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Derivation of a macroscopic model for nutrient uptake by hairy-roots☆

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ABSTRACT

In this article the process of nutrient uptake by a single root branch is studied. We consider diffusion and active transport of nutrients dissolved in water. The uptake happens on the surface of thin root hairs distributed periodically and orthogonal to the root surface. Water velocity is defined by the Stokes equations. We derive a macroscopic model for nutrient uptake by a hairy-root from microscopic descriptions using homogenization techniques. The macroscopic model consists of a reaction-diffusion equation in the domain with hairs and a diffusion-convection equation in the domain without hairs. The macroscopic water velocity is described by the Stokes system in the domain without hairs, with no-slip condition on the boundary between domain with hairs and free fluid.

1. Introduction

As a result of root damage, certain species of plant can be genetically transformed by the bacterium *Agrobacterium rhizogenes*. This transformation causes the plant to produce "hairy-roots" — dense, highly branched root structures. Of particular interest is that hairy-roots can produce certain metabolites, which have beneficial pharmaceutical properties. In an attempt to intensify the production of these metabolites, experiments concerning the growth of hairy-roots in bioreactors are now underway. In order to optimize this process, it is necessary to obtain a better understanding the metabolism and growth of these root structures. Here, as a first step, we develop and analyse a mathematical model for the nutrient uptake by a single branch of a hairy-root. The surface of a hairy-root is covered with fine "hairs" (micro-scale roots), which enlarge the active surface area of roots and thus increase the uptake of nutrients. However, due to their high density, the hairs are a significant obstacle to the flow of water. The model we propose is defined in a partially perforated domain. We consider water flow around the root structure and diffusion of nutrient molecules dissolved in water. Substrates diffuse and are transported by the flow in the fluid part and are absorbed on the surface of the hairs, i.e. on the boundary of the microstructure. Flow velocity of the water can be defined by the Stokes system. The scale of hairs is too small for accurate numerical computation of the full problem and the derivation of a macroscopic model is required.

The derivation of macroscopic equations for the fluid flow in partially perforated domains was considered in [1–3]. As a zero order approximation, a solution of Stokes or Navier–Stokes system in a free fluid domain with no-slip boundary conditions on the interface between two domains was obtained. Higher order approximations and effective boundary conditions at the interface between homogeneous and perforated domains were derived using boundary layers. In this work, these ideas are applied to a more general geometry. To derive macroscopic equations for the velocity field we have to assume C^2 regularity of the interface between free fluid and perforated domain, which implies the regularity of a Stokes solution needed for the analysis. As a macroscopic model, we obtain Stokes equations in the domain without hairs with no-slip condition on the interface between two domains. A better approximation for the water velocity requires the construction of boundary layers, see [3]. For the more complicated geometry considered here, boundary layer correction can be constructed

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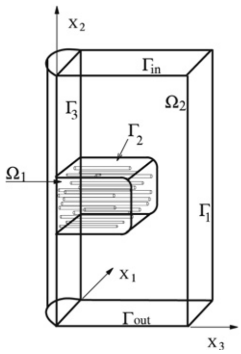
only locally and hence will not be considered further here. A macroscopic model for the nutrient concentration consists of a diffusion equation with a reaction describing the uptake process on hair surfaces in the perforated domain and a diffusion-convection equation in the homogeneous domain. Both the partial heterogeneity of the domain and the convective term make the analysis of the equations for the concentration proposed here, non-standard. In the estimates for the convective term, the regularity of the velocity field and the error estimate for the difference of microscopic and macroscopic velocities are used. To derive a macroscopic equation for the nutrient concentration we use the technique of two-scale convergence, which was introduced in [4,5] and extended to sequences of functions defined on surfaces in [6,7]. This extension and a compactness argument are used here to obtain the convergence of the nonlinear function defined on the surface of the microstructure. There are many results on homogenization of parabolic equations defined in completely perforated domains. The two-scale convergence was used in [7] to homogenize diffusion-reaction processes in a catalyst consisting of periodic distributed bars. A similar model with convection defined in a porous medium was studied in [8] using an energy method. A macroscopic model describing diffusion, convection and nonlinear reaction in a periodic array of cells was derived in [9]. Two-scale convergence coupled with monotonicity methods and compensated compactness were used there to show the convergence in the nonlinear terms. Homogenization of reaction-diffusion and reaction-diffusion-convection equations coupled with linear or nonlinear ordinary differential equations on the surface of the microstructure was studied in [10,11]. Macroscopic equations for reaction-diffusion between periodic distributed soil grain with nonlinear monotone kinetics on the grain surface and for reaction-diffusion processes both inside and outside grains were derived in [12,13]. The effective behavior of solutions of Laplace equation in a partially perforated domain and the contact problem between a porous medium and a non-perforated domain were studied in [14,15]. Derivation of macroscopic equations in a domain with a microstructure consisting of thick junctions is based on the construction of a proper extension operator, [16].

The paper is organized as follows: First, we present a description of the considered geometry, define a microscopic model, and formulate existence and uniqueness results for solutions of the microscopic model. In Section 3 we show a priori estimates for the water velocity and derive macroscopic equations for the velocity field. In Section 4 we prove a priori estimates for the nutrient concentration and, after extension of the solutions from the perforated domain to the whole domain, using there estimates, we show the convergence of solutions of the microscopic problem to a solution of a macroscopic model.

2. Problem formulation

We consider a single root with hairs orthogonal to the root surface and distributed periodically. For the sake of simplicity we replace the cylindrical geometry of a root surface by a rectangle and pose periodic boundary conditions on the sides. We define a domain $\Omega = (0,1) \times (0,M)^2$ with inflow boundary $\Gamma_{\text{in}} = (0,1) \times \{M\} \times (0,M)$, outflow boundary $\Gamma_{\text{out}} = (0,1) \times \{0\} \times (0,M)$, and $\Gamma_1 = (0,1) \times (0,M) \times \{M\}$, $\Gamma_3 = (0,1) \times (0,M) \times \{0\}$. For $0 < m_1 < m_2 < M$ and a smooth (C^2), positive, 1-periodic in x_1 function $G : (0,1) \times (m_1, m_2) \to \mathbb{R}$ with $\sup_{x_1,x_2} G < M$, $G = 0$ for $x_2 = m_1$, $x_2 = m_2$, we define $\Omega_1 = \{(x_1,x_2,x_3) \in (0,1) \times (m_1,m_2) \times (0, G(x_1,x_2))\}$, $\Omega_2 = \Omega \setminus \Omega_1$, $\Gamma_2 = \partial\Omega_1 \setminus \Gamma_3$. We can extend G to $\mathbb{R}^2$ by zero in x_2 and periodically in x_1. We define also



- Unit cell $Z = [0,1]^2$, repeated periodically over $\mathbb{R}^2$, $Y_0 \subset Z$, an open compactly included in Z subset with a smooth boundary $R = \partial Y_0$, $Y = Z \setminus \bar{Y}_0$.
- $Z^k = Z + \sum_{i=1}^2 k_i e_i$, $Y_0^k = Y_0 + \sum_{i=1}^2 k_i e_i$, $Y^k = Y + \sum_{i=1}^2 k_i e_i$, $R^k = R + \sum_{i=1}^2 k_i e_i$ for $k \in \mathbb{Z}^2$; $Z^* = \cup\{Z^k,\ k \in \mathbb{Z}^2\}$; $\Gamma^* = \cup\{R^k \times (0, L^k),\ k \in \mathbb{Z}^2\}$, L^k are the lengths of the hairs, $L^k = \inf_{(x_1,x_2)\in\varepsilon Z^k} G(x_1,x_2) - \varepsilon$, and $\varepsilon > 0$ is the ratio between the radius of a root hair and the size of Ω_1.
- $Q = \cup\{\varepsilon Z^k | \varepsilon Z^k \subset \Omega_1 \cap \{x_3 = 0\}\}$; $Q^\varepsilon = \cup\{\varepsilon Y^k | \varepsilon Z^k \subset Q\}$;
  $R^\varepsilon = \cup\{\varepsilon R^k | \varepsilon Z^k \subset Q\}$; $\mathcal{R}^\varepsilon = \cup\{\varepsilon R^k \times \{x_3 = L^k\} | \varepsilon Z^k \subset Q\}$;
  $\Gamma^\varepsilon = \cup\{\varepsilon R^k \times (0, L^k) | \varepsilon Z^k \times (0, L^k) \subset \Omega_1\}$; $\Upsilon^\varepsilon = \cup\{\varepsilon \bar{Y}_0^k \times \{L^k\} | \varepsilon Z^k \subset Q\}$.
- $\Omega_0^\varepsilon = \cup\{\varepsilon Y_0^k \times (0, L^k) | \varepsilon Z^k \times (0, L^k) \subset \Omega_1\}$, $\Omega_1^\varepsilon = \Omega_1 \setminus \bar{\Omega}_0^\varepsilon$ and $\Omega^\varepsilon = \Omega_1^\varepsilon \cup \Omega_2$.

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We consider water flow and diffusion and active transport of nutrients along the single root. The water velocity is given by the Stokes equations

$$\begin{array}{ll} -\Delta u^\varepsilon + \nabla p^\varepsilon = 0 & \text{in } \Omega^\varepsilon, \\ \operatorname{div} u^\varepsilon = 0 & \text{in } \Omega^\varepsilon, \\ p^\varepsilon = p_i,\ u^\varepsilon \times \nu = 0 & \text{on } \Gamma_{\text{in}}, \\ p^\varepsilon = p_o,\ u^\varepsilon \times \nu = 0 & \text{on } \Gamma_{\text{out}}, \\ u^\varepsilon = 0 & \text{on } \Gamma^\varepsilon \cup \Upsilon^\varepsilon \text{ and } \Gamma_1 \cup \Gamma_3, \\ u^\varepsilon,\ p^\varepsilon \text{are 1-periodic} & \text{in } x_1, \end{array} \tag{1}$$

where p_i and p_o are given constants.

Remark. For the flat boundary $\operatorname{div} u^\varepsilon = 0, p^\varepsilon = p_i,\ u^\varepsilon \times \nu = 0$ on Γ_{in} is equivalent to $(\nabla u^\varepsilon - p^\varepsilon I)\nu \cdot \nu = p_i, u^\varepsilon \times \nu = 0$ on Γ_{in}. The same holds for Γ_{out}.

For the nutrient concentration we have

$$\begin{array}{ll} \partial_t c^\varepsilon - \nabla \cdot (D^\varepsilon \nabla c^\varepsilon) + u^\varepsilon \cdot \nabla c^\varepsilon = 0 & \text{in } (0,T) \times \Omega^\varepsilon, \\ c^\varepsilon = c_D & \text{on } (0,T) \times \Gamma_{\text{in}}, \\ (D^\varepsilon \nabla c^\varepsilon - u^\varepsilon c^\varepsilon) \cdot \nu = 0 & \text{on } (0,T) \times \Gamma_{\text{out}}, \\ \nabla c^\varepsilon \cdot \nu = 0 & \text{on } (0,T) \times \Gamma_1 \cup \Gamma_3 \cup \Upsilon^\varepsilon, \\ -D^\varepsilon \nabla c^\varepsilon \cdot \nu^\varepsilon = \varepsilon f^\varepsilon(t,x,c^\varepsilon) & \text{on } (0,T) \times \Gamma^\varepsilon, \\ c^\varepsilon \quad \text{is 1-periodic} & \text{in } x_1, \\ c^\varepsilon(0) = c_0 & \text{in } \Omega^\varepsilon. \end{array} \tag{2}$$

The diffusion coefficient and the reaction term are defined by Z-periodic functions $D^\varepsilon_{ij}(t,x) = D_{ij}(t,x,\frac{\bar{x}}{\varepsilon})$ in $\Omega_1 \times Z^*$, $D^\varepsilon_{ij}(t,x) = D_{ij}(t,x)$ in Ω_2, and D^ε_{ij} are 1-periodic in x_1, $f^\varepsilon(t,x,\xi) = f(t,\frac{\bar{x}}{\varepsilon},x_3,\xi)$ on Γ^*, where $\bar{x} = (x_1,x_2)$. In general applications diffusion coefficients are constant. However, we will consider here a general case, because we can conduct our analysis and also we can envisage a case when diffusion depends on nonhomogeneous chemical properties of the medium.

Assumption 2.1. (1) The diffusion coefficient D, $\partial_t D \in L^\infty((0,T) \times \Omega \times Z)^{3\times 3}$, is uniformly elliptic, i.e. $D(t,x,y)\xi \cdot \xi \geq d_0|\xi|^2, d_0 > 0, \xi \in \mathbb{R}^3, y = (y_1,y_2)$.

(2) The reaction term $f(t,y,x_3,\xi) : (0,T) \times \Gamma^* \times \mathbb{R} \to \mathbb{R}$ is sublinear, i.e $|f(t,y,x_3,\xi)| \leq \mu_f(1+|\xi|)$, Lipschitz continuous in ξ, differentiable in t, and measurable in $(y,x_3) \in \Gamma^*$.

(3) The boundary condition $c_D \in H^1(0,T;H^2(\Omega)) \cap H^2(0,T;L^2(\Omega))$, the initial condition $c_0 \in H^2(\Omega)$, c_D and c_0 are 1-periodic in x_1, and $c_0|_{\partial\Omega} = c_D(0,x)$.

We define the spaces

$$\begin{aligned} V(\Omega^\varepsilon) &= \{v \in H^1(\Omega^\varepsilon)^3,\ v = 0 \text{ on } \Gamma_1 \cup \Gamma_3,\ v \times \nu = 0 \text{ on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}}, \\ &\quad v = 0 \text{ on } \Gamma^\varepsilon \cup \Upsilon^\varepsilon,\ v \text{ is 1-periodic in } x_1\}; \\ V_d(\Omega^\varepsilon) &= \{v \in V(\Omega^\varepsilon),\ \ \operatorname{div} v = 0\}; \\ W &= \{w \in H^1(\Omega^\varepsilon),\ w = 0 \text{ on } \Gamma_{\text{in}},\ w \text{ is 1-periodic in } x_1\}. \end{aligned}$$

Definition 2.2. A weak solution of (1), (2) is a triple $(u^\varepsilon, p^\varepsilon, c^\varepsilon)$ such that $u^\varepsilon \in V_d(\Omega^\varepsilon)$, $p^\varepsilon \in L^2(\Omega^\varepsilon)$, $c^\varepsilon - c_D \in L^2(0,T;W)$, $c^\varepsilon \in H^1(0,T;L^2(\Omega^\varepsilon))$ and

$$\int_{\Omega^\varepsilon} \nabla u^\varepsilon\, \nabla\phi\, \mathrm{d}x - \int_{\Omega^\varepsilon} p^\varepsilon \operatorname{div}\phi\, \mathrm{d}x = -\int_{\Gamma_{\text{in}}} p_i\, \phi \cdot \nu\, \mathrm{d}\sigma - \int_{\Gamma_{\text{out}}} p_o\, \phi \cdot \nu\, \mathrm{d}\sigma, \tag{3}$$

$$\int_0^T \int_{\Omega^\varepsilon} (\partial_t c^\varepsilon\, \psi + D^\varepsilon \nabla c^\varepsilon \nabla\psi - u^\varepsilon\, c^\varepsilon \nabla\psi)\, \mathrm{d}x\mathrm{d}t = -\varepsilon \int_0^T \int_{\Gamma^\varepsilon} f^\varepsilon(t,x,c^\varepsilon)\psi\, \mathrm{d}\sigma\, \mathrm{d}t, \tag{4}$$

for all functions $\phi \in V(\Omega^\varepsilon)$ and $\psi \in L^2(0,T;W)$.

Theorem 2.3. *Let Assumption 2.1 be satisfied. Then, for any ε there exists a unique weak solution of the problem (1)–(2) such that $u^\varepsilon \in V_d(\Omega^\varepsilon) \cap H^2(\Omega^\varepsilon_\delta)$, $p^\varepsilon \in L^2(\Omega^\varepsilon) \cap H^1(\Omega^\varepsilon_\delta)$, $c^\varepsilon - c_D \in L^2(0,T;W)$, $c^\varepsilon \in H^1(0,T;L^2(\Omega^\varepsilon))$, where $\Omega^\varepsilon_\delta$ is the domain without a neighbourhood of $\mathcal{R}^\varepsilon$.*

Proof idea. The existence of a solution of the Stokes or Navier–Stokes system with boundary conditions for the pressure is shown in [17,18,1,2]. The existence and regularity theory for the Stokes and Navier–Stokes equations with Dirichlet boundary conditions can be found in [19]. The proof relies on using Lax Milgram and DeRham theorems and thus a solution $u^\varepsilon \in V_d(\Omega^\varepsilon)$ and $p^\varepsilon \in L^2(\Omega^\varepsilon)$ is obtained. The solution is uniquely defined due to the boundary conditions for u^ε and p^ε.

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The regularity of the solution follows from the regularity for elliptic equations and boundary condition $u^\varepsilon = 0$ on $\Gamma_1 \cup \Gamma_3$ (such a boundary condition allows the extension of the solution across the boundaries) [19,1].

The existence of a solution, c^ε, of the parabolic equation for a give u^ε can be shown using the Galerkin method based on a priori estimates, similar to those in Lemma 4.1, [20]. The a priori estimates imply also the regularity $c^\varepsilon \in H^1(0,T;L^2(\Omega^\varepsilon))$. The only deviation from the standard situation is the presence of the convective term, which can be estimated in the following way

$$\begin{aligned}\int_0^T\int_{\Omega^\varepsilon} u^\varepsilon c^\varepsilon \nabla(c^\varepsilon - c_D)\,dxdt &\le \int_0^T\int_{\Gamma_{\text{out}}} u^\varepsilon (c^\varepsilon - c_D)^2\cdot\nu\,d\sigma\,dt + \int_0^T\int_{\Omega^\varepsilon} u^\varepsilon c_D \nabla(c^\varepsilon - c_D)\,dxdt\\ &\le \mu_1 \sup_{\Gamma_{\text{out}}}|u^\varepsilon|\,\|c^\varepsilon\|_{L^2((0,T)\times\Omega^\varepsilon)}\|c^\varepsilon\|_{L^2(0,T;H^1(\Omega^\varepsilon))}\\ &\quad + \mu_2 \sup_{(0,T)\times\Omega^\varepsilon}|c_D|\,\|u^\varepsilon\|_{L^2(\Omega^\varepsilon)}\|c^\varepsilon - c_D\|_{L^2(0,T;H^1(\Omega^\varepsilon))}\\ &\le \mu_3\delta\|c^\varepsilon\|^2_{L^2(0,T;H^1(\Omega^\varepsilon))} + \mu_4/\delta\left(1 + \|c^\varepsilon\|^2_{L^2((0,T)\times\Omega^\varepsilon)} + \|u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)}\right).\end{aligned}$$

Here we use the regularity of $u^\varepsilon \in V_d \cap H^2(\Omega^\varepsilon_\delta)$, $c_D \in H^1(0,T;H^2(\Omega))$ and the embedding $H^2(\Omega^\varepsilon_\delta) \subset C^\alpha(\bar\Omega^\varepsilon_\delta)$, $\alpha \in [0,1/2)$, for $\dim(\Omega^\varepsilon_\delta) \le 3$, [20]. The same calculations hold for $\int_0^T\int_{\Omega^\varepsilon} u^\varepsilon \partial_t c^\varepsilon \nabla \partial_t (c^\varepsilon - c_D)\,dxdt$ in estimates for the time derivative $\partial_t c^\varepsilon$. The uniqueness of the solution c^ε follows from the Lipschitz continuity of f and can be shown by considering the equation for the difference of two solutions c_1^ε and c_2^ε. ■

3. Macroscopic equations for the fluid flow

We assume the following macroscopic model for the water flow

$$\begin{array}{lll} -\Delta u^0 + \nabla\pi^0 = 0, & \operatorname{div} u^0 = 0 & \text{in } \Omega_2,\\ u^0 = 0 & & \text{on } \Gamma_1\cup\Gamma_2\cup\Gamma_{3,2},\\ u^0\times\nu = 0, & \pi^0 = p_i & \text{on } \Gamma_{\text{in}},\\ u^0\times\nu = 0, & \pi^0 = p_o & \text{on } \Gamma_{\text{out}},\\ u^0,\ \pi^0 - \text{1-periodic} & & \text{in } x_1.\end{array}$$

Here $\Gamma_{3,2} = \Gamma_3 \cap \overline{\Omega_2}$ and Γ_2 is the boundary between Ω_1 and Ω_2. There exists a unique solution $u^0 \in V_d(\Omega_2)\cap H^2(\Omega_2)$, $\pi^0 \in H^1(\Omega_2)$, [17–19,1,2].

We extend u^0 by zero into Ω_1. To show that u^0 is a macroscopic approximation of the microscopic velocity u^ε we need to estimate the difference $u^0 - u^\varepsilon$. For this we will make use of the following estimates in the porous medium.

Lemma 3.1 *([14,2,10]). Let $\phi \in H^1(\Omega_1^\varepsilon)$ be such that $\phi = 0$ on $\Gamma^\varepsilon \cup \Upsilon^\varepsilon$. Then, the following estimates hold*

$$\|\phi\|_{L^2(\Omega_1^\varepsilon)} \le C\varepsilon\|\nabla\phi\|_{L^2(\Omega_1^\varepsilon)}, \qquad \|\phi\|_{L^2(\Gamma_2)} \le C\varepsilon^{1/2}\|\nabla\phi\|_{L^2(\Omega_1^\varepsilon)}.$$

Proof idea. The first estimate follows from Poincaré's inequality. The second inequality hold true due to the estimate

$$\operatorname{dist}(\Gamma_2, \Gamma^\varepsilon\cup\Upsilon^\varepsilon) \le \max_{\varepsilon Z^k\subset Q}|\sup_{(x_1,x_2)\in\varepsilon Z^k} G - L^k| + C\varepsilon \le C(\sup|\nabla G| + 1)\varepsilon \le C\varepsilon.$$

Lemma 3.2. *For the solutions of the Stokes problems u^ε and u^0 we have*

$$\|\nabla(u^\varepsilon - u^0)\|_{L^2(\Omega^\varepsilon)^{3\times3}} \le C\sqrt{\varepsilon}, \qquad \|u^\varepsilon\|_{L^2(\Omega_1^\varepsilon)^3} \le C\varepsilon\sqrt{\varepsilon}, \qquad \|u^\varepsilon\|_{L^2(\Gamma_2)^3} \le C\varepsilon,$$
$$\|u^\varepsilon - u^0\|_{L^2(\Omega_2)^3} \le C\varepsilon, \qquad \|p^\varepsilon - \pi^0\|_{L^2(\Omega_2)} \le C\sqrt{\varepsilon},$$

where C is a constant independent of ε.

Proof. For the proof we use the ideas from [2]. We consider the equation for the difference $u^\varepsilon - u^0$ and use the estimates in Lemma 3.1

$$\begin{aligned}\int_{\Omega^\varepsilon}\nabla(u^\varepsilon - u^0)\nabla\phi\,dx - \int_{\Omega^\varepsilon}(p^\varepsilon - \pi^0\chi(\Omega_2))\nabla\cdot\phi\,dx = \int_{\Gamma_2}(\nabla u^0 - \pi^0 I)\,\nu\,\phi\,d\gamma\\ \le \frac{1}{2}\|\nabla u^0 - \pi^0 I\|_{L^2(\Gamma_2)}\|\phi\|_{L^2(\Gamma_2)} \le C\varepsilon^{1/2}\|\nabla u^0 - \pi^0 I\|_{L^2(\Gamma_2)}\|\nabla\phi\|_{L^2(\Omega_1^\varepsilon)}\end{aligned}$$

for $\phi \in V(\Omega^\varepsilon)$. The estimate $\|\nabla u^0 - \pi^0 I\|_{L^2(\Gamma_2)} \le C$ follows from the regularity of u^0 and π^0 in the domain Ω_2. Then using $u^\varepsilon - u^0$ as test function, $\operatorname{div} u^\varepsilon = 0$ and $\operatorname{div} u^0 = 0$, the Poincaré's inequality and the trace inequality in Ω_1^ε from Lemma 3.1

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yield the first three estimates of the lemma. To obtain the last two estimates we consider the equations for $w^\varepsilon = u^\varepsilon - u^0$ and $\pi^\varepsilon = p^\varepsilon - \pi^0$

$$\begin{array}{lll} -\Delta w^\varepsilon + \nabla \pi^\varepsilon = 0, & \operatorname{div} w^\varepsilon = 0 & \text{in } \Omega_2, \\ w^\varepsilon = u^\varepsilon & & \text{on } \Sigma = \Gamma_2, \\ w^\varepsilon = 0 & & \text{on } \Gamma_1 \cup \Gamma_{3,2}, \\ w^\varepsilon \times \nu = 0, & \pi^\varepsilon = 0 & \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}}, \\ w^\varepsilon, \pi^\varepsilon \text{ are 1-periodic} & & \text{in } x_1. \end{array} \tag{5}$$

Now, we use the estimate for a very weak solution w^ε of the Stokes system, [21,22]. We seek a solution $(w^\varepsilon, \pi^\varepsilon) \in L^2(\Omega_2) \times H^{-1}(\Omega_2)$ using the transposition method (for the definition of very weak solution see Appendix). Thus,

$$\|w^\varepsilon\|_{L^2(\Omega_2)} \le C\|u^\varepsilon\|_{L^2(\Gamma_2)} \le C\varepsilon.$$

The first equation in (5), the estimate for the velocity, and the Nečas's inequality, i.e. $\|\pi^\varepsilon\|_{L^2(\Omega_2)} \le C(\Omega_2)\|\nabla \pi^\varepsilon\|_{H^{-1}(\Omega_2)}$, [2,23], imply

$$\|\pi^\varepsilon\|_{L^2(\Omega_2)} \le C\|\nabla w^\varepsilon\|_{L^2(\Omega_2)} \le C\varepsilon^{1/2}. \quad \blacksquare$$

4. Derivation of macroscopic model for concentration

Macroscopic equations are derived using the method of two-scale convergence, [4,6,7,5].

4.1. A priori estimates for solutions of microscopic model

Lemma 4.1. *For $\varepsilon \le d_0^2/4$ solutions of the microscopic model (2) satisfy*

$$\|c^\varepsilon\|_{L^\infty(0,T;L^2(\Omega^\varepsilon))} + \|\nabla c^\varepsilon\|_{L^2(0,T;L^2(\Omega^\varepsilon))} \le \mu,$$
$$\|\partial_t c^\varepsilon\|_{L^\infty(0,T;L^2(\Omega^\varepsilon))} + \|\partial_t \nabla c^\varepsilon\|_{L^2(0,T;L^2(\Omega^\varepsilon))} \le \mu,$$

where μ is a constant independent of ε.

Proof. We choose $c^\varepsilon - c_D$ as a test function in (4) and calculate for $\tau \in [0, T]$

$$\int_0^\tau \int_{\Omega^\varepsilon} \partial_t c^\varepsilon (c^\varepsilon - c_D)\, dxdt + \int_0^\tau \int_{\Omega^\varepsilon} D^\varepsilon \nabla c^\varepsilon \nabla (c^\varepsilon - c_D)\, dxdt - \int_0^\tau \int_{\Omega^\varepsilon} u^\varepsilon c^\varepsilon \nabla(c^\varepsilon - c_D)\, dxdt$$
$$= -\varepsilon \int_0^\tau \int_{\Gamma^\varepsilon} f^\varepsilon(t, x, c^\varepsilon)(c^\varepsilon - c_D)\, d\sigma_x dt.$$

Applying the Young inequality we estimate the above integrals as

$$\int_0^\tau \int_{\Omega^\varepsilon} D^\varepsilon \nabla c^\varepsilon \nabla c_D\, dxdt \le \delta \int_0^\tau \int_{\Omega^\varepsilon} |\nabla c^\varepsilon|^2 dxdt + \frac{\mu}{\delta} \int_0^\tau \int_{\Omega^\varepsilon} |\nabla c_D|^2 dxdt,$$
$$\begin{aligned} \int_0^\tau \int_{\Omega^\varepsilon} \partial_t c^\varepsilon c_D dxdt &= \int_{\Omega^\varepsilon} (c^\varepsilon(\tau) c_D(\tau) - c_0 c_D(0))\, dx - \int_0^\tau \int_{\Omega^\varepsilon} c^\varepsilon \partial_t c_D dxdt \\ &\le \frac{1}{2} \int_{\Omega^\varepsilon} \left(\frac{1}{2}|c^\varepsilon(\tau)|^2 + 8|c_D(\tau)|^2 + |c_0|^2 + |c_D(0)|^2 \right) dx + \frac{1}{2} \int_0^\tau \int_{\Omega^\varepsilon} \left(|c^\varepsilon|^2 + |\partial_t c_D|^2 \right) dxdt. \end{aligned}$$

Using the estimate for $\nabla(u^\varepsilon - u^0)$, the regularity of $u^0 \in H^2(\Omega_2)$ and the embedding $H^2(\Omega_2) \subset C^\alpha(\bar{\Omega}_2)$, $\alpha \in [0, 1/2)$, for $\dim(\Omega_2) \le 3$, [20], we obtain

$$\begin{aligned} \int_0^\tau \int_{\Omega^\varepsilon} u^\varepsilon c^\varepsilon \nabla(c^\varepsilon - c_D)\, dxdt &= \int_0^\tau \int_{\Omega^\varepsilon} (u^\varepsilon - u^0 + u^0) c^\varepsilon \nabla (c^\varepsilon - c_D)\, dxdt \\ &\le \mu \varepsilon^{1/2} \|c^\varepsilon\|^2_{L^2(0,\tau;H^1(\Omega^\varepsilon))} + \sup_{\Omega_2} |u^0| \int_0^\tau \int_{\Omega^\varepsilon} \left(\frac{1}{\delta}|c^\varepsilon|^2 + \delta |\nabla(c^\varepsilon - c_D)|^2 \right) dxdt. \end{aligned}$$

In estimates for boundary integrals we apply

$$\|c^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} \le \mu \|c^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + \varepsilon^2 \mu \|\nabla c^\varepsilon\|^2_{L^2(\Omega^\varepsilon)}. \tag{6}$$

The inequality (6) follows from the trace theorem in $H^1(Y)$ and the scaling argument

$$\varepsilon \int_{\Gamma^\varepsilon} |c^\varepsilon|^2 d\gamma \le \mu_1 \sum_{\varepsilon Z^k \subset Q} \int_0^{L^k} \int_{\varepsilon Y^k} \left(|c^\varepsilon|^2 + \varepsilon^2 |\nabla_{\bar{x}} c^\varepsilon|^2 \right) dx \le \mu_2 \int_{\Omega^\varepsilon} \left(|c^\varepsilon|^2 + \varepsilon^2 |\nabla c^\varepsilon|^2 \right) dx.$$

Then, the sublinearity of f^ε and (6) imply

$$\varepsilon \int_0^\tau \int_{\Gamma^\varepsilon} f^\varepsilon(t,x,c^\varepsilon)(c^\varepsilon - c_D)\, d\sigma_x dt \le \mu_1 \int_0^\tau \int_{\Gamma^\varepsilon} \varepsilon \left(1 + |c^\varepsilon|^2 + |c^\varepsilon||c_D|\right) d\sigma_x dt$$
$$\le \mu_2 \int_0^\tau \int_{\Omega^\varepsilon} \left(1 + |c^\varepsilon|^2 + \varepsilon^2 |\nabla c^\varepsilon|^2 + |c_D|^2 + \varepsilon^2|\nabla c_D|^2\right) dx dt.$$

Using the ellipticity assumption on D^ε, regularity of the initial and boundary conditions and Gronwall's Lemma, we obtain the first estimate in the lemma. To derive the estimate for the time derivative, we differentiate the equation with respect to t and choose $\partial_t(c^\varepsilon - c_D)$ as a test function. Similar calculations as above and the inequality

$$\int_0^\tau \int_{\Omega^\varepsilon} \partial_t^2 c^\varepsilon \partial_t c_D \, dx dt \le \int_{\Omega^\varepsilon} \left(\frac{1}{4}|\partial_t c^\varepsilon(\tau)|^2 + 4|\partial_t c_D(\tau)|^2\right) dx$$
$$+ \frac{1}{2}\int_{\Omega^\varepsilon} \left(|\partial_t c^\varepsilon(0)|^2 + |\partial_t c_D(0)|^2\right) dx + \frac{1}{2}\int_0^\tau \int_{\Omega^\varepsilon} \left(|\partial_t c^\varepsilon|^2 + |\partial_t^2 c_D|^2\right) dx dt$$

imply the required estimate. Here we used, due to the assumptions $c_0 \in H^2(\Omega)$, $c_0|_{\partial\Omega} = c_D(0)$, and $c_D \in H^1(0,T;H^2(\Omega))$, that

$$\int_{\Omega^\varepsilon} \left(|\partial_t c^\varepsilon(0)|^2 + |\partial_t c_D(0)|^2\right) dx \le \mu \left(\|c_0\|_{H^2(\Omega)} + \|c_D\|_{H^1(0,T;H^2(\Omega))}\right). \quad \blacksquare$$

4.2. *Convergence of solutions of microscopic model*

To obtain a priori estimates for functions defined in the domain independent of ε we extend c^ε defined on Ω^ε to a function defined on the whole Ω. The main ideas of the extension are presented in [24,25,8].

Lemma 4.2. 1. *For $c \in H^1(F^i)$, where $F^i = Y \times (0, L^i] \cup Z \times (L^i, L^i + \delta)$ for some $0 < \delta < (M - \sup_{(x_1,x_2)} G)/4$, there exists an extension $\tilde{c}$ from F^i to $Z \times (0, L^i + \delta)$*

$$\|\tilde{c}\|_{L^2(Z\times(0,L^i+\delta))} \le \mu \|c\|_{L^2(F^i)} \quad \text{and} \quad \|\nabla \tilde{c}\|_{L^2(Z\times(0,L^i+\delta))} \le \mu \|\nabla c\|_{L^2(F^i)}.$$

2. *For $c^\varepsilon \in H^1(\tilde{\Omega}^\varepsilon)$ there exists an extension $\tilde{c}^\varepsilon$ into $\tilde{\Omega}$,*

$$\|\tilde{c}^\varepsilon\|_{L^2(\tilde{\Omega})} \le \mu \|c^\varepsilon\|_{L^2(\tilde{\Omega}^\varepsilon)}, \qquad \|\nabla \tilde{c}^\varepsilon\|_{L^2(\tilde{\Omega})} \le \mu \|\nabla c^\varepsilon\|_{L^2(\tilde{\Omega}^\varepsilon)}.$$

Here $\tilde{\Omega} = (0,1) \times (m_1, m_2) \times (0, G + \delta)$, $\tilde{\Omega}^\varepsilon = \tilde{\Omega} \setminus \bar{\Omega}_0^\varepsilon$, and μ is a constant dependent on G, Y, Y_0, but independent of ε, c^ε.

Proof. For a connected, with Lipschitz-continuous boundary domain F^i, we can use the extension result from [24,25,8]. The proof is based on the reflection technique. We extend c from F^i into neighbourhood U of $R \times (0, L^i) \cup Y_0 \times \{L^i\}$ by reflection and further into all $Y_0 \times (0, L^i)$ in any smooth manner, and define this extension by c^*. For $x_3 < L^i$ we have to extend only in $\bar{x}$ direction and the Neumann boundary conditions on Γ_3 does not cause any problems. Then we consider $\tilde{c} = (1 - \psi)(c^* - m) + m$, where ψ is a smooth function on $Z \times (0, L^i + \delta)$ with compact support in the interior of $Y_0 \times (0, L^i)$, such that $\psi \equiv 1$ in $Y_0 \setminus U$, and $m = \frac{1}{|F^i|}\int_{F^i} c^*(y)dy$. The estimate for the L^2-norm follows directly. Since $\|\nabla \tilde{c}\|_{L^2(Z\times(0,L^i+\delta))} = \|\nabla c\|_{L^2(F^i)} + \|\nabla \tilde{c}\|_{L^2(Y_0\times(0,L^i))}$ we have to estimate the last term. Using Poincaré's inequality we obtain

$$\int_{Y_0\times(0,L^i)} |\nabla \tilde{c}|^2 dy \le \mu_1 \|\nabla(1-\psi)\|_{L^\infty(Y_0\times(0,L^i))} \int_{Y_0\times(0,L^i)\cap U} |c^* - m|^2 dy$$
$$+ \mu_2 \|(1-\psi)\|_{L^\infty(Y_0\times(0,L^i))} \int_{Y_0\times(0,L^i)\cap U} |\nabla c^*|^2 dy \le \mu_3 \int_{F^i} |\nabla c|^2 dy.$$

To obtain the estimates in $\tilde{\Omega}$ we use the scaling argument

$$\|\nabla \tilde{c}^\varepsilon\|_{L^2(\tilde{\Omega})} = \sum_{\varepsilon Z^k \subset Q} \int_0^{G+\delta} \int_{\varepsilon Z^k} |\nabla \tilde{c}^\varepsilon|^2 dx$$
$$\le \sum_{\varepsilon Z^k \subset Q} \int_{L^k+\delta}^{G+\delta} \int_{\varepsilon Z^k} |\nabla c^\varepsilon|^2 dx + \sum_{\varepsilon Z^k \subset Q} \int_0^{L^k+\delta} \int_{\varepsilon Z^k} |\nabla \tilde{c}^\varepsilon|^2 dx$$
$$\le \sum_{\varepsilon Z^k \subset Q} \int_{L^k+\delta}^{G+\delta} \int_{\varepsilon Z^k} |\nabla c^\varepsilon|^2 dx + \sum_{\varepsilon Z^k \subset Q} \int_0^{L^k+\delta} \int_{Z^k} (\varepsilon^{-2}|\nabla_y \tilde{c}^\varepsilon|^2(\varepsilon y, x_3) + |\partial_{x_3}\tilde{c}^\varepsilon|^2) dy dx_3$$

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$$\le \sum_{\varepsilon Z^k \subset Q} \left(\mu_1 \int_0^{L^k+\delta} \int_{Y^k} (\varepsilon^{-2}|\nabla_y c^\varepsilon|^2 + |\partial_{x_3} c^\varepsilon|^2) \mathrm{d}y \mathrm{d}x_3 + \int_{L^k+\delta}^{G+\delta} \int_{\varepsilon Z^k} |\nabla c^\varepsilon|^2 \mathrm{d}x \right)$$
$$\le \sum_{\varepsilon Z^k \subset Q} \left(\mu_1(Y, Y_0, L^k) \int_0^{L^k+\delta} \int_{\varepsilon Y^k} |\nabla c^\varepsilon|^2 \mathrm{d}x + \int_{L^k+\delta}^{G+\delta} \int_{\varepsilon Z^k} |\nabla c^\varepsilon|^2 \mathrm{d}x \right)$$
$$\le \mu(\nabla G, Y, Y_0) \, \|\nabla c^\varepsilon\|_{L^2(\tilde{\Omega}^\varepsilon)}.$$

The estimate for $\|\tilde{c}^\varepsilon\|_{L^2(\tilde{\Omega})}$ follows from similar calculations. ■

The extension into $\tilde{\Omega}$ implies the extension into all Ω.

Remark. For $c^\varepsilon \in L^2(0, T; H^1(\Omega^\varepsilon))$ we define $\bar{c}^\varepsilon(\cdot, t) := \tilde{c}^\varepsilon(\cdot, t)$. Since the extension operator is linear, $\bar{c}^\varepsilon \in L^2(0, T; H^1(\Omega))$. We identify c^ε with $\bar{c}^\varepsilon$.

To show the compactness of c^ε in $L^2((0, T) \times \Gamma^\varepsilon)$ we use the following space

Definition 4.3 (*[26]*). Let $W^{\beta,2}(\Omega)$ with $\beta \in \mathbb{R}$, $\beta > 0$ be a Hilbert space defined as the completion of $C^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W^{\beta,2}(\Omega)} = \|u\|_{W^{k,2}(\Omega)} + \left\{ \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{n+2(\beta-k)}} \mathrm{d}x\mathrm{d}y \right\}^{\frac{1}{2}}, \quad k = [\beta],\ n = \dim(\Omega).$$

Lemma 4.4. *For a function $v^\varepsilon \in W^{\beta,2}(\Omega_1^\varepsilon)$, $\frac{1}{2} < \beta < 1$*

$$\varepsilon \int_{\Gamma^\varepsilon} |v^\varepsilon|^2 \, \mathrm{d}\sigma_x \le \mu_1 \left(\int_{\Omega_1^\varepsilon} |v^\varepsilon|^2 \mathrm{d}x + \varepsilon^{2\beta} \int_{\Omega_1^\varepsilon} \int_{Q^\varepsilon} \frac{|v^\varepsilon(\bar{x}^1, x_3) - v^\varepsilon(\bar{x}^2, x_3)|^2}{|\bar{x}^1 - \bar{x}^2|^{2+2\beta}} \mathrm{d}\bar{x}^1 \mathrm{d}\bar{x}^2 \mathrm{d}x_3 \right)$$
$$\le \mu_2 \|v^\varepsilon\|_{W^{\beta,2}(\Omega_1^\varepsilon)}.$$

Proof. For a function $v \in W^{\beta,2}(Y)$ the trace theorem implies

$$\int_R |v|^2 \, \mathrm{d}\sigma_y \le \mu_1 \int_Y |v|^2 \mathrm{d}y + \mu_2 \int_Y \int_Y \frac{|v(y^1) - v(y^2)|^2}{|y^1 - y^2|^{2+2\beta}} \mathrm{d}y^1 \mathrm{d}y^2.$$

Changing variables, $y = \bar{x}/\varepsilon$ with $\bar{x} = (x_1, x_2)$, we obtain

$$\int_{\varepsilon R^i} |v^\varepsilon|^2 \frac{\mathrm{d}\sigma_{\bar{x}}}{\varepsilon} \le \mu_1 \int_{\varepsilon Y^i} |v^\varepsilon|^2 \frac{\mathrm{d}\bar{x}}{\varepsilon^2} + \mu_2 \int_{\varepsilon Y^i} \int_{\varepsilon Y^i} \frac{|v^\varepsilon(\bar{x}^1, x_3) - v^\varepsilon(\bar{x}^2, x_3)|^2}{|\bar{x}^1 - \bar{x}^2|^{2+2\beta}} \varepsilon^{2+2\beta} \frac{\mathrm{d}\bar{x}^1}{\varepsilon^2} \frac{\mathrm{d}\bar{x}^2}{\varepsilon^2}.$$

Integrating the inequality over x_3 from 0 to L^i, multiplying by ε^2 and summing up over i, $\varepsilon Z^i \in Q$, we obtain the estimate in the lemma. ■

Lemma 4.5. *There exist functions c and c_1 such that*

$$c^\varepsilon \to c \quad \text{weakly in } L^2(0, T; H^1(\Omega)),\ \text{weakly-}*\ \text{in } L^\infty(0, T; L^2(\Omega))$$
$$\partial_t c^\varepsilon \to \partial_t c \quad \text{weakly in } L^2(0, T; H^1(\Omega)),\ \text{weakly-}*\ \text{in } L^\infty(0, T; L^2(\Omega)),$$
$$c^\varepsilon \to c \quad \text{strongly in } L^2(0, T; W^{\beta,2}(\Omega)), \qquad \frac{1}{2} < \beta < 1,$$
$$\lim_{\varepsilon \to 0} \varepsilon \int_0^T \int_{\Gamma^\varepsilon} |c^\varepsilon - c|^2 \mathrm{d}\sigma_x \mathrm{d}t = \lim_{\varepsilon \to 0} \|c^\varepsilon - c\|^2_{L^2((0,T)\times\Gamma^\varepsilon)} = 0,$$
$$c^\varepsilon \to c,\ \partial_t c^\varepsilon \to \partial_t c \quad \text{in two-scale sense in } \Omega_1,$$
$$\nabla c^\varepsilon \to \nabla_x c + \nabla_y c_1 \quad \text{in two-scale sense, } c_1 \in L^2((0, T) \times \Omega_1; H^1_{per}(Z)/\mathbb{R}).$$

Proof. From the a priori estimates in Lemma 4.1, which imly the correspondent estimates for the extension in Ω, we obtain a weak convergence $c^\varepsilon \rightharpoonup c$ and $\partial_t c^\varepsilon \rightharpoonup \partial_t c$ in $L^2(0, T; H^1(\Omega))$ and weak-$*$ convergence in $L^\infty(0, T; L^2(\Omega))$. The strong convergence of c^ε in $L^2(0, T; W^{\beta,2}(\Omega))$, $\frac{1}{2} < \beta < 1$ follows from the compact embedding of $W^{\beta,2}(\Omega)$ in $H^1(\Omega)$ and the Lions–Aubin Lemma, [27], with $B = W^{\beta,2}(\Omega)$. The Lemma 4.4 implies $\|c^\varepsilon - c\|_{L^2((0,T)\times\Gamma^\varepsilon)} \le \mu_1 \|c^\varepsilon - c\|^2_{L^2(0,T;W^{\beta,2}(\Omega_1^\varepsilon))} \le \mu_2 \|c^\varepsilon - c\|^2_{L^2(0,T;W^{\beta,2}(\Omega))} \to 0$ for $\varepsilon \to 0$.

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The boundedness of c^ε in $L^2(0,T;H^1(\Omega))$ and the Compactness Theorem (see Theorem A.2 in Appendix) imply the two-scale convergence of c^ε to c and existence of a function $c_1 \in L^2((0,T)\times\Omega_1; H^1_{per}(Z)/\mathbb{R})$ such that, up to a subsequence, ∇c^ε two-scale converges to $\nabla_x c(t,x) + \nabla_y c_1(t,x,y)$, $\nabla_y c_1 = (\partial_{y_1} c_1, \partial_{y_2} c_1, 0)$. Using the a priori estimates and applying again the Compactness Theorem (see Theorem A.2) we obtain the two-scale convergence of $\partial_t c^\varepsilon$ to $\partial_t c$. ■

4.3. Macroscopic equations

We define the space

$$\mathcal{K} = \{(\phi,\psi),\ \phi\in H^1(\Omega_1),\ \psi\in H^1(\Omega_2),\ \psi = 0 \text{ on } \Gamma_{\text{in}},\ \phi=\psi \text{ on } \Gamma_2, \phi,\psi \text{ are 1-periodic in } x_1\}.$$

Theorem 4.6. *The solutions of the microscopic problem c^ε converge as $\varepsilon\to 0$ to the solution $(c_1 - c_D, c_2 - c_D)\in L^2(0,T;\mathcal{K})$, $\partial_t c_1 \in L^2((0,T)\times\Omega_1)$, $\partial_t c_2 \in L^2((0,T)\times\Omega_2)$, of the following macroscopic problem*

$$\begin{aligned}
&|Y|\partial_t c_1 - \nabla\cdot(D^{\text{hom}}\nabla c_1) + \int_R f(t,y,x_3,c_1)\,d\sigma_y = 0 && \text{in } (0,T)\times\Omega_1,\\
&\partial_t c_2 - \nabla\cdot(D\nabla c_2) + u^0\cdot\nabla c_2 = 0 && \text{in } (0,T)\times\Omega_2,\\
&D^{\text{hom}}\nabla c_1\cdot\nu = D\nabla c_2\cdot\nu, \qquad c_1 = c_2 && \text{on } (0,T)\times\Gamma_2,\\
&c_2 = c_D && \text{on } (0,T)\times\Gamma_{\text{in}},\\
&(D\nabla c_2 - u^0 c_2)\cdot\nu = 0 && \text{on } (0,T)\times\Gamma_{\text{out}},\\
&\nabla c_1\cdot\nu = 0 \quad \text{on } (0,T)\times\Gamma_{3,1}, \qquad \nabla c_2\cdot\nu = 0 && \text{on } (0,T)\times\Gamma_1\cup\Gamma_{3,2},\\
&c_1, c_2 \text{ are } 1-\text{periodic} && \text{in } x_1,\\
&c_1(0) = c_0 \quad \text{in } \Omega_1, \qquad c_2(0) = c_0 && \text{in } \Omega_2,
\end{aligned}$$

where $D^{\text{hom}}_{ij} = \sum_{k=1}^2 \int_Y (D_{ij}(t,x,y) + D_{ik}(t,x,y)\partial_{y_k} s_j)\,dy$ and s_j are the solutions of the cell problems

$$-\nabla_y\cdot(\bar D\,\nabla_y s_j) = \sum_{k=1}^2 \partial_{y_k} D_{kj} \quad \text{in } Y, \qquad -\bar D\nabla_y s_j\cdot\nu = \sum_{k=1}^2 D_{kj}\nu_k \quad \text{on } R,$$

and $\bar D = (D_{ij})$ for $i,j = 1,2$.

Proof. First we choose $\phi\in C(0,T;C_0^\infty(\Omega_2))$ as a test function in the Eq. (4). The estimate $\|u^\varepsilon - u^0\|_{L^2(\Omega_2)} \le \mu\,\varepsilon$ and the convergence of c^ε imply

$$\int_0^T\int_{\Omega_2} u^\varepsilon c^\varepsilon \nabla\phi\,dxdt \to \int_0^T\int_{\Omega_2} u^0 c\nabla\phi\,dxdt.$$

Due to the weak convergence of c^ε in Ω_2 we obtain

$$\int_0^T\int_{\Omega_2} c_t\,\phi\,dxdt + \int_0^T\int_{\Omega_2} D\nabla c\nabla\phi\,dxdt - \int_0^T\int_{\Omega_2} u^0 c\nabla\phi\,dxdt = 0.$$

To show the convergence in Ω_1^ε we use the extension of c^ε from Ω^ε to Ω and the two-scale limit with a test function $\phi = \phi_1 + \varepsilon\phi_2$, where $\phi_1\in C(0,T;C_0^\infty(\Omega_1))$, $\phi_2\in C(0,T;C_0^\infty(\Omega_1);C^\infty_{per}(Z))$ and obtain

$$\begin{aligned}
&\int_0^T\int_{\Omega_1} \chi_\varepsilon\, c_t^\varepsilon\,(\phi_1+\varepsilon\phi_2)dx\,dt \to \int_0^T\int_{\Omega_1} |Y|\,c_t\,\phi_1\,dx\,dt,\\
&\int_0^T\int_{\Omega_1} \chi_\varepsilon\, u^\varepsilon\, c^\varepsilon\nabla(\phi_1+\varepsilon\phi_2)dx\,dt \to 0,\\
&\int_0^T\int_{\Omega_1} \chi_\varepsilon\, D^\varepsilon\nabla c^\varepsilon\nabla(\phi_1+\varepsilon\phi_2)dxdt \to \int_0^T\int_{\Omega_1}\int_Y D\,(\nabla c + \nabla_y c_1)(\nabla\phi_1 + \nabla_y\phi_2)dxdtdy.
\end{aligned}$$

Here χ_ε is the characteristic function of Ω_1^ε. Using strong convergence of c^ε and two-scale convergence of $f^\varepsilon(t,x,c)$ on Γ_ε, and Lipschitz continuity of f, we obtain

$$\begin{aligned}
\varepsilon\int_0^T\int_{\Gamma_\varepsilon} f^\varepsilon(t,x,c^\varepsilon)(\phi_1+\varepsilon\phi_2)d\sigma_x dt &= \varepsilon\int_0^T\int_{\Gamma_\varepsilon} (f^\varepsilon(c^\varepsilon) - f^\varepsilon(c))(\phi_1+\varepsilon\phi_2)d\sigma_x dt\\
&+\varepsilon\int_0^T\int_{\Gamma_\varepsilon} f^\varepsilon(t,x,c)(\phi_1+\varepsilon\phi_2)d\sigma_x dt \le \mu_1\|c^\varepsilon - c\|_{L^2((0,T)\times\Gamma^\varepsilon)}\|\phi\|_{L^2((0,T)\times\Gamma^\varepsilon)}\\
&+\varepsilon\int_0^T\int_{\Gamma_\varepsilon} f^\varepsilon(t,x,c)(\phi_1+\varepsilon\phi_2)d\sigma_x dt \to \int_0^T\int_{\Omega_1}\int_R f(t,y,x_3,c)\,\phi_1\,d\sigma_y dxdt.
\end{aligned}$$

Then, the limit equation reads

$$\int_0^T\int_{\Omega_2} c_t\phi \mathrm{d}x\mathrm{d}t + \int_0^T\int_{\Omega_2} D\nabla c\nabla\phi \mathrm{d}x\mathrm{d}t - \int_0^T\int_{\Omega_2} u^0 c\nabla\phi \mathrm{d}x\mathrm{d}t + |Y|\int_0^T\int_{\Omega_1} c_t\phi_1 \mathrm{d}x\mathrm{d}t$$
$$+\int_0^T\int_{\Omega_1}\int_Y D(\nabla c+\nabla_y c_1)(\nabla\phi_1+\nabla_y\phi_2)\mathrm{d}x\mathrm{d}t\mathrm{d}y = -\int_0^T\int_{\Omega_1}\int_R f(c)\mathrm{d}\sigma_y\phi_1\mathrm{d}x\mathrm{d}t.$$

To determinate the unknown function $c_1 \in L^2((0,T)\times\Omega_1; H^1_{per}(Z)/\mathbb{R})$ we set in the last equation $\phi = 0, \phi_1 = 0$ and obtain for all ϕ_2

$$\int_0^T\int_{\Omega_1}\int_Y D(t,y)(\nabla c(t,x)+\nabla_y c_1(t,x,y))\nabla_y\phi_2(t,x,y)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y = 0.$$

From the structure of the equation follows that c_1 depends linearly on $\nabla_x c$ and can be written in the form $c_1 = \sum_{j=1}^3 s_j\partial_{x_j}c$, where s_j are solutions of

$$-\nabla_y(\bar{D}\nabla_y s_j) = \sum_{k=1}^2 \partial_{y_k}D_{kj} \quad \text{in } Y, \qquad -\bar{D}\nabla_y s_j\cdot\nu = \sum_{k=1}^2 D_{kj}\nu_k \quad \text{on } R,$$

s_j are periodic in Z. Then the macroscopic equation for c reads

$$\int_0^T\int_{\Omega_2} c_t\,\varphi\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\Omega_2}(D\nabla c - u^0\,c)\nabla\varphi\,\mathrm{d}x\mathrm{d}t + |Y|\int_0^T\int_{\Omega_1} c_t\,\varphi\,\mathrm{d}x\mathrm{d}t$$
$$+\int_0^T\int_{\Omega_1} D^{\mathrm{hom}}\nabla c\nabla\varphi\,\mathrm{d}x\mathrm{d}t = -\int_0^T\int_{\Omega_1}\int_R f(t,y,x_3,c)\,\mathrm{d}\sigma_y\,\varphi\,\mathrm{d}x\mathrm{d}t,$$

for $\varphi \in L^2(0,T;\mathcal{K})$ and $D^{\mathrm{hom}}_{ij} = \sum_{k=1}^2\int_Y(D_{ij}+D_{ik}\partial_{y_k}s_j)\,\mathrm{d}y$. We denote the concentration of nutrients in Ω_1 and Ω_2 by c_1 and c_2 respectively and obtain the continuity condition in the weak sense $c_1 = c_2$ and $D^{\mathrm{hom}}\nabla c_1\cdot\nu = D\nabla c_2\cdot\nu$ on Γ_2. ■

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Appendix

We recall the compactness results for two-scale convergence of functions dependent on parameters, [7], the proofs of which are straight-forward modifications of the proofs for the two-scale convergence presented in [4,6,7,5].

Definition A.1. 1. A sequence $\{v^\varepsilon\}\subset L^2(\Lambda\times\Omega)$ converges two-scale to $v\in L^2(\Lambda\times\Omega\times Z)$ iff for any $\phi\in\mathcal{D}(\Lambda\times\Omega, C^\infty_{per}(Z))$

$$\lim_{\varepsilon\to 0}\int_\Lambda\int_\Omega v^\varepsilon(\lambda,x)\phi\left(\lambda,x,\frac{x}{\varepsilon}\right)\,\mathrm{d}x = \int_\Lambda\int_\Omega\int_Z v(\lambda,x,y)\phi(\lambda,x,y)\,\mathrm{d}x\mathrm{d}y.$$

2. A sequence $\{v^\varepsilon\}\subset L^2(\Lambda\times\Gamma^\varepsilon)$ converges two-scale to $v\in L^2(\Lambda\times\Omega\times\Gamma)$ iff for $\psi\in\mathcal{D}(\Lambda\times\Omega, C^\infty_{per}(\Gamma))$

$$\lim_{\varepsilon\to 0}\varepsilon\int_\Lambda\int_{\Gamma^\varepsilon} v^\varepsilon(\lambda,x)\psi\left(\lambda,x,\frac{x}{\varepsilon}\right)\mathrm{d}\gamma_x = \int_\Lambda\int_\Omega\int_\Gamma v(\lambda,x,y)\psi(\lambda,x,y)\mathrm{d}x\mathrm{d}\gamma_y.$$

Theorem A.2 ([4,5]).

1. *Let $\{v_\varepsilon\}$ be a bounded sequence in $L^2(\Lambda, H^1(\Omega))$, which converges weakly to a limit function $v\in L^2(\Lambda, H^1(\Omega))$. Then there exists $v_1\in L^2(\Lambda\times\Omega, H^1_{per}(Z)/\mathbb{R})$ such that, up to a subsequence, v_ε two-scale converges to v and ∇v_ε two-scale converges to $\nabla v(\lambda,x)+\nabla_y v_1(\lambda,x,y)$.*
2. *Let $\{v_\varepsilon\}$ and $\varepsilon\nabla v_\varepsilon$ be bounded sequences in $L^2(\Lambda\times\Omega)$. Then there exists $v_0\in L^2(\Lambda\times\Omega, H^1_{per}(Z)/\mathbb{R})$ such that, up to a subsequence, v_ε and $\varepsilon\nabla v_\varepsilon$ two-scale converge to $v_0(\lambda,x,y)$ and $\nabla_y v_0(\lambda,x,y)$ respectively.*

Theorem A.3 ([6,7]). *From each bounded sequence $\{v^\varepsilon\}$ in $L^2(\Lambda\times\Gamma^\varepsilon)$ we can extract a subsequence which two-scale converges to $v\in L^2(\Lambda\times\Omega\times\Gamma)$.*

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For very weak solution we seek a solution $(w, \pi) \in L^2(\Omega_2) \times H^{-1}(\Omega_2)$ of

$$\begin{array}{lll} -\Delta w + \nabla \pi = f, & \operatorname{div} w = 0 & \text{in } \Omega_2, \\ w = \xi & & \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \\ w \times \nu = \zeta_1, & \pi = \pi_i & \text{on } \Gamma_{\text{in}}, \\ w \times \nu = \zeta_2, & \pi = \pi_o & \text{on } \Gamma_{\text{out}}, \\ w, \pi \text{ is 1-periodic} & & \text{in } x_1. \end{array}$$

Let $\{\phi, q\}$ be given by

$$\begin{array}{lll} -\Delta \phi + \nabla q = g, & \operatorname{div} \phi = h & \text{in } \Omega_2, \\ \phi = 0 & & \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \Gamma, \\ \phi \times \nu = 0, & q = 0 & \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}}, \\ \phi, q \text{ is 1-periodic} & & \text{in } x_1. \end{array}$$

For $g \in L^2(\Omega_2)^3$, $h \in H = \{h \in H_0^1(\Omega_2), \int_{\Omega_2} h\, dx = 0\}$ we have the solution $\phi \in H^2(\Omega_2)^3,\ q \in H^1(\Omega_2)$. Now we test the equations for w and π by ϕ and using $\int_{\Omega_2} w \nabla q\, dx = \int_\Gamma q I \nu w\, d\sigma$ obtain

$$\int_{\Omega_2} f\phi\, dx = \int_{\Omega_2} (-\Delta w + \nabla \pi)\phi\, dx = \int_{\Omega_2} (-w\Delta\phi + w\nabla q - \pi \operatorname{div} \phi)\, dx$$
$$+ \int_\Gamma (\nabla\phi - qI)\nu w\, d\sigma - \int_{\Gamma_{\text{in}}} (\zeta_1 \nabla\phi\nu + \pi_i \phi\nu)\, d\sigma - \int_{\Gamma_{\text{out}}} (\zeta_2 \nabla\phi\nu + \pi_o\phi\nu)\, d\sigma.$$

We consider the linear continuous form $l : L^2(\Omega_2)^3 \times H \to \mathbb{R}$

$$l(g, h) = \langle f, \phi\rangle - \int_\Gamma (\nabla\phi - qI)\nu\xi\, d\sigma + \int_{\Gamma_{\text{in}}} (\zeta_1 \nabla\phi\nu + \phi_i\phi\nu)\, d\sigma + \int_{\Gamma_{\text{out}}} (\zeta_2 \nabla\phi\nu + \phi_o\phi\nu)\, d\sigma.$$

Definition A.4 ([21]). We define a pair (w, π) as a very weak solution if $(w, \pi) \in L^2(\Omega_2)^3 \times H^*$ and

$$\int_{\Omega_2} wg\, dx - \langle \pi, h\rangle_{H^*,H} = l(g, h) \quad \text{for all } (g, h) \in L^2(\Omega_2)^3 \times H.$$

Due to the linearity and continuity of l, the Riesz theorem implies

Proposition A.5 ([21]). *There exists a unique very weak solution (w, π),*

$$\|w\|_{L^2(\Omega_2)^3} \le C\left(\|f\|_{L^2(\Omega_2)^3} + \|\xi\|_{L^2(\Gamma_2)^3} + \|\zeta_1\|_{L^2(\Gamma_{\text{in}})} + \|\zeta_2\|_{L^2(\Gamma_{\text{out}})} + \|\pi_i\|_{L^2(\Gamma_{\text{in}})} + \|\pi_o\|_{L^2(\Gamma_{\text{out}})}\right).$$

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