^{\varepsilon}_{Y_i}(\phi_2)-\phi_2\|_{L^2((0,T)\times\Omega_i^\varepsilon)}\\
&\quad\le\varepsilon C\|\nabla\phi_2\|_{L^2((0,T)\times\tilde\Omega_i^{\varepsilon,2})}.
\end{aligned}
$$

For $\phi \in H^1(\Omega)$ applying the trace theorem to a function in $L^2(\Gamma_i)$ yields

$$
\begin{aligned}
&\int_{\Omega\times\Gamma_i} |\phi - \mathcal{T}^\varepsilon_{\Gamma_i}(\phi)|^2 \mathrm{d}\gamma\, \mathrm{d}x \\
&\le \int_{\Omega\times\Gamma_i} \big(|\phi - \mathcal{M}^\varepsilon_{Y_i}(\phi)|^2 + |\mathcal{M}^\varepsilon_{Y_i}(\phi) - \mathcal{T}^\varepsilon_{\Gamma_i}(\phi)|^2\big)\mathrm{d}\gamma\, \mathrm{d}x \\
&\le C\varepsilon^2|\Gamma_i| \int_\Omega |\nabla\phi|^2 \mathrm{d}x + C\int_{\Omega\times Y_i} \big(|\mathcal{M}^\varepsilon_{Y_i}(\phi) - \mathcal{T}^\varepsilon_{Y_i}(\phi)|^2 + |\nabla_y(\mathcal{M}^\varepsilon_{Y_i}(\phi) - \mathcal{T}^\varepsilon_{Y_i}(\phi))|^2\big)\mathrm{d}y\, \mathrm{d}x \\
&\le C\varepsilon^2|\Gamma_i| \int_\Omega |\nabla\phi|^2 \mathrm{d}x + C\int_{\Omega_i^\varepsilon} |\mathcal{M}^\varepsilon_{Y_i}(\phi) - \phi|^2 \mathrm{d}x + \int_{\Omega\times Y_i} |\nabla_y \mathcal{T}^\varepsilon_{Y_i}(\phi))|^2 \mathrm{d}y\, \mathrm{d}x \\
&\le \varepsilon^2 C\bigg(\int_\Omega |\nabla\phi|^2\mathrm{d}x + \int_{\Omega_i^\varepsilon} |\nabla\phi|^2 \mathrm{d}x\bigg).
\end{aligned}
$$

To obtain an estimate for the gradient of $\mathcal{Q}^\varepsilon_{Y_i}(\phi_2)$, with $\phi_2 \in L^2(0,T; H^1(\tilde{\Omega}^\varepsilon))$, we define $\tilde{k}^j = (k_1, \dots, k_{j-1}, k_{j+1}, \dots, k_n)$, $\tilde{k}^j_1 = (k_1, \dots, k_{j-1}, 1, k_{j+1}, \dots, k_n)$, $\tilde{k}^j_0 = (k_1, \dots, k_{j-1}, 0, k_{j+1}, \dots, k_n)$ and calculate

$$
\frac{\partial \mathcal{Q}^\varepsilon_{Y_i}(\phi_2)}{\partial x_j} = \sum_{\tilde{k}^j} \frac{\mathcal{Q}^\varepsilon_{Y_i}(\phi_2)(\varepsilon\xi + \varepsilon\tilde{k}^j_1) - \mathcal{Q}^\varepsilon_{Y_i}(\phi_2)(\varepsilon\xi + \varepsilon\tilde{k}^j_0)}{\varepsilon} \bar{x}_1^{k_1} \dots \bar{x}_{j-1}^{k_{j-1}} \dots \bar{x}_{j+1}^{k_{j+1}} \bar{x}_n^{k_n}.
$$

Now, applying (19) we obtain the estimates for $\nabla \mathcal{Q}^\varepsilon_{Y_i}(\phi_2)$ in $L^2((0,T)\times\Omega)$.

The estimate for $\mathcal{M}^\varepsilon_{Y_i}(\omega) - \omega$ follows directly by applying Poincaré's inequality. To derive the last estimate, we consider

$$
\begin{aligned}
&\|\mathcal{T}^\varepsilon_{Y_i}(\mathcal{Q}^\varepsilon_{Y_i}(\phi_2)) - \mathcal{Q}^\varepsilon_{Y_i}(\phi_2))\|_{L^2(\Omega\times Y_i)} \\
&\quad \le \|\mathcal{T}^\varepsilon_{Y_i}(\mathcal{Q}^\varepsilon_{Y_i}(\phi_2)) - \mathcal{M}^\varepsilon_{Y_i}(\mathcal{Q}^\varepsilon_{Y_i}(\phi_2))\|_{L^2(\Omega\times Y_i)} + \|\mathcal{M}^\varepsilon_{Y_i}(\mathcal{Q}^\varepsilon_{Y_i}(\phi_2)) - \mathcal{Q}^\varepsilon_{Y_i}(\phi_2)\|_{L^2(\Omega\times Y_i)} \\
&\quad \le C\|\mathcal{Q}^\varepsilon_{Y_i}(\phi_2) - \mathcal{M}^\varepsilon_{Y_i}(\mathcal{Q}^\varepsilon_{Y_i}(\phi_2))\|_{L^2(\Omega_i^\varepsilon)} + C\|\mathcal{M}^\varepsilon_{Y_i}(\mathcal{Q}^\varepsilon_{Y_i}(\phi_2)) - \mathcal{Q}^\varepsilon_{Y_i}(\phi_2)\|_{L^2(\Omega)} \\
&\quad \le \varepsilon C\|\nabla \mathcal{Q}^\varepsilon_{Y_i}(\phi_2)\|_{L^2(\Omega)} \le \varepsilon C\|\nabla\phi_2\|_{L^2(\tilde{\Omega}_i^{\varepsilon,2})}. \qquad \blacksquare
\end{aligned}
$$

3.2. *Periodicity defect*

In the derivation of error estimates we use a generalization of Theorem 3.4 proved in [8] for functions defined in a perforated domain:

Theorem 3.5 *For any $\phi \in H^1(\Omega_i^\varepsilon)$, $i = 1, 2$, there exists $\hat{\psi}^\varepsilon \in L^2(\Omega; H^1_{\text{per}}(Y_i))$:*

$$
\|\hat{\psi}^\varepsilon\|_{L^2(\Omega; H^1(Y_i))} \le C\|\nabla\phi\|_{L^2(\Omega_i^\varepsilon)^n},
$$

$$
\|\mathcal{T}^\varepsilon_{Y_i}(\nabla\phi) - \nabla\phi^\varepsilon - \nabla_y\hat{\psi}^\varepsilon\|_{H^{-1}(\Omega; L^2(Y_i))} \le C\varepsilon\|\nabla\phi\|_{L^2(\Omega_i^\varepsilon)^n}.
$$

Here $\phi^\varepsilon = \mathcal{Q}^\varepsilon_{Y_i}(\phi)$.

The proof of Theorem 3.5 goes the same lines as in [8, Theorem 3.4], using the estimates

$$
\|\mathcal{T}^\varepsilon_{Y_i}(\phi)\|_{L^2(\Omega\times Y_i)} \le C\|\phi\|_{L^2(\Omega_i^\varepsilon)}, \quad \|\nabla \mathcal{Q}^\varepsilon_{Y_i}(\phi)\|_{L^2(\Omega)} \le C\|\nabla\phi\|_{L^2(\tilde{\Omega}_i^{\varepsilon,2})}.
$$

For more details we refer to the appendix.

3.3. *Error estimates*

Under additional regularity assumptions on the solution of the macroscopic problem, we obtain a set of error estimates. We emphasize here again that only H^1-regularity for the solutions of the microscopic model and of the cell problems is required.

THEOREM 3.6 *Suppose* $(u^\varepsilon, v^\varepsilon, w^\varepsilon, r^\varepsilon)$ *are solutions of the microscopic problem* (1)–(4) *and* $u, v, w \in L^2(0,T; H^2(\Omega)) \cap H^1((0,T)\times\Omega))$, $r \in H^1(0,T; L^2(\Omega\times\Gamma_1))$ *are non-negative and bounded solutions of the macroscopic equations* (14). *Then we have the following corrector estimates*:

$$\|u^\varepsilon - u\|_{L^2((0,T)\times\Omega_1^\varepsilon)} + \left\|\nabla u^\varepsilon - \nabla u - \sum_{j=1}^n \mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j} u)\nabla_y \omega_u^j\right\|^2_{L^2((0,T)\times\Omega_1^\varepsilon)} \le C\varepsilon^{\frac{1}{2}},$$

$$\|v^\varepsilon - v\|_{L^2((0,T)\times\Omega_1^\varepsilon)} + \left\|\nabla v^\varepsilon - \nabla v - \sum_{j=1}^n \mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j} v)\nabla_y \omega_v^j\right\|^2_{L^2((0,T)\times\Omega_1^\varepsilon)} \le C\varepsilon^{\frac{1}{2}},$$

$$\|w^\varepsilon - w\|_{L^2((0,T)\times\Omega_2^\varepsilon)} + \left\|\nabla w^\varepsilon - \nabla w - \sum_{j=1}^n \mathcal{Q}^\varepsilon_{Y_2}(\partial_{x_j} w)\nabla_y \omega_w^j\right\|^2_{L^2((0,T)\times\Omega_2^\varepsilon)} \le C\varepsilon^{\frac{1}{2}},$$

$$\varepsilon^{\frac{1}{2}}\|r^\varepsilon - \mathcal{U}^\varepsilon_{\Gamma_1}(r)\|_{L^2((0,T)\times\Gamma_1^\varepsilon)} \le C\varepsilon^{\frac{1}{2}}.$$

4. Proof of Theorem 3.6

We define distance function $\rho(x) = \mathrm{dist}(x, \partial\Omega)$, domains $\hat{\Omega}^\varepsilon_{\rho,\mathrm{in}} = \{x\in\Omega, \rho(x) < \varepsilon\}$ and $\hat{\Omega}^\varepsilon_{i,\rho,\mathrm{in}} = \{x \in \Omega_i^\varepsilon, \rho(x) < \varepsilon\}$, and $\rho^\varepsilon(\cdot) = \inf\{\frac{\rho(\cdot)}{\varepsilon}, 1\}$. Definition of ρ^ε yields

$$\|\nabla\rho^\varepsilon\|_{L^\infty(\Omega)^n} = \|\nabla\rho^\varepsilon\|_{L^\infty(\hat{\Omega}^\varepsilon_{\rho,\mathrm{in}})^n} = \varepsilon^{-1}. \tag{20}$$

Then, for $\Phi \in H^2(\Omega)$ and $\omega \in H^1(Y_i)$, where $i = 1, 2$, we obtain the following estimates [8]:

$$\begin{aligned}
\|\nabla\Phi\|_{L^2(\hat{\Omega}^\varepsilon_{\rho,\mathrm{in}})^n} + \|\mathcal{Q}^\varepsilon_{Y_i}(\nabla\Phi)\|_{L^2(\hat{\Omega}^\varepsilon_{\rho,\mathrm{in}})^n} + \|\mathcal{M}^\varepsilon_{Y_i}(\nabla\Phi)\|_{L^2(\hat{\Omega}^\varepsilon_{\rho,\mathrm{in}})^n} &\le C\varepsilon^{\frac{1}{2}}\|\Phi\|_{H^2(\Omega)},\\
\left\|\omega\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2(\hat{\Omega}^\varepsilon_{i,\rho,\mathrm{in}})} + \left\|\nabla\omega\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2(\hat{\Omega}^\varepsilon_{i,\rho,\mathrm{in}})^n} &\le C\varepsilon^{\frac{1}{2}}\|\nabla_y\omega\|_{L^2(Y_i)^n},\\
\|(1-\rho_\varepsilon)\nabla_x\Phi\|_{L^2(\Omega)^n} \le \|\nabla_x\Phi\|_{L^2(\hat{\Omega}^\varepsilon_{\rho,\mathrm{in}})^n} &\le C\varepsilon^{\frac{1}{2}}\|\Phi\|_{H^2(\Omega)},\\
\|\nabla_x(\rho_\varepsilon\partial_{x_j}\Phi)\|_{L^2(\Omega)^n} &\le C(\varepsilon^{-\frac{1}{2}} + 1)\|\Phi\|_{H^2(\Omega)},\\
\left\|\varepsilon\partial_{x_i}\rho_\varepsilon\mathcal{Q}^\varepsilon_{Y_i}(\partial_{x_j}\Phi)\omega\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2(\Omega_i^\varepsilon)} &\le C\varepsilon^{\frac{1}{2}}\|\Phi\|_{H^2(\Omega)}\|\omega\|_{L^2(Y_i)},\\
\left\|\varepsilon\rho_\varepsilon\partial_{x_i}\mathcal{Q}^\varepsilon_{Y_i}(\partial_{x_j}\Phi)\omega\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2(\Omega_i^\varepsilon)} &\le C\varepsilon\|\Phi\|_{H^2(\Omega)}\|\omega\|_{L^2(Y_i)}.
\end{aligned} \tag{21}$$

Note that for a bounded Lipschitz domain Ω there exists an extension of Φ from Ω into $\Omega^{\varepsilon,2}$, such that $\|\mathcal{P}(\Phi)\|_{L^2(\tilde{\Omega}^{\varepsilon,2})} \le C\big(\|\Phi\|_{L^2(\Omega)} + \varepsilon\|\nabla\Phi\|_{L^2(\Omega)^n}\big)$ and $\|\nabla\mathcal{P}(\Phi)\|_{L^2(\tilde{\Omega}^{\varepsilon,2})^n} \le C\|\nabla\Phi\|_{L^2(\Omega)^n}$.

Now, for $\phi_1 \in L^2(0, T; H^1_{\partial\Omega}(\Omega_1^\varepsilon))$ given by

$$\phi_1(x) = u^\varepsilon(x) - u(x) - \varepsilon\rho^\varepsilon(x)\sum_{j=1}^n \mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j}u)(x)\omega_u^j\left(\frac{x}{\varepsilon}\right)$$

we consider an extension $\tilde{\phi}_1^\varepsilon$ of ϕ_1 from $(0, T) \times \Omega_1^\varepsilon$ into $(0, T) \times \Omega$, such that

$$\|\tilde{\phi}_1^\varepsilon\|_{L^2((0,T)\times\Omega)} \le C\|\phi_1\|_{L^2((0,T)\times\Omega_1^\varepsilon)} \quad \text{and} \quad \|\nabla\tilde{\phi}_1^\varepsilon\|_{L^2((0,T)\times\Omega)} \le C\|\nabla\phi_1\|_{L^2((0,T)\times\Omega_1^\varepsilon)}.$$

Due to the presence of zero boundary conditions and since all phases are connected, standard extension results apply [18]. We consider $\tilde{\phi}_1^\varepsilon \in L^2(0, T; H^1_0(\Omega))$ and $\hat{\psi}_1^\varepsilon \in L^2((0, T) \times \Omega, H^1_{\mathrm{per}}(Y_1))$, given by Theorem 3.5 applied to ϕ_1, as test functions in the macroscopic Equation (14) for u:

$$\int_0^\tau\int_{\Omega\times Y_1} \partial_t u\tilde{\phi}_1^\varepsilon + D_u(y)\left(\nabla u + \sum_{j=1}^n \frac{\partial u}{\partial x_j}\nabla_y\omega_u^j\right)\left(\nabla\tilde{\phi}_1^\varepsilon + \nabla_y\hat{\psi}_1^\varepsilon\right)\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^\tau\int_{\Omega\times Y_1} f(u, v)\tilde{\phi}_1^\varepsilon\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_{\Omega\times\Gamma_1} \eta(u, r)\tilde{\phi}_1^\varepsilon\mathrm{d}\gamma\,\mathrm{d}x\,\mathrm{d}t = 0.$$

In the first term and in the last two integrals, we replace $\tilde{\phi}_1^\varepsilon$ by $\mathcal{M}^\varepsilon_{Y_1}(\phi_1)$, $\tilde{\phi}_1^\varepsilon$ by $\mathcal{T}^\varepsilon_{\Gamma_1}(\phi_1)$, and u by $\mathcal{T}^\varepsilon_{Y_1}(u)$. As next step, we introduce ρ^ε in front of ∇u and $\partial_{x_j}u$, and replace $\nabla\tilde{\phi}_1^\varepsilon$ by $\nabla\mathcal{Q}^\varepsilon_{Y_1}(\phi_1)$. Notice that $\mathcal{Q}^\varepsilon_{Y_i}(\partial_{x_j}u)$ and ∇u are in $L^2(0, T; H^1(\Omega))$, but not in $L^2(0, T; H^1_0(\Omega))$. Now, using Theorem 3.5, we replace $\nabla\phi_1^\varepsilon + \nabla_y\hat{\psi}_1^\varepsilon$, by $\mathcal{T}^\varepsilon_{Y_1}(\nabla\phi_1)$, where $\phi_1^\varepsilon = \mathcal{Q}^\varepsilon_{Y_1}(\phi_1)$, and obtain

$$\int_0^\tau\int_{\Omega\times Y_1} \partial_t\mathcal{T}^\varepsilon_{Y_1}(u)\mathcal{M}^\varepsilon_{Y_1}(\phi_1) + D_u(y)\rho^\varepsilon\left(\nabla u + \sum_{j=1}^n \frac{\partial u}{\partial x_j}\nabla_y\omega_u^j\right)\mathcal{T}^\varepsilon_{Y_1}(\nabla\phi_1)\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^\tau\int_{\Omega\times Y_1} \mathcal{T}^\varepsilon_{Y_1}(f(u, v))\mathcal{M}^\varepsilon_{Y_1}(\phi_1)\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_{\Omega\times\Gamma_1} \eta(u, r)\mathcal{T}^\varepsilon_{\Gamma_1}(\phi_1)\mathrm{d}\gamma\,\mathrm{d}x\,\mathrm{d}t = R_u^1,$$

where

$$R_u^1 = \int_0^\tau\int_{\Omega\times Y_1}\Bigg[\partial_t(u - \mathcal{T}^\varepsilon_{Y_1}(u))\mathcal{M}^\varepsilon_{Y_1}(\phi_1) + \partial_t u(\tilde{\phi}_1^\varepsilon - \mathcal{M}^\varepsilon_{Y_1}(\phi_1))$$
$$+\rho^\varepsilon D_u\left(\nabla u + \sum_{j=1}^n \frac{\partial u}{\partial x_j}\nabla_y\omega_u^j\right)\left(\nabla(\mathcal{Q}^\varepsilon_{Y_i}(\phi_1) - \tilde{\phi}_1^\varepsilon) + (\mathcal{T}^\varepsilon_{Y_1}(\nabla\phi_1) - \nabla\phi_1^\varepsilon - \nabla_y\hat{\psi}_1^\varepsilon)\right)$$
$$+(\rho^\varepsilon - 1)D_u\left(\nabla u + \sum_{j=1}^n \frac{\partial u}{\partial x_j}\nabla_y\omega_u^j\right)(\nabla\tilde{\phi}_1^\varepsilon + \nabla_y\hat{\psi}_1^\varepsilon) + f(u, v)(\mathcal{M}^\varepsilon_{Y_1}(\phi_1) - \tilde{\phi}_1^\varepsilon)$$
$$+(\mathcal{T}^\varepsilon_{Y_1}(f) - f)\mathcal{M}^\varepsilon_{Y_1}(\phi_1)\Bigg]\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_{\Omega\times\Gamma_1} \eta(u, r)(\mathcal{T}^\varepsilon_{\Gamma_1}(\phi_1) - \tilde{\phi}_1^\varepsilon)\mathrm{d}\gamma\,\mathrm{d}x\,\mathrm{d}t.$$

Then we remove ρ^ε, replace ∇u by $\mathcal{M}^\varepsilon_{Y_1}(\nabla u)$, $\partial_{x_j}u$ by $\mathcal{M}^\varepsilon_{Y_1}(\partial_{x_j}u)$ and, using $\mathcal{M}^\varepsilon_{Y_1}(\phi) = \mathcal{T}^\varepsilon_{Y_1}\circ\mathcal{M}^\varepsilon_{Y_1}(\phi)$, we apply the inverse unfolding

$$\int_0^\tau\int_{\Omega_1^\varepsilon}\left(\partial_t u\mathcal{M}^\varepsilon_{Y_1}(\phi_1) + D_u^\varepsilon\left(\mathcal{M}^\varepsilon_{Y_1}(\nabla u) + \sum_{j=1}^n \mathcal{M}^\varepsilon_{Y_1}(\partial_{x_j}u)\nabla_y\omega_u^j\left(\frac{x}{\varepsilon}\right)\right)\nabla\phi_1\right)\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^\tau\int_{\Omega_1^\varepsilon} f(u, v)\mathcal{M}^\varepsilon_{Y_1}(\phi_1)\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_{\Omega\times\Gamma_1} \eta(u, r)\mathcal{T}^\varepsilon_{\Gamma_1}(\phi_1)\mathrm{d}\gamma\,\mathrm{d}x\,\mathrm{d}t = R_u^1 + R_u^2,$$

where

$$R_u^2 = \int_0^\tau \int_{\Omega\times Y_1} \Big[(1-\rho^\varepsilon)D_u(y)\Big(\nabla u + \sum_{j=1}^n \partial_{x_j} u \nabla_y \omega_u^j(y)\Big)\mathcal{T}^\varepsilon_{Y_1}(\nabla\phi_1)$$
$$+ D_u(y)\Big(\mathcal{M}^\varepsilon_{Y_1}(\nabla u) - \nabla u + \sum_{j=1}^n \big(\mathcal{M}^\varepsilon_{Y_1}(\partial_{x_j}u) - \partial_{x_j}u\big)\nabla_y\omega_u^j(y)\Big)\mathcal{T}^\varepsilon_{Y_1}(\nabla\phi_1)\Big]\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t.$$

Introducing ρ^ε in front of $\mathcal{M}^\varepsilon_{Y_i}(\partial_{x_j}u)$ and replacing $\mathcal{M}^\varepsilon_{Y_1}(\phi_1)$ by ϕ_1, $\mathcal{M}^\varepsilon_{Y_1}(\nabla u)$ by ∇u, $\mathcal{M}^\varepsilon_{Y_1}(\partial_{x_j}u)$ by $\mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j}u)$ yield

$$\int_0^\tau \int_{\Omega_1^\varepsilon} \Big[\partial_t u\phi_1 + D_u^\varepsilon\Big(\nabla u + \sum_{j=1}^n \rho^\varepsilon \mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j}u)\nabla_y\omega_u^j\big(\frac{x}{\varepsilon}\big)\Big)\nabla\phi_1 + f(u,v)\phi_1\Big]\mathrm{d}x\,\mathrm{d}t$$
$$= -\int_0^\tau \int_{\Omega\times\Gamma_1} \eta(u,r)\mathcal{T}^\varepsilon_{Y_1}(\phi_1)\mathrm{d}\gamma\,\mathrm{d}x\,\mathrm{d}t + R_u^1 + R_u^2 + R_u^3, \tag{22}$$

where

$$R_u^3 = \int_0^\tau \int_{\Omega_1^\varepsilon} \Big[\big(\partial_t u + f\big)(\phi_1 - \mathcal{M}^\varepsilon_{Y_1}(\phi_1)) + (\rho^\varepsilon - 1)D_u^\varepsilon \sum_{j=1}^n \mathcal{M}^\varepsilon_{Y_i}(\partial_{x_j}u)\nabla_y\omega_u^j\Big(\frac{x}{\varepsilon}\Big)\nabla\phi_1$$
$$+ D_u^\varepsilon\Big(\nabla u - \mathcal{M}^\varepsilon_{Y_1}(\nabla u) + \sum_{j=1}^n \rho^\varepsilon\big(\mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j}u) - \mathcal{M}^\varepsilon_{Y_1}(\partial_{x_j}u)\big)\nabla_y\omega_u^j\Big(\frac{x}{\varepsilon}\Big)\Big)\nabla\phi_1\Big]\mathrm{d}x\,\mathrm{d}t.$$

Now, we subtract Equation (22) from Equation (5) for u^ε and obtain for the test function $\phi_1 = u^\varepsilon - u - \varepsilon\rho^\varepsilon\sum_{j=1}^n \mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j}u)\omega_u^j$ the equality

$$\int_0^\tau \int_{\Omega_1^\varepsilon} \Big[\partial_t(u^\varepsilon - u)\Big(u^\varepsilon - u - \varepsilon\rho^\varepsilon \sum_{j=1}^n \mathcal{Q}^\varepsilon_{Y_i}(\partial_{x_j}u)\omega_u^j\Big)$$
$$+ D_u^\varepsilon\Big(\nabla(u^\varepsilon - u) - \rho^\varepsilon \sum_{j=1}^n \mathcal{Q}^\varepsilon_{Y_i}\big(\partial_{x_j}u\big)\nabla_y\omega_u^j\Big)\Big(\nabla(u^\varepsilon - u) - \varepsilon\sum_{j=1}^n \nabla_x\big(\rho^\varepsilon \mathcal{Q}^\varepsilon_{Y_i}\big(\partial_{x_j}u\big)\omega_u^j\big)\Big)$$
$$+ \big(f(u^\varepsilon, v^\varepsilon) - f(u,v)\big)\Big(u^\varepsilon - u - \varepsilon\rho^\varepsilon \sum_{j=1}^n \mathcal{Q}^\varepsilon_{Y_i}(\partial_{x_j}u)\omega_u^j\Big)\Big]\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_0^\tau \int_{\Omega\times\Gamma_1} \big(\eta(\mathcal{T}_\varepsilon u^\varepsilon, \mathcal{T}_\varepsilon r^\varepsilon) - \eta(u,r)\big)\mathcal{T}^\varepsilon_{\Gamma_1}\Big(u^\varepsilon - u - \varepsilon\rho^\varepsilon \sum_{j=1}^n \mathcal{Q}^\varepsilon_{Y_i}(\partial_{x_j}u)\omega_u^j\Big)\mathrm{d}\gamma\,\mathrm{d}x\,\mathrm{d}t = R_u,$$

where $R_u = R_u^1 + R_u^2 + R_u^3$.

We consider $\psi^\varepsilon = \mathcal{T}^\varepsilon_{\Gamma_1} r^\varepsilon - r$ as a test function in the equations for r in (14) and for $\mathcal{T}^\varepsilon_{\Gamma_1}(r^\varepsilon)$, obtained from (8) by applying the unfolding operator. Using the local Lipschitz continuity of η and the boundedness of u^ε, u, r^ε and r, we obtain

$$\int_0^\tau \int_{\Omega\times\Gamma_1} \partial_t|\mathcal{T}^\varepsilon_{\Gamma_1} r^\varepsilon - r|^2 \mathrm{d}\gamma\,\mathrm{d}x\,\mathrm{d}t \le C\int_0^\tau \int_{\Omega\times\Gamma_1} \big(|\mathcal{T}^\varepsilon_{\Gamma_1} r^\varepsilon - r|^2 + |\mathcal{T}^\varepsilon_{\Gamma_1} u^\varepsilon - u|^2\big)\mathrm{d}\gamma\,\mathrm{d}x\,\mathrm{d}t.$$

Applying Gronwall's inequality and considering $\mathcal{T}^\varepsilon_{\Gamma_1}(r_0^\varepsilon)(x,y) = r_0(y)$ yields

$$\|(\mathcal{T}^\varepsilon_{\Gamma_1}(r^\varepsilon) - r)(t)\|^2_{L^2(\Omega\times\Gamma_1)} \le C\|\mathcal{T}^\varepsilon_{\Gamma_1}(u^\varepsilon) - u\|^2_{L^2((0,\tau)\times\Omega\times\Gamma_1)} + \|\mathcal{T}^\varepsilon_{\Gamma_1}(r_0^\varepsilon) - r_0\|^2_{L^2(\Omega\times\Gamma_1)}$$
$$\le C\big(\|\mathcal{T}^\varepsilon_{\Gamma_1}(u^\varepsilon - u)\|^2_{L^2((0,\tau)\times\Omega\times\Gamma_1)} + \|\mathcal{T}^\varepsilon_{\Gamma_1}(u) - u\|^2_{L^2((0,\tau)\times\Omega\times\Gamma_1)}\big).$$

Then, for the boundary integral, using the estimate in Lemma 3.4, follows

$$\begin{aligned}
&\int_0^\tau \int_{\Omega\times\Gamma_1} \big(\eta(\mathcal{T}^\varepsilon_{\Gamma_1}(r^\varepsilon), \mathcal{T}^\varepsilon_{\Gamma_1}(u^\varepsilon)) - \eta(r,u)\big)\mathcal{T}^\varepsilon_{\Gamma_1}(\phi_1)\mathrm{d}\gamma\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le C\big(\|\mathcal{T}^\varepsilon_{\Gamma_1}(r^\varepsilon) - r\|_{L^2((0,\tau)\times\Omega\times\Gamma_1)} + \|\mathcal{T}^\varepsilon_{\Gamma_1}(u^\varepsilon) - u\|_{L^2((0,\tau)\times\Omega\times\Gamma_1)}\big)\varepsilon^{\frac{1}{2}}\|\phi_1\|_{L^2((0,\tau)\times\Gamma_1^\varepsilon)} \\
&\quad \le C\big(\|u^\varepsilon - u\|_{L^2((0,\tau)\times\Omega_1^\varepsilon)} + \varepsilon\|\nabla(u^\varepsilon - u)\|_{L^2((0,\tau)\times\Omega_1^\varepsilon)} + \varepsilon\|\nabla u\|_{L^2((0,\tau)\times\Omega)}\big) \\
&\qquad \times \big(\|\phi_1\|_{L^2((0,\tau)\times\Omega_1^\varepsilon)} + \varepsilon\|\nabla\phi_1\|_{L^2((0,\tau)\times\Omega_1^\varepsilon)}\big).
\end{aligned} \tag{23}$$

Therefore, the ellipticity assumption, the Lipschitz continuity of f, the estimate (23) and Young's inequality imply

$$\begin{aligned}
&\int_0^\tau \int_{\Omega_1^\varepsilon} \Big(\partial_t\big|\hat{u}^\varepsilon - \varepsilon\rho^\varepsilon \sum_{j=1}^n \mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j}u)\omega_u^j\big|^2 + \big|\nabla\hat{u}^\varepsilon - \rho^\varepsilon \sum_{j=1}^n \mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j}u)\nabla_y\omega_u^j\big|^2\Big)\mathrm{d}x\,\mathrm{d}t \\
&\quad \le C\int_0^\tau \int_{\Omega_1^\varepsilon} \Big(\big|\hat{u}^\varepsilon - \varepsilon\rho^\varepsilon \sum_{j=1}^n \mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j}u)\omega_u^j\big|^2 + \big|\hat{v}^\varepsilon - \varepsilon\rho^\varepsilon \sum_{j=1}^n \mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j}v)\omega_v^j\big|^2\Big)\mathrm{d}x\,\mathrm{d}t \\
&\qquad + C\varepsilon^2\|\nabla u\|^2_{L^2((0,T)\times\Omega)} + R_u + C_u^\varepsilon,
\end{aligned}$$

where $\hat{u}^\varepsilon := u^\varepsilon - u$, $\hat{v}^\varepsilon := v^\varepsilon - v$, and

$$\begin{aligned}
C_u^\varepsilon &:= C\varepsilon^2 \int_0^\tau \int_{\Omega_1^\varepsilon} \sum_{j=1}^n \big(|\mathcal{Q}^\varepsilon_{Y_1}(\partial_t\partial_{x_j}u)\omega_u^j|^2 + (1+\varepsilon^2)|\nabla\mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j}u)\omega_u^j|^2 + |\mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j}u)\omega_u^j|^2 \\
&\qquad + |\mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j}v)\omega_v^j|^2 + |\mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j}u)\nabla_y\omega_u^j|^2\big)\mathrm{d}x\,\mathrm{d}t + C\int_0^\tau \int_{\hat{\Omega}^\varepsilon_{1,\rho,\mathrm{in}}} \sum_{j=1}^n \big|\mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j}u)\omega_u^j\big|^2\mathrm{d}x\,\mathrm{d}t \\
&\le C\big(\varepsilon^2\|u\|^2_{L^2(0,T;H^2(\Omega))} + \varepsilon^2\|u\|^2_{H^1((0,T)\times\Omega)} + \varepsilon\|u\|^2_{L^2(0,T;H^2(\Omega))}\big)\|\omega_u\|^2_{H^1(Y_1)^n} \\
&\quad + C\varepsilon^2\|v\|^2_{L^2(0,T;H^1(\Omega))}\|\omega_v\|^2_{L^2(Y_1)^n}.
\end{aligned}$$

Here we used that

$$\varepsilon^2 \int_{\Omega_1^\varepsilon} |\nabla\big(\rho^\varepsilon\mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j}u)\big)\omega_u^j|^2\mathrm{d}x \le \varepsilon^2 \int_{\Omega_1^\varepsilon} |\nabla\mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j}u)\omega_u^j|^2\mathrm{d}x + \int_{\hat{\Omega}^\varepsilon_{1,\rho,\mathrm{in}}} |\mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j}u)\omega_u^j|^2\mathrm{d}x.$$

The estimates of the error terms in Section 4.1 imply

$$\begin{aligned}
|R_u| = |R_u^1 + R_u^2 + R_u^3| &\le \varepsilon^{1/2}C\big(1 + \|u\|_{H^1((0,T)\times\Omega)} + \|u\|_{L^2(0,T;H^2(\Omega))} \\
&\quad + \|v\|_{L^2(0,T;H^1(\Omega))} + \|r\|_{L^2((0,T)\times\Omega\times\Gamma_1)}\big)\|\phi_1\|_{L^2(0,T;H^1(\Omega_1^\varepsilon))}.
\end{aligned}$$

Then, applying Young's inequality, we obtain

$$\begin{aligned}
&\int_0^\tau \int_{\Omega_1^\varepsilon} \Big(\partial_t|\hat{u}^\varepsilon - \varepsilon\rho^\varepsilon \sum_{j=1}^n \mathcal{Q}^\varepsilon_{Y_i}(\partial_{x_j}u)\omega_u^j|^2 + |\nabla\hat{u}^\varepsilon - \rho^\varepsilon \sum_{j=1}^n \mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j}u)\nabla_y\omega_u^j|^2\Big)\mathrm{d}x\,\mathrm{d}t \\
&\quad \le C\int_0^\tau \int_{\Omega_1^\varepsilon} \Big(|\hat{u}^\varepsilon - \varepsilon\rho^\varepsilon \sum_{j=1}^n \mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j}u)\omega_u^j|^2 + |\hat{v}^\varepsilon - \varepsilon\rho^\varepsilon \sum_{j=1}^n \mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j}v)\omega_v^j|^2\Big)\mathrm{d}x\,\mathrm{d}t \\
&\qquad + C(\varepsilon + \varepsilon^2)(1 + \|u\|^2_{H^1((0,T)\times\Omega)} + \|u\|^2_{L^2(0,T;H^2(\Omega))})\big(1 + \|\omega_u\|^2_{H^1(Y_1)^n}\big) \\
&\qquad + C\varepsilon^2\|v\|^2_{L^2(0,T;H^1(\Omega)}\big(1 + \|\omega_v\|^2_{H^1(Y_1)^n}\big) + \varepsilon^2\|r\|^2_{L^\infty((0,T)\times\Omega\times\Gamma_1)}.
\end{aligned}$$

Similarly, estimates for $v^\varepsilon - v - \varepsilon\sum_{j=1}^n Q^\varepsilon_{Y_1}(\partial_{x_j}v)\omega_v^j$ and $w^\varepsilon - w - \varepsilon\sum_{j=1}^n Q^\varepsilon_{Y_2}(\partial_{x_j}w)\omega_w^j$ are obtained. The only difference is the boundary term. Applying the trace theorem and estimates in Lemma 3.4, the boundary term can be estimated by

$$\begin{aligned}
&\int_{\Omega\times\Gamma_2}\big((a(y)w-b(y)v)\tilde\phi_2^\varepsilon-(a(y)\mathcal{T}^\varepsilon_{\Gamma_2}(w)-b(y)\mathcal{T}^\varepsilon_{\Gamma_2}(v))\mathcal{T}^\varepsilon_{\Gamma_2}(\phi_2)\big)\mathrm{d}\gamma\,\mathrm{d}x\\
&\quad\le C\int_{\Omega\times\Gamma_2}\big(|w-\mathcal{T}^\varepsilon_{\Gamma_2}(w)|+|v-\mathcal{T}^\varepsilon_{\Gamma_2}(v)|\big)\mathcal{T}^\varepsilon_{\Gamma_2}(\phi_2)+(w+v)|\tilde\phi_2^\varepsilon-\mathcal{M}^\varepsilon_{Y_1}(\phi_2)|\\
&\qquad+(w+v)|\mathcal{M}^\varepsilon_{Y_1}(\phi_2)-\mathcal{T}^\varepsilon_{\Gamma_2}(\phi_2)|\big)\mathrm{d}\gamma\,\mathrm{d}x\\
&\quad\le \varepsilon C\big(\|v\|_{H^1(\Omega)}+\|w\|_{H^1(\Omega)}\big)\|\phi_2\|_{H^1(\Omega_1^\varepsilon)},
\end{aligned}$$

where $\phi_2 = v^\varepsilon - v - \varepsilon\rho^\varepsilon\sum_{j=1}^n Q^\varepsilon_{Y_1}(\partial_{x_j}v)\omega_v^j$.

Thus, for $\hat v^\varepsilon = v^\varepsilon - v$ and $\hat{\mathrm{w}}^\varepsilon = w^\varepsilon - w$ we have

$$\begin{aligned}
&\int_0^\tau\int_{\Omega_1^\varepsilon}\left(\partial_t\left|\hat v^\varepsilon-\varepsilon\rho^\varepsilon\sum_{j=1}^n Q^\varepsilon_{Y_1}\left(\frac{\partial v}{\partial x_j}\right)\omega_v^j\right|^2+\left|\nabla\hat v^\varepsilon-\rho^\varepsilon\sum_{j=1}^n Q^\varepsilon_{Y_1}\left(\frac{\partial v}{\partial x_j}\right)\nabla_y\omega_v^j\right|^2\right)\mathrm{d}x\,\mathrm{d}t\\
&\le C\int_0^\tau\int_{\Omega_1^\varepsilon}\left(\left|\hat u^\varepsilon-\varepsilon\rho^\varepsilon\sum_{j=1}^n Q^\varepsilon_{Y_1}\left(\frac{\partial u}{\partial x_j}\right)\omega_u^j\right|^2+\left|\hat v^\varepsilon-\varepsilon\rho^\varepsilon\sum_{j=1}^n Q^\varepsilon_{Y_1}\left(\frac{\partial v}{\partial x_j}\right)\omega_v^j\right|^2\right)\mathrm{d}x\,\mathrm{d}t\\
&+C\int_0^\tau\int_{\Omega_2^\varepsilon}\left(\left|\hat w^\varepsilon-\varepsilon\rho^\varepsilon\sum_{j=1}^n Q^\varepsilon_{Y_2}\left(\frac{\partial w}{\partial x_j}\right)\omega_w^j\right|^2+\varepsilon^2\left|\nabla\hat w^\varepsilon-\rho^\varepsilon\sum_{j=1}^n Q^\varepsilon_{Y_2}\left(\frac{\partial w}{\partial x_j}\right)\nabla_y\omega_w^j)\right|^2\right)\mathrm{d}x\,\mathrm{d}t\\
&+C(\varepsilon+\varepsilon^2)\big(1+\|v\|^2_{L^2(0,T;H^2(\Omega))}+\|v\|^2_{H^1((0,T)\times\Omega)}\big)\big(1+\|\omega_v\|^2_{H^1(Y_1)^n}\big)\\
&+C\big(\varepsilon^2\|u\|^2_{L^2(0,T;H^1(\Omega))}+\varepsilon^2\|w\|^2_{L^2(0,T;H^1(\Omega))}+C_v\big),
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\tau\int_{\Omega_2^\varepsilon}\left(\partial_t\left|\hat w^\varepsilon-\varepsilon\rho^\varepsilon\sum_{j=1}^n Q^\varepsilon_{Y_2}\left(\frac{\partial w}{\partial x_j}\right)\omega_w^j\right|^2+\left|\nabla\hat w^\varepsilon-\rho^\varepsilon\sum_{i=j}^n Q^\varepsilon_{Y_2}\left(\frac{\partial w}{\partial x_j}\right)\nabla_y\omega_w^j\right|^2\right)\mathrm{d}x\,\mathrm{d}t\\
&\le C\int_0^\tau\int_{\Omega_1^\varepsilon}\left(\left|\hat v^\varepsilon-\varepsilon\rho^\varepsilon\sum_{j=1}^n Q^\varepsilon_{Y_1}\left(\frac{\partial v}{\partial x_j}\right)\omega_v^j\right|^2+\varepsilon^2\left|\nabla\hat v^\varepsilon-\rho^\varepsilon\sum_{j=1}^n Q^\varepsilon_{Y_1}\left(\frac{\partial v}{\partial x_j}\right)\nabla_y\omega_v^j\right|^2\right)\mathrm{d}x\,\mathrm{d}t\\
&+C\int_0^\tau\int_{\Omega_2^\varepsilon}\left(\left|\hat w^\varepsilon-\varepsilon\rho^\varepsilon\sum_{i=1}^n Q^\varepsilon_{Y_2}\left(\frac{\partial w}{\partial x_j}\right)\omega_w^j\right|^2+\varepsilon^2\left|\nabla\hat w^\varepsilon-\rho^\varepsilon\sum_{j=1}^n Q^\varepsilon_{Y_2}\left(\frac{\partial w}{\partial x_j}\right)\nabla_y\omega_w^j\right|^2\right)\mathrm{d}x\,\mathrm{d}t\\
&+C(\varepsilon+\varepsilon^2)\big(1+\|w\|^2_{L^2(0,T;H^2(\Omega))}+\|w\|^2_{H^1((0,T)\times\Omega)}\big)\big(1+\|\omega_v\|^2_{H^1(Y_1)^n}\big)\\
&+C\big(\varepsilon^2\|v\|^2_{L^2(0,T;H^1(\Omega)}+C_w\big),
\end{aligned}$$

where

$$\begin{aligned}
C_v:=\varepsilon^2\int_0^\tau\int_{\Omega_1^\varepsilon}\sum_{j=1}^n&\left(\left|Q^\varepsilon_{Y_1}\left(\frac{\partial^2 v}{\partial t\partial x_j}\right)\omega_v^j\right)\right|^2+(1+\varepsilon^2)\left|\nabla Q^\varepsilon_{Y_1}\left(\frac{\partial v}{\partial x_j}\right)\omega_v^j\right|^2+\left|Q^\varepsilon_{Y_1}\left(\frac{\partial u}{\partial x_j}\right)\omega_u^j\right|^2\\
&+\left|Q^\varepsilon_{Y_1}\left(\frac{\partial v}{\partial x_j}\right)\omega_v^j\right|^2+\left|Q^\varepsilon_{Y_1}\left(\frac{\partial v}{\partial x_j}\right)\nabla_y\omega_v^j\right|^2\Bigg)\mathrm{d}x\,\mathrm{d}t+\int_0^\tau\int_{\hat\Omega^\varepsilon_{1,\rho,\mathrm{in}}}\left|Q^\varepsilon_{Y_1}\left(\frac{\partial v}{\partial x_j}\right)\omega_v^j\right|^2\mathrm{d}x\,\mathrm{d}t\\
&+\varepsilon^2\int_0^\tau\int_{\Omega_2^\varepsilon}\sum_{i=1}^n\left(\left|Q^\varepsilon_{Y_2}\left(\frac{\partial w}{\partial x_j}\right)\omega_w^j\right|^2+\left|Q^\varepsilon_{Y_2}\left(\frac{\partial w}{\partial x_j}\right)\nabla_y\omega_w^j\right|^2\right)\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

$$\leq C\big(\varepsilon^2\|v\|^2_{L^2(0,T;H^2(\Omega))}+\varepsilon^2\|v\|^2_{H^1((0,T)\times\Omega)}+\varepsilon\|v\|^2_{L^2(0,T;H^2(\Omega))}\big)\|\omega_v\|^2_{H^1(Y_1)^n}$$
$$+\varepsilon^2 C\big(\|u\|^2_{L^2(0,T;H^1(\Omega))}\|\omega_u\|^2_{L^2(Y_1)^n}+\|w\|^2_{L^2(0,T;H^1(\Omega))}\|\omega_w\|^2_{H^1(Y_2)^n}\big).$$

For C_w we have the same expression as for C_v, with v replaced by w, ω_v^j by ω_w^j, Ω_1^ε by Ω_2^ε, and without the term $|\mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j}u)\omega_u^j|^2$. Thus, we can estimate

$$C_w\leq \varepsilon^2 C\|v\|^2_{L^2(0,T;H^1(\Omega))}\|\omega_v\|^2_{H^1(Y_1)^n}$$
$$+C\big(\varepsilon\|w\|^2_{L^2(0,T;H^2(\Omega))}+\varepsilon^2\|w\|^2_{L^2(0,T;H^2(\Omega))}+\varepsilon^2\|w\|^2_{H^1((0,T)\times\Omega)}\big)\|\omega_w\|^2_{H^1(Y_2)^n}.$$

For sufficiently small ε, adding all the estimates, removing ρ^ε by using the estimates (21), applying Gronwall's inequality and considering that $u^\varepsilon(0)=u_0$, $v^\varepsilon(0)=v_0$, $v^\varepsilon(0)=v_0$ we obtain the estimates for u^ε, v^ε, w^ε, stated in the theorem.

To obtain the estimate for $r^\varepsilon-\mathcal{U}^\varepsilon_{\Gamma_1}(r)$, we consider the equations for $\mathcal{T}^\varepsilon_{\Gamma_1}r^\varepsilon$, obtained from (8) by applying the unfolding operator, and the equation for r in (14) with the test function $\mathcal{T}^\varepsilon_{\Gamma_1}r^\varepsilon-r$. Using the properties of $\mathcal{U}^\varepsilon_{\Gamma_1}$, the local Lipschitz continuity of η, and Gronwall's inequality, yields

$$\varepsilon\int_{\Gamma_i^\varepsilon}|r^\varepsilon-\mathcal{U}^\varepsilon_{\Gamma_1}(r)|^2\mathrm{d}\gamma$$
$$\leq C\int_{\Omega\times\Gamma_1}|\mathcal{T}^\varepsilon_{\Gamma_1}(r^\varepsilon)-r|^2\mathrm{d}\gamma\,\mathrm{d}x$$
$$\leq C\int_0^\tau\int_{\Omega\times\Gamma_1}|\mathcal{T}^\varepsilon_{\Gamma_1}(u^\varepsilon)-u|^2\mathrm{d}\gamma\,\mathrm{d}x\,\mathrm{d}t+\int_{\Omega\times\Gamma_1}|\mathcal{T}^\varepsilon_{\Gamma_1}(r_0)-r_0|^2\mathrm{d}\gamma_y\,\mathrm{d}x$$
$$\leq\int_0^\tau\int_{\Omega\times\Gamma_1}\big(|\mathcal{T}^\varepsilon_{\Gamma_1}(u)-\mathcal{M}^\varepsilon_{Y_1}(u)|^2+|\mathcal{M}^\varepsilon_{Y_1}(u)-u|^2\big)\mathrm{d}\gamma\,\mathrm{d}x\,\mathrm{d}t$$
$$+C\int_0^\tau\int_{\Omega_1^\varepsilon}\Bigg[\bigg|\hat{u}^\varepsilon-\varepsilon\sum_{j=1}^n\mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j}u)\omega_u^j\bigg|^2+\varepsilon^2\bigg|\nabla\hat{u}^\varepsilon-\sum_{j=1}^n\mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j}u)\nabla_y\omega_u^j\bigg|^2$$
$$+\varepsilon^2\sum_{j=1}^n\Big(\big|\mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j}u)\omega_u^j\big|^2+\varepsilon^2\big|\nabla\mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j}u)\omega_u^j\big|^2+\big|\mathcal{Q}^\varepsilon_{Y_1}(\partial_{x_j}u)\nabla_y\omega_u^j\big|^2\Big)\Bigg]\mathrm{d}x\,\mathrm{d}t$$
$$\leq C(\varepsilon+\varepsilon^2)\big(\|u\|^2_{L^2(0,T;H^2(\Omega))}+\|u\|^2_{H^1((0,T)\times\Omega)}+\|v\|^2_{L^2(0,T;H^2(\Omega))}$$
$$+\|v\|^2_{H^1((0,T)\times\Omega)}+\|w\|^2_{L^2(0,T;H^2(\Omega))}+\|w\|^2_{H^1((0,T)\times\Omega)}+\|r\|^2_{L^\infty((0,T)\times\Omega\times\Gamma_1)}\big).$$

4.1. *Estimates of the error terms*

Now, we proceed to estimating the error terms R_u^1, R_u^2 and R_u^3. Using the definition of ρ^ε, the extension properties of $\tilde{\phi}_1^\varepsilon$, Theorem 3.5 and the estimates (21) we obtain

$$\int_0^\tau\int_{\Omega\times Y_1}\bigg|D_u(y)(\rho^\varepsilon-1)\bigg(\nabla u+\sum_{j=1}^n\frac{\partial u}{\partial x_j}\nabla_y\omega_u^j\bigg)\big(\nabla\tilde{\phi}_1^\varepsilon+\nabla\hat{\psi}_1^\varepsilon\big)\bigg|\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq C\|\nabla u\|_{L^2((0,T)\times\hat{\Omega}_{1,\rho,\mathrm{in}})}\bigg(1+\sum_{j=1}^n\|\nabla_y\omega_u^j\|_{L^2(Y_1)}\bigg)\big(\|\nabla\tilde{\phi}_1^\varepsilon\|_{L^2(\Omega_\tau)}+\|\nabla\hat{\psi}_1^\varepsilon\|_{L^2(\Omega_\tau\times Y_1)}\big)$$
$$\leq C\varepsilon^{1/2}\|u\|_{L^2(0,T;H^2(\Omega))}\bigg(1+\sum_{j=1}^n\|\nabla_y\omega_u^j\|_{L^2(Y_1)}\bigg)\|\nabla\phi_1\|_{L^2((0,\tau)\times\Omega_1^\varepsilon)},$$

where $\Omega_\tau = (0,\tau)\times\Omega$. Theorem 3.5 and the estimates (20) and (21) imply

$$\int_0^\tau \int_{\Omega\times Y_1} \rho^\varepsilon D_u(y)\Big(\nabla u + \sum_{j=1}^n \partial_{x_j} u \nabla_y \omega_u^j\Big)\Big(\mathcal{T}^\varepsilon_{Y_1}(\nabla\phi_1) - \nabla\phi_1^\varepsilon - \nabla_y \hat{\psi}_1^\varepsilon\Big)\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t$$
$$\le C(\varepsilon^{1/2}+\varepsilon)\|u\|_{L^2(0,T;H^2(\Omega))}\Big(1+\sum_{j=1}^n \|\nabla_y\omega_u^j\|_{L^2(Y_1)}\Big)\|\nabla\phi_1\|_{L^2((0,T)\times\Omega_1^\varepsilon)}.$$

We notice $\mathcal{M}^\varepsilon_{Y_1}(\tilde{\phi}_1^\varepsilon) = \mathcal{M}^\varepsilon_{Y_1}(\phi_1)$ and using estimates (20) and (21), Lemma 3.4, the fact that $\tilde{\phi}_1^\varepsilon$ is an extension of ϕ_1 from Ω_1^ε into Ω and $\phi_1 = \tilde{\phi}_1$ a.e in $(0,T)\times\Omega_1^\varepsilon$, implies

$$\int_0^\tau \int_{\Omega\times Y_1} \rho^\varepsilon D_u\Big(\nabla u + \sum_{j=1}^n \frac{\partial u}{\partial x_j}\nabla_y\omega_u^j\Big)\nabla\big(\mathcal{Q}^\varepsilon_{Y_i}(\phi_1) - \tilde{\phi}_1^\varepsilon\big)\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t$$
$$\le \Big\|\nabla\Big(\rho^\varepsilon D_u\Big(\nabla u + \sum_{j=1}^n \frac{\partial u}{\partial x_j}\nabla_y\omega_u^j\Big)\Big)\Big\|_{L^2((0,\tau)\times\Omega\times Y_1)} \|\mathcal{Q}^\varepsilon_{Y_i}(\tilde{\phi}_1^\varepsilon) - \tilde{\phi}_1^\varepsilon\|_{L^2(\Omega_\tau)}$$
$$\le C\varepsilon\big(\varepsilon^{-1}\|\nabla u\|_{L^2((0,T)\times\hat{\Omega}_{1,\rho,\mathrm{in}})} + \|\nabla^2 u\|_{L^2}\big)\Big(1+\sum_{j=1}^n\|\nabla\omega_u^j\|_{L^2(Y_1)}\Big)\|\nabla\tilde{\phi}_1^\varepsilon\|_{L^2(\Omega_\tau)}$$
$$\le C(\varepsilon^{1/2}+\varepsilon)\|u\|_{L^2(0,T;H^2(\Omega))}\Big(1+\sum_{j=1}^n\|\nabla_y\omega_u^j\|_{L^2(Y_1)}\Big)\|\nabla\phi_1\|_{L^2((0,\tau)\times\Omega_1^\varepsilon)}.$$

Applying the estimates in Lemma 3.4 yields

$$\int_0^\tau \int_{\Omega\times Y_1} \big(\partial_t\big(u - \mathcal{T}^\varepsilon_{Y_1}(u)\big)\mathcal{M}^\varepsilon_{Y_1}(\phi_1) + \partial_t u\big(\tilde{\phi}_1^\varepsilon - \mathcal{M}^\varepsilon_{Y_1}(\phi_1)\big)\big)\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t$$
$$\le C\varepsilon\big(\|\partial_t\nabla u\|_{L^2(\Omega_T)}\|\phi_1\|_{L^2(0,\tau)\times\Omega_1^\varepsilon)} + \|\partial_t u\|_{L^2(\Omega_T)}\|\nabla\phi_1\|_{L^2((0,\tau)\times\Omega_1^\varepsilon)}\big),$$

where $\Omega_T = (0,T)\times\Omega$. Due to Lipschitz continuity of f, we can estimate

$$\int_0^\tau \int_{\Omega\times Y_1} \big(\big(\mathcal{T}^\varepsilon_{Y_1}(f) - f(u,v)\big)\mathcal{M}^\varepsilon_{Y_1}(\phi_1) + f(u,v)\big(\mathcal{M}^\varepsilon_{Y_1}(\phi_1) - \tilde{\phi}_1^\varepsilon\big)\big)\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t$$
$$\le \varepsilon C\big(\|\nabla u\|_{L^2(\Omega_T)} + \|\nabla v\|_{L^2(\Omega_T)}\big)\|\phi_1\|_{L^2((0,\tau)\times\Omega_1^\varepsilon)}$$
$$+ \varepsilon C\big(1 + \|u\|_{L^2(\Omega_T)} + \|v\|_{L^2(\Omega_T)}\big)\|\nabla\phi_1\|_{L^2((0,\tau)\times\Omega_1^\varepsilon)}.$$

For the boundary integral we have

$$\int_0^\tau \int_{\Omega\times\Gamma_1} \eta(u,r)(\mathcal{T}^\varepsilon_{\Gamma_1}(\phi_1) - \tilde{\phi}_1^\varepsilon))\mathrm{d}\gamma\,\mathrm{d}x\,\mathrm{d}t$$
$$\le \|\eta(u,r)\|_{L^2((0,\tau)\times\Omega\times\Gamma_1)}\big(\|\mathcal{T}^\varepsilon_{\Gamma_1}(\phi_1) - \mathcal{M}^\varepsilon_{Y_1}(\phi_1)\|_{L^2((0,\tau)\times\Omega\times\Gamma_1)}$$
$$+ \|\mathcal{M}^\varepsilon_{Y_1}(\phi_1) - \tilde{\phi}_1^\varepsilon)\|_{L^2((0,\tau)\times\Omega\times\Gamma_1)}\big)$$
$$\le C\big(1 + \|u\|_{L^2(\Omega_T)} + \|r\|_{L^\infty(\Omega_T\times\Gamma_1)}\big)\|\mathcal{T}^\varepsilon_{Y_1}(\phi_1) - \mathcal{M}^\varepsilon_{Y_1}(\phi_1)\|_{L^2(\Omega_\tau;H^1(Y_1))}$$
$$+ C\big(1 + \|u\|_{L^2(\Omega_T)} + \|r\|_{L^\infty(\Omega_T\times\Gamma_1)}\big)\|\mathcal{M}^\varepsilon_{Y_1}(\phi_1) - \tilde{\phi}_1^\varepsilon)\|_{L^2(\Omega_\tau)}$$
$$\le \varepsilon C\big(1 + \|u\|_{L^2(\Omega_T)} + \|r\|_{L^\infty(\Omega_T\times\Gamma_1)}\big)\|\nabla\phi_1\|_{L^2((0,\tau)\times\Omega_1^\varepsilon)}.$$

Thus, collecting all estimates from above we obtain for R_u^1:

$$|R_u^1| \le C(\varepsilon^{\frac{1}{2}} + \varepsilon)\|u\|_{L^2(0,T;H^2(\Omega))}\left(1 + \sum_{j=1}^n \|\nabla_y \omega_u^j\|_{L^2(Y_1)}\right)\|\nabla\phi_1\|_{L^2((0,\tau)\times\Omega_i^\varepsilon)}$$
$$+ C\varepsilon\big(\|u\|_{H^1((0,T)\times\Omega)} + \|v\|_{L^2(0,T;H^1(\Omega))}\big)\|\phi_1\|_{L^2(0,\tau;H^1(\Omega_1^\varepsilon))}.$$

Using the estimates (21) implies

$$\int_0^\tau \int_{\Omega_1^\varepsilon} (1-\rho^\varepsilon) D_u^\varepsilon \sum_{j=1}^n \mathcal{M}_{Y_i}^\varepsilon(\partial_{x_j} u)\nabla_y \omega_u^j\left(\frac{x}{\varepsilon}\right)\nabla\phi_1 \,\mathrm{d}x\,\mathrm{d}t$$
$$\le \sum_{j=1}^n \|\mathcal{M}_{Y_i}^\varepsilon(\partial_{x_j} u)\|_{L^2((0,\tau)\times\hat{\Omega}^\varepsilon_{1,\rho,\mathrm{in}})}\left\|\nabla_y \omega_u^j\left(\frac{x}{\varepsilon}\right)\right\|_{L^2(\hat{\Omega}^\varepsilon_{1,\rho,\mathrm{in}})}\|\nabla\phi_1\|_{L^2((0,\tau)\times\Omega_1^\varepsilon)}$$
$$\le \varepsilon C \sum_{j=1}^n \|u\|_{L^2(0,T;H^2(\Omega_1^\varepsilon))}\|\nabla_y \omega_u^j\|_{L^2(Y_1)}\|\nabla\phi_1\|_{L^2((0,\tau)\times\Omega_1^\varepsilon)}.$$

Thus, the last estimate and the estimates (18), (21) yield

$$|R_u^2| \le \|\nabla u\|_{L^2((0,\tau)\times\hat{\Omega}^\varepsilon_{1,\rho,\mathrm{in}})}(1 + \|\nabla_y \omega_u\|_{L^2(Y_1)^{n\times n}})\|\mathcal{T}_{Y_1}^\varepsilon(\nabla\phi_1)\|_{L^2((0,\tau)\times\Omega\times Y_1)}$$
$$+ C\varepsilon\|u\|_{L^2(0,\tau;H^2(\Omega))}(1 + \|\nabla_y \omega_u\|_{L^2(Y_1)^{n\times n}})\|\mathcal{T}_{Y_1}^\varepsilon(\nabla\phi_1)\|_{L^2((0,\tau)\times\Omega\times Y_1)}$$
$$\le (\varepsilon^{1/2} + \varepsilon)C\|u\|_{L^2(0,T;H^2(\Omega))}(1 + \|\nabla_y \omega_u\|_{L^2(Y_1)^{n\times n}})\|\nabla\phi_1\|_{L^2((0,\tau)\times\Omega_1^\varepsilon)}.$$

Due to estimates in (21) and in Lemma 3.4 we also obtain

$$|R_u^3| \le \varepsilon C\big(\|\partial_t u\|_{L^2((0,T)\times\Omega_1^\varepsilon)} + \|f\|_{L^2((0,T)\times\Omega_1^\varepsilon)} + \|u\|_{L^2(0,T;H^2(\Omega_1^\varepsilon))}\|\nabla_y \omega_u\|_{L^2(Y_1)^{n\times n}}$$
$$+ \|\nabla^2 u\|_{L^2((0,T)\times\Omega_1^\varepsilon)} + \|\nabla^2 u\|_{L^2((0,T)\times\Omega_1^\varepsilon)}\|\nabla_y \omega_u\|_{L^2(Y_1)^{n\times n}}\big)\|\nabla\phi_1\|_{L^2((0,\tau)\times\Omega_1^\varepsilon)}.$$

In the similar way we show the estimates for the error terms in the equations for v and w:

$$|R_v| \le C\varepsilon^{\frac{1}{2}}\big(1 + \|v\|_{L^2(0,T;H^2(\Omega))} + \|v\|_{H^1(\Omega_T)}$$
$$+ \|u\|_{L^2(0,T;H^1(\Omega))} + \|w\|_{L^2(0,T;H^1(\Omega))}\big)\|\phi_2\|_{L^2(0,\tau;H^1(\Omega_1^\varepsilon))},$$
$$|R_w| \le C\varepsilon^{\frac{1}{2}}\big(1 + \|w\|_{L^2(0,T;H^2(\Omega))} + \|w\|_{H^1(\Omega_T)} + \|v\|_{L^2(0,T;H^1(\Omega))}\big)\|\phi_3\|_{L^2(0,\tau;H^1(\Omega_2^\varepsilon))}.$$

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Appendix

LEMMA A.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. If $z \in H^1(\Omega) \cap L^\infty(\Omega)$, then $z \in L^\infty(\partial\Omega)$.*

Proof Let $z \in H^1(\Omega) \cap L^\infty(\Omega)$. Since $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$, we consider a sequence of smooth functions $\{f_n\} \subset C^\infty(\overline{\Omega})$, such that $f_n \to z$ in $H^1(\Omega)$ and $\|f_n\|_{L^\infty(\Omega)} \leq \|z\|_{L^\infty(\Omega)}$. Applying the trace theorem, see [10], we obtain $f_n \to z$ in $L^2(\partial\Omega)$. Thus, there exists a subsequence $\{f_{n_i}\} \subset \{f_n\}$ converging pointwise, i.e. $f_{n_i}(x) \to z(x)$ a.e. $x \in \partial\Omega$, and $|f_{n_i}(x)| \leq \|z\|_{L^\infty(\Omega)}$ for a.a. $x \in \partial\Omega$. The pointwise convergence and the estimate $\|f_{n_i}\|_{L^\infty(\partial\Omega)} \leq \|z\|_{L^\infty(\Omega)}$ imply $\|z\|_{L^\infty(\partial\Omega)} \leq \|z\|_{L^\infty(\Omega)}$.

LEMMA 5.2 [12,14] (1) *For $w \in L^p(\Omega_i^\varepsilon)$, $p \in [1, \infty)$, we have*

$$\|\mathcal{T}^\varepsilon_{Y_i} w\|_{L^p(\Omega \times Y_i)} = |Y|^{1/p} \|w\|_{L^p(\tilde{\Omega}^\varepsilon_{i,\mathrm{int}})} \leq |Y|^{1/p} \|w\|_{L^p(\Omega_i^\varepsilon)}.$$

(2) *For $u \in L^p(\Gamma_i^\varepsilon)$, $p \in [1, \infty)$, we have*

$$\|\mathcal{T}^\varepsilon_{\Gamma_i} u\|_{L^p(\Omega \times \Gamma_i)} = \varepsilon^{1/p} |Y|^{1/p} \|u\|_{L^p(\tilde{\Gamma}^\varepsilon_{i,\mathrm{int}})} \leq \varepsilon^{1/p} |Y|^{1/p} \|u\|_{L^p(\Gamma_i^\varepsilon)}.$$

(3) *If $w \in L^p(\Omega)$, $p \in [1, \infty)$, then $\mathcal{T}^\varepsilon_{Y_i} w \to w$ strongly in $L^p(\Omega \times Y_i)$ as $\varepsilon \to 0$.*

(4) *For $w \in W^{1,p}(\Omega_i^\varepsilon)$, $1 < p < +\infty$,*

$$\|\mathcal{T}^\varepsilon_{\Gamma_i} w\|_{L^p(\Omega \times \Gamma_i)} \leq C\big(\|w\|_{L^p(\Omega_i^\varepsilon)} + \varepsilon \|\nabla w\|_{L^p(\Omega_i^\varepsilon)^n}\big).$$

(5) *For $w \in W^{1,p}(\Omega_i^\varepsilon)$ holds $\mathcal{T}^\varepsilon_{Y_i}(w) \in L^p(\Omega, W^{1,p}(Y_i))$ and $\nabla_y \mathcal{T}^\varepsilon_{Y_i}(w) = \varepsilon \mathcal{T}^\varepsilon_{Y_i}(\nabla w)$.*

(6) *Let* $v \in L^p_{\text{per}}(Y_i)$ *and* $v^\varepsilon(x) = v(\frac{x}{\varepsilon})$, *then* $\mathcal{T}^\varepsilon_{Y_i}(v^\varepsilon)(x, y) = v(y)$.

(7) *For* $v, w \in L^p(\Omega^\varepsilon_i)$ *and* $\phi, \psi \in L^p(\Gamma^\varepsilon_i)$ *holds* $\mathcal{T}^\varepsilon_{Y_i}(v\,w) = \mathcal{T}^\varepsilon_{Y_i}(v)\mathcal{T}^\varepsilon_{Y_i}(w)$ *and* $\mathcal{T}^\varepsilon_{\Gamma_i}(\phi\,\psi) = \mathcal{T}^\varepsilon_{\Gamma_i}(\phi)\mathcal{T}^\varepsilon_{\Gamma_i}(\psi)$.

Theorem A.3 [13,16] *Let* $p \in (1, \infty)$ *and* $i = 1, 2$.

(1) *For* $\{\phi_\varepsilon\} \subset W^{1,p}(\Omega^\varepsilon_i)$, *satisfies* $\|\phi_\varepsilon\|_{W^{1,p}(\Omega^\varepsilon_i)} \leq C$, *there exists a subsequence of* $\{\phi^\varepsilon\}$ *(still denoted by* ϕ_ε*), and* $\phi \in W^{1,p}(\Omega)$, $\hat\phi \in L^p(\Omega; W^{1,p}_{\text{per}}(Y_i))$, *such that*

$$\begin{aligned} \mathcal{T}^\varepsilon_{Y_1}\phi_\varepsilon &\to \phi \quad \textit{strongly in } L^p_{\text{loc}}(\Omega; W^{1,p}(Y_i)),\\ \mathcal{T}^\varepsilon_{Y_1}\phi_\varepsilon &\rightharpoonup \phi \quad \textit{weakly in } L^p(\Omega; W^{1,p}(Y_i)),\\ \mathcal{T}^\varepsilon_{Y_1}(\nabla\phi_\varepsilon) &\rightharpoonup \nabla\phi + \nabla_y\hat\phi \quad \textit{weakly in } L^p(\Omega \times Y_i). \end{aligned}$$

(2) *For* $\{\phi^\varepsilon\} \subset W^{1,p}_0(\Omega^\varepsilon_i)$, *such that* $\|\phi^\varepsilon\|_{W^{1,p}_0(\Omega^\varepsilon_i)} \leq C$, *there exists a subsequence of* $\{\phi^\varepsilon\}$ *(still denoted by* ϕ_ε*) and* $\phi \in W^{1,p}_0(\Omega)$, $\hat\phi \in L^p(\Omega; W^{1,p}_{\text{per}}(Y_i))$, *such that*

$$\begin{aligned} \mathcal{T}^\varepsilon_{Y_i}\phi^\varepsilon &\to \phi \quad \textit{strongly in } L^p(\Omega; W^{1,p}(Y_i)),\\ \mathcal{T}^\varepsilon_{Y_i}(\nabla\phi^\varepsilon) &\rightharpoonup \nabla\phi + \nabla_y\hat\phi \quad \textit{weakly in } L^p(\Omega \times Y_i). \end{aligned}$$

(3) *For* $\{\psi^\varepsilon\} \subset L^p(\Gamma^\varepsilon_i)$, *such that* $\varepsilon^{1/p}\|\psi^\varepsilon\|_{L^p(\Gamma^\varepsilon_i)} \leq C$, *there exists a subsequence of* $\{\psi^\varepsilon\}$ *and* $\psi \in L^p(\Omega \times \Gamma_i)$ *such that*

$$\mathcal{T}^\varepsilon_{\Gamma_i}(\psi^\varepsilon) \rightharpoonup \psi \quad \textit{weakly in } L^p(\Omega \times \Gamma_i).$$

Proposition A.4 [13,16] (1) *The operator* $\mathcal{U}^\varepsilon_{Y_i}$ *is formal adjoint and left inverse of* $\mathcal{T}^\varepsilon_{Y_i}$, *i.e for* $\phi \in L^p(\Omega^\varepsilon_i)$, *where* $p \in [1, \infty)$,

$$\mathcal{U}^\varepsilon_{Y_i}(\mathcal{T}^\varepsilon_{Y_i}(\phi))(x) = \begin{cases} \phi(x) & \textit{a.e. for } \in \tilde\Omega^\varepsilon_{i,\text{int}},\\ 0 & \textit{a.e. for } \in \Omega^\varepsilon_i \setminus \tilde\Omega^\varepsilon_{i,\text{int}}. \end{cases}$$

(2) *For* $\phi \in L^p(\Omega \times Y_i)$ *holds* $\|\mathcal{U}^\varepsilon_{Y_i}(\phi)\|_{L^p(\Omega^\varepsilon_i)} \leq |Y|^{-1/p}\|\phi\|_{L^p(\Omega\times Y_i)}$.

Theorem A.5 [8] *For any* $\phi \in H^1(\Omega)$, *there exists* $\hat\phi_\varepsilon \in H^1_{\text{per}}(Y; L^2(\Omega))$:

$$\begin{aligned} \|\hat\phi_\varepsilon\|_{H^1(Y;L^2(\Omega))} &\leq C\|\nabla\phi\|_{L^2(\Omega)^n},\\ \|\mathcal{T}_\varepsilon(\nabla_x\phi) - \nabla\phi - \nabla_y\hat\phi_\varepsilon\|_{L^2(Y;H^{-1}(\Omega))^n} &\leq C\varepsilon\|\nabla\phi\|_{L^2(\Omega)^n}. \end{aligned}$$

The proofs of Theorems A.5 and 3.5 are based on the following results:

Theorem A.6 [8] *For any* $\phi \in W^{1,p}(Y_i)$, *with* $i = 1, 2$ *and* $p \in (1, \infty]$, *there exists* $\hat\phi \in W^{1,p}_{\text{per}}(Y_i)$ *such that*

$$\|\phi - \hat\phi\|_{W^{1,p}(Y_i)} \leq C\sum_{j=1}^n \|\phi|_{e_j+Y^j_i} - \phi_{Y^j_i}\|_{W^{1-\frac{1}{p},p}(Y^j_i)},$$

where $Y^j_i = \{y \in \bar Y_i \mid y_j = 0\}$, *for* $j = 1, \ldots, n$, *and C depends only on n.*

Lemma A.7 [8] *For any* $\phi \in W^{1,p}(Y_i)$, *where* $p \in (1, \infty]$, $i = 1, 2$, *and for* $k \in \{1, \ldots, n\}$, *there exists* $\hat\phi_k \in W_k = \{\phi \in W^{1,p}(Y_i), \phi(\cdot) = \phi(\cdot + e_j), j \in \{1, \ldots, k\}\}$, *whereas* $W_0 = W^{1,p}(Y_i)$, *such that*

$$\|\phi - \hat\phi_k\|_{W^{1,p}(Y_i)} \leq C\sum_{j=1}^k \|\phi|_{e_j+Y^j_i} - \phi|_{Y^j_i}\|_{W^{1-1/p,p}(Y^j_i)}.$$

The constant C is independent on n.

Theorem A.8 [8] *For any* $\phi \in H^1(Y_i, X)$ *and X separable Hilbert space, there exists a unique* $\hat\phi \in H^1_{\text{per}}(Y_i, X)$, $i = 1, 2$, *such that* $\phi - \hat\phi \in (H^1_{\text{per}}(Y_i, X))^\perp$ *and*

$$\|\hat\phi\|_{H^1(Y_i,X)} \leq \|\phi\|_{H^1(Y_i,X)}, \quad \|\phi - \hat\phi\|_{H^1(Y_i,X)} \leq C\sum_{j=1}^n \|\phi|_{e_j+Y^j_i} - \phi|_{Y^j_i}\|_{H^{1/2}(Y^j_i,X)}.$$

Lemma A.7, Theorems A.6 and A.8 follow directly from the corresponding Lemma 2.2., Theorems 2.1 and 2.3 in [8], replacing Y by Y_1 and Y_2.

The proof of Theorem 3.5 relies also on the following generalization of the Proposition 3.3 in [8].

PROPOSITION A.9 *For $\phi \in H^1(\Omega_i^\varepsilon)$ there exists a unique $\hat{\psi}^\varepsilon \in L^2(\Omega; H^1_{\rm per}(Y_i))$, such that*

$$\|\hat{\psi}_\varepsilon\|_{L^2(\Omega;H^1(Y_i))} \le C\big(\|\phi\|_{L^2(\Omega_i^\varepsilon)} + \varepsilon\|\nabla\phi\|_{L^2(\Omega_i^\varepsilon)^n}\big),$$

$$\|\mathcal{T}^\varepsilon_{Y_i}(\phi) - \hat{\psi}_\varepsilon\|_{H^{-1}(\Omega;H^1(Y_i))} \le C\varepsilon\big(\|\phi\|_{L^2(\Omega_i^\varepsilon)} + \varepsilon\|\nabla\phi\|_{L^2(\Omega_i^\varepsilon)^n}\big).$$

Proof The proof follows the same lines as [8]. We consider $K_j = {\rm Int}(Y_i \cup (Y_i + e_j))$ and $\varepsilon(K_i + [x/\varepsilon]_{Y_i}) \subset \tilde{\Omega}_i^{\varepsilon,2}$ for $x \in \Omega_i^\varepsilon$, where $\tilde{\Omega}_i^{\varepsilon,k}$ are introduced in Section 2.3. Then for all $\phi \in L^2(\tilde{\Omega}_i^{\varepsilon,2})$ we define

$$\mathcal{T}^{\varepsilon,j}_{Y_i}(\phi)(x,y) = \phi\left(x, \varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) \quad \text{for } x \in \Omega \text{ and a.e. } y \in K_j.$$

For a.e. $y \in Y_i$ and $\phi \in H^1_0(\Omega)$, extended by zero to $\mathbb{R}^n \setminus \Omega$, we obtain

$$\int_\Omega \mathcal{T}^{\varepsilon,j}_{Y_i}(\phi)(x, y + e_j)\psi(x)\mathrm{d}x = \int_{\Omega+\varepsilon e_j} \mathcal{T}^{\varepsilon,j}_{Y_i}(\phi)(x,y)\psi(x - \varepsilon e_j)\mathrm{d}x.$$

Notice that $\mathcal{T}^{\varepsilon,j}_{Y_i}(\phi)(x, y+e_j) = \phi\big(x, \varepsilon[\frac{x}{\varepsilon}]_Y + \varepsilon y + \varepsilon e_j\big) = \phi\big(x, \varepsilon\big[\frac{x+\varepsilon e_j}{\varepsilon}\big]_Y + \varepsilon y\big) = \mathcal{T}^{\varepsilon,j}_{Y_i}(\phi)(x + \varepsilon e_j, y)$ for $x \in \Omega$ and $y \in K_j$ and $\mathcal{T}^{\varepsilon,j}_{Y_i}(\phi)|_{\Omega\times Y_i} = \mathcal{T}^\varepsilon_{Y_i}(\phi)$. Thus

$$\left|\int_\Omega \big(\mathcal{T}^{\varepsilon,j}_{Y_i}(\phi)(\cdot, y+e_j) - \mathcal{T}^{\varepsilon,j}_{Y_i}(\phi)(\cdot,y)\big)\psi\,\mathrm{d}x - \int_\Omega \mathcal{T}^{\varepsilon,j}_{Y_i}(\phi)(\cdot,y)\big(\psi(\cdot - \varepsilon e_j) - \psi\big)\mathrm{d}x\right|$$
$$\le C\|\mathcal{T}^{\varepsilon,j}_{Y_i}(\phi)(\cdot,y))\|_{L^2(\tilde{\Omega}^{\varepsilon,1})}\|\psi\|_{L^2(\Omega\setminus(\Omega+\varepsilon e_j))},$$

where $\tilde{\Omega}^{\varepsilon,k} = \{x \in \mathbb{R}^n, {\rm dist}(\Omega, x) < k\varepsilon\sqrt{n}\}$. The Lipschitz continuity of $\partial\Omega$ and $\psi \in H^1_0(\Omega)$ imply, for $j = 1, \ldots, n$,

$$\|\psi\|_{L^2(\Omega\setminus(\Omega+\varepsilon e_j))} \le C\varepsilon\|\nabla\psi\|_{L^2(\Omega)^n}, \;\; \|\psi(\cdot - \varepsilon e_j) - \psi\|_{L^2(\Omega)} \le C\varepsilon\|\partial_{x_j}\psi\|_{L^2(\Omega)}.$$

Due to Lipschitz continuity of $\partial\Omega_i^\varepsilon$ a function $\phi \in H^1(\Omega_i^\varepsilon)$ can be extended into $H^1(\tilde{\Omega}_i^{\varepsilon,2})$, such that $\|\mathcal{P}(\phi)\|_{L^2(\tilde{\Omega}_i^{\varepsilon,2})} \le C\big(\|\phi\|_{L^2(\Omega_i^\varepsilon)} + \varepsilon\|\nabla\phi\|_{L^2(\Omega_i^\varepsilon)^n}\big)$ and $\|\nabla\mathcal{P}(\phi)\|_{L^2(\tilde{\Omega}_i^{\varepsilon,2})^n} \le C\|\nabla\phi\|_{L^2(\Omega_i^\varepsilon)^n}$.

Hence for $\phi \in H^1(\Omega_i^\varepsilon)$ and $\psi \in H^1_0(\Omega)$ it follows, for a.a. $y \in Y_i$,

$$\langle \mathcal{T}^{\varepsilon,j}_{Y_i}(\phi)(\cdot, \cdot + e_j) - \mathcal{T}^{\varepsilon,j}_{Y_i}(\phi)(\cdot,\cdot), \psi\rangle_{H^{-1}(\Omega), H^1_0(\Omega)}$$
$$= \int_\Omega (\mathcal{T}^{\varepsilon,j}_{Y_i}(\phi)(\cdot, y+e_j) - \mathcal{T}^{\varepsilon,j}_{Y_i}(\phi)(\cdot,y))\psi\,\mathrm{d}x \le C\varepsilon\|\nabla\psi\|_{L^2(\Omega)^n}\|\mathcal{T}^{\varepsilon,j}_{Y_i}(\phi)(\cdot,y)\|_{L^2(\tilde{\Omega}^{\varepsilon,1})}.$$

The last estimate, the definition of $\mathcal{T}^{\varepsilon,j}_{Y_i}$ and the extension properties yield

$$\|\mathcal{T}^{\varepsilon,j}_{Y_i}(\phi)(\cdot, \cdot + e_j) - \mathcal{T}^{\varepsilon,j}_{Y_i}(\phi)(\cdot,\cdot)\|_{H^{-1}(\Omega;L^2(Y_i))}$$
$$\le C\varepsilon\|\mathcal{T}^{\varepsilon,j}_{Y_i}(\phi)\|_{L^2(\tilde{\Omega}^{\varepsilon,1}\times Y_i)}$$
$$\le C\varepsilon\|\phi\|_{L^2(\tilde{\Omega}_i^{\varepsilon,2})} \le C\varepsilon\big(\|\phi\|_{L^2(\Omega_i^\varepsilon)} + \varepsilon\|\nabla\phi\|_{L^2(\Omega_i^\varepsilon)}\big),$$
$$\|\mathcal{T}^{\varepsilon,j}_{Y_i}(\nabla\phi)(\cdot, \cdot + e_j) - \mathcal{T}^{\varepsilon,j}_{Y_i}(\nabla\phi)(\cdot,\cdot)\|_{H^{-1}(\Omega;L^2(Y_i))} \le \varepsilon C\|\nabla\phi\|_{L^2(\Omega_i^\varepsilon)}.$$

Applying $\nabla_y\mathcal{T}^{\varepsilon,j}_{Y_i}(\phi) = \varepsilon\mathcal{T}^{\varepsilon,j}_{Y_i}(\nabla\phi)$, we obtain the estimate in $H^1(Y_i)$, i.e.

$$\|\mathcal{T}^{\varepsilon,j}_{Y_i}(\phi)(\cdot, \cdot + e_j) - \mathcal{T}^{\varepsilon,j}_{Y_i}(\phi)(\cdot,\cdot)\|_{H^{-1}(\Omega;H^1(Y_i))} \le C\varepsilon\big(\|\phi\|_{L^2(\Omega_i^\varepsilon)} + \varepsilon\|\nabla\phi\|_{L^2(\Omega_i^\varepsilon)}\big).$$

This implies also the estimate for the traces of $y \to \mathcal{T}^\varepsilon_{Y_i}(\phi)$ on Y_i^j and $e_j + Y_i^j$

$$\|\mathcal{T}^{\varepsilon,j}_{Y_i}(\phi)(\cdot,\cdot+e_j) - \mathcal{T}^{\varepsilon,j}_{Y_i}(\phi)(\cdot,\cdot)\|_{H^{-1}(\Omega;H^{1/2}(Y_i^j))} \le C\varepsilon\big(\|\phi\|_{L^2(\Omega_i^\varepsilon)} + \varepsilon\|\nabla\phi\|_{L^2(\Omega_i^\varepsilon)}\big).$$

Using Theorem A.8, we decompose $\mathcal{T}^\varepsilon_{Y_i}(\phi) = \hat{\psi}^\varepsilon + \bar{\phi}^\varepsilon$, where $\hat{\psi}^\varepsilon \in L^2(\Omega; H^1_{\text{per}}(Y_i))$ and $\bar{\phi}^\varepsilon \in (L^2(\Omega; H^1_{\text{per}}(Y_i)))^\perp$ such that

$$\begin{aligned}
\|\bar{\phi}^\varepsilon\|_{L^2(\Omega;H^1(Y_i))} + \|\hat{\psi}^\varepsilon\|_{L^2(\Omega;H^1(Y_i))} &\le C\Big(\|\phi\|_{L^2(\Omega_i^\varepsilon)} + \varepsilon\|\nabla\phi\|_{L^2(\Omega_i^\varepsilon)}\Big),\\
\|\bar{\phi}^\varepsilon\|_{H^{-1}(\Omega;H^1(Y_i))} &\le C\sum_{j=1}^n \|\mathcal{T}^\varepsilon_{Y_i}(\cdot,\cdot+e_j) - \mathcal{T}^\varepsilon_{Y_i}(\cdot,\cdot)\|_{H^{-1}(\Omega;H^{1/2}(Y_i^j))}\\
&\le C\varepsilon\Big(\|\phi\|_{L^2(\Omega_i^\varepsilon)} + \varepsilon\|\nabla\phi\|_{L^2(\Omega_i^\varepsilon)}\Big).
\end{aligned}$$

■

Proof (of Theorem 3.5) The proof is similar to the proof of Theorem A.5, see [8, Theorem A.4]. For $\phi \in H^1(\Omega_i^\varepsilon)$ we consider $\phi = \phi^\varepsilon + \varepsilon\underline{\phi}$, where $\phi^\varepsilon = Q^\varepsilon_{Y_i}(\phi)$ and $\underline{\phi} = \frac{1}{\varepsilon}(\phi - Q^\varepsilon_{Y_i}(\phi))$. Then, it follows

$$\|\nabla\phi^\varepsilon\|_{L^2(\Omega_i^\varepsilon)} + \|\underline{\phi}\|_{L^2(\Omega_i^\varepsilon)} + \varepsilon\|\nabla\underline{\phi}\|_{L^2(\Omega_i^\varepsilon)} \le C\|\nabla\phi\|_{L^2(\Omega_i^\varepsilon)}. \tag{24}$$

For $\underline{\phi} \in H^1(\Omega_i^\varepsilon)$, using Proposition A.9 and (24), there exists $\hat{\psi}^\varepsilon \in L^2(\Omega; H^1_{\text{per}}(Y_i))$

$$\|\mathcal{T}^\varepsilon_{Y_i}(\underline{\phi}) - \hat{\psi}^\varepsilon\|_{H^{-1}(\Omega;H^1(Y_i))} \le C\varepsilon\|\nabla\phi\|_{L^2(\Omega_i^\varepsilon)},\ \ \|\hat{\psi}^\varepsilon\|_{L^2(\Omega;H^1(Y_i))} \le C\|\nabla\phi\|_{L^2(\Omega_i^\varepsilon)}. \tag{25}$$

Definition and properties of $\mathcal{M}^\varepsilon_{Y_i}$ and $Q^\varepsilon_{Y_i}$, see Section 3, imply

$$\|\partial_{x_j}\phi^\varepsilon - \mathcal{M}^\varepsilon_{Y_i}(\partial_{x_j}\phi^\varepsilon)\|_{H^{-1}(\Omega)} \le C\varepsilon\|\nabla\phi^\varepsilon\|_{L^2(\Omega)^n} \le C\varepsilon\|\nabla\phi\|_{L^2(\Omega_i^\varepsilon)^n}. \tag{26}$$

For $\psi \in H^1_0(\Omega)$ we consider

$$\begin{aligned}
&\langle \mathcal{T}^\varepsilon_{Y_i}(\partial_{x_j}\phi^\varepsilon) - \mathcal{M}^\varepsilon_{Y_i}(\partial_{x_j}\phi^\varepsilon)), \psi\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}\\
&\quad = \int_\Omega \Big(\mathcal{T}^\varepsilon_{Y_i}(\partial_{x_j}\phi^\varepsilon)) - \mathcal{M}^\varepsilon_{Y_i}(\partial_{x_j}\phi^\varepsilon)\Big)\psi \mathrm{d}x\\
&\quad = \int_\Omega \Big(\mathcal{T}^\varepsilon_{Y_i}(\partial_{x_j}\phi^\varepsilon)(\cdot,y) - \mathcal{M}^\varepsilon_{Y_i}(\partial_{x_j}\phi^\varepsilon))\Big)\mathcal{M}^\varepsilon_{Y_i}(\psi)\mathrm{d}x.
\end{aligned}$$

Then, due to the definition of $\mathcal{T}^\varepsilon_{Y_i}(\partial_{x_j}Q^\varepsilon_{Y_i}(\phi))$, it follows, in the same way as in [8],

$$\begin{aligned}
&\int_\Omega \Big(\mathcal{T}^\varepsilon_{Y_i}(\partial_{x_j}\phi^\varepsilon)(\cdot,y) - \mathcal{M}^\varepsilon_{Y_i}(\partial_{x_j}\phi^\varepsilon)\Big)\mathcal{M}^\varepsilon_{Y_i}(\psi)\mathrm{d}x\\
&\quad = \varepsilon^n \sum_\xi \frac{\mathcal{M}^\varepsilon_{Y_i}(\phi)(\varepsilon\xi + \varepsilon e_j) - \mathcal{M}^\varepsilon_{Y_i}(\phi)(\varepsilon\xi)}{\varepsilon}\\
&\quad\quad \times \sum_{\tilde{k}^j}\Big(\mathcal{M}^\varepsilon_{Y_i}(\psi)(\varepsilon\xi - \varepsilon\tilde{k}_0^j) - \frac{1}{2^{n-1}}\sum_{\tilde{k}^j}\mathcal{M}^\varepsilon_{Y_i}(\psi)(\varepsilon\xi - \varepsilon\tilde{k}_0^j)\Big)\bar{y}^j,
\end{aligned}$$

where $\tilde{k}^j = (k_1,\dots,k_{j-1},k_{j+1},\dots,k_n)$, $\tilde{k}_1^j = (k_1,\dots,k_{j-1},1,k_{j+1},\dots,k_n)$, $\tilde{k}_0^j = (k_1,\dots,k_{j-1},0,k_{j+1},\dots,k_n)$, $\bar{y}^j = \bar{y}_1^{k_1}\cdots\bar{y}_{j-1}^{k_{j-1}}\bar{y}_{j+1}^{k_{j+1}}\cdots\bar{y}_n^{k_n}$. This, applying the estimates for $\mathcal{M}^\varepsilon_{Y_i}$, see Section 3, implies, for every $y \in Y_i$, the inequality

$$\langle \mathcal{T}^\varepsilon_{Y_i}(\partial_{x_j}\phi^\varepsilon)(\cdot,y) - \mathcal{M}^\varepsilon_{Y_i}(\partial_{x_j}\phi^\varepsilon), \psi\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} \le C\varepsilon|\bar{y}^j|\,\|\nabla\phi\|_{L^2(\Omega_i^\varepsilon)^n}\|\nabla\psi\|_{L^2(\Omega)^n}$$

and the estimate

$$\|\mathcal{T}^\varepsilon_{Y_i}(\partial_{x_j}\phi^\varepsilon) - \mathcal{M}^\varepsilon_{Y_i}(\partial_{x_j}\phi^\varepsilon)\|_{H^{-1}(\Omega;L^2(Y_i))} \le C\varepsilon\|\nabla\phi\|_{L^2(\Omega_i^\varepsilon)}.$$

Using the last estimate, and the estimate (25) and (26), together with $\nabla\phi = \nabla\phi^\varepsilon + \varepsilon\nabla\underline{\phi}^\varepsilon$, yield

$$\begin{aligned}
&\|\mathcal{T}^{\varepsilon}_{Y_i}(\nabla\phi) - \nabla\phi^\varepsilon - \nabla_y\hat{\psi}^\varepsilon\|_{H^{-1}(\Omega;L^2(Y_i))} \\
&\quad \le \|\mathcal{T}^{\varepsilon}_{Y_i}(\nabla\phi^\varepsilon) - \nabla\phi^\varepsilon\|_{H^{-1}(\Omega;L^2(Y_i))} + \|\nabla_y(\mathcal{T}^{\varepsilon}_{Y_i}(\underline{\phi}^\varepsilon)) - \nabla_y\hat{\psi}^\varepsilon\|_{H^{-1}(\Omega;L^2(Y_i))} \\
&\quad \le C\varepsilon\|\nabla\phi\|_{L^2(\Omega_i^\varepsilon)}.
\end{aligned}$$