Mathematical modelling, finance, and the recession

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15:40 - 16:00, room 2.14
UK GDP (at 2005 prices) mid-2006 to end-2009
Some ingredients for a model

- Gross Domestic Product
  - GDP as expenditure
  - GDP as income
- Money, finance
  - The demand for money
  - The supply of money
- International trade
  - The exchange rate
  - The balance of payments
Gross Domestic Product evaluated as expenditure

The GDP $Y$ can be defined as the rate at which money is being spent for providing goods and services.

The UK Office of National Statistics breaks down this total into various sub-totals:

- $Y := C + I + G + E - M$
- $C :=$ spending by consumers (households and non-profit institutions) on goods and services
- $I :=$ private sector new investment (capital formation plus changes in inventories)
- $G :=$ government spending on goods and services (excludes interest on Gov’t bonds, pensions, unemployment benefit)
- $E :=$ exports
- $M :=$ imports
GDP as expenditure: UK data /£10^9 per annum

<table>
<thead>
<tr>
<th></th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consumption by households and NPIs</td>
<td>893</td>
<td>928</td>
<td>911</td>
</tr>
<tr>
<td>Investment: capital formation, inventory changes</td>
<td>255</td>
<td>244</td>
<td>194</td>
</tr>
<tr>
<td>Gov’t spending on goods and services</td>
<td>295</td>
<td>314</td>
<td>330</td>
</tr>
<tr>
<td>Exports</td>
<td>372</td>
<td>422</td>
<td>387</td>
</tr>
<tr>
<td>Imports</td>
<td>416</td>
<td>460</td>
<td>421</td>
</tr>
<tr>
<td>(Y = C + I + G + E - M)</td>
<td>1399</td>
<td>1448</td>
<td>1401</td>
</tr>
</tbody>
</table>
GDP evaluated as income

Alternatively, the GDP $Y$ can be defined as the rate at which money is being *received* in exchange for providing goods and services

- $Y = W + P + T$
- $W :=$ wages and salaries (less taxes)
- $P :=$ profits and rents
- $T :=$ taxes less subsidies
GDP as income: UK data / £10^9 per annum

<table>
<thead>
<tr>
<th>year</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wages and salaries of employees</td>
<td>746</td>
<td>772</td>
<td>764</td>
</tr>
<tr>
<td>Profits of corporations and businesses</td>
<td>484</td>
<td>510</td>
<td>480</td>
</tr>
<tr>
<td>Taxes less subsidies</td>
<td>168</td>
<td>166</td>
<td>152</td>
</tr>
</tbody>
</table>

\[ Y = W + P + T \]

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<td>1399</td>
<td>1448</td>
<td>1396</td>
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</table>
The accounting identity

The expenditure and income GDPs are equal:

\[ Y = C + I + G + E - M = W + P + T \]
GDP and unemployment

unemployment U vs. "real" GDP
A ‘phase transition’

The two phases:

- Full employment phase: GDP increases and unemployment stays constant at its upper bound. If GDP increases too rapidly, the result is an increase in wages and prices.

- Partial employment phase: unemployment is below the upper bound and varies with GDP. Wages and prices do not increase.

\[ Y = wN \]

\( w := \) average wage, \( N := \) number employed, \( N \leq N_{\text{max}} \).

Prices assumed proportional (in the short term) to \( w \).
Ways to influence the GDP

- \( Y = C + I + G + E - M \)
- increase \( C \) : persuade householders to spend more 
  (reduce taxes: ‘fiscal stimulus’)
- increase \( I \) : persuade firms to invest more 
  (reduce interest rate)
- increase \( G \) : spend more on goods and services
- increase \( E - M \) : export more and/or import less
“Keynes’ hypothesis”: that $C$ depends mainly on $Y - T$.

The straight lines are $C = 0.72(Y - T)$ and $C = 0.735(Y - T)$. 
Whereas $C - M$ does not depend mainly on $Y - T$. 

Domestically produced consumption vs disposable income

- Axis labels: $0.25(C - M)$, $0.25(Y - T)$
- Values range from 295 to 320 on the $0.25(Y - T)$ axis, and from 114 to 124 on the $0.25(C - M)$ axis.
Using Keynes’ hypothesis: the ‘fiscal simulus’

The data on $C$ vs $Y - T$ can be summarized by the formula

$$C \approx k(Y - T)$$

where $k$ is about $3/4$. i.e. people spend about $3/4$ of their income and save about $1/4$.

Using this in the expenditure equation,

$$Y \approx 0.75(Y - T) + I + G + E - M$$

i.e.

$$0.25Y \approx -0.75T + I + G + E - M$$

Whence, if $I$, $E$ and $M$ don’t change,

$$\partial Y/\partial T = -3 \quad \text{and} \quad \partial Y/\partial G = 4$$

So a tax cut should increase $Y$ by 3 times as much; an increase of $G$ should increase $Y$ by 4 times as much.
Efficacy of a tax cut in increasing \( Y \): observations

<table>
<thead>
<tr>
<th>quarter</th>
<th>( Y/4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2008 Q1</td>
<td>363.4</td>
</tr>
<tr>
<td>2008 Q2</td>
<td>364.0</td>
</tr>
<tr>
<td>2008 Q3</td>
<td>361.7</td>
</tr>
<tr>
<td>2008 Q4</td>
<td>359.3</td>
</tr>
<tr>
<td>2009 Q1</td>
<td>348.8</td>
</tr>
<tr>
<td>2009 Q2</td>
<td>346.0</td>
</tr>
<tr>
<td>2009 Q3</td>
<td>348.9</td>
</tr>
<tr>
<td>2009 Q4</td>
<td>352.8</td>
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During 2009, VAT was cut by £10, i.e. \( dT = -12 \) whence the "fiscal stimulus" theory would predict \( dY = 3 \times 12 = 36 \), \( d(Y/4) = 9 \). The data show that \( d(Y/4) \) did not even have the hoped-for sign. This is because, although \( T \) decreased, the other variables \( I, E, M \) did not stay constant as assumed in the simple theory.
Efficacy of a tax cut in increasing $Y$: observations.

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During 2009, VAT was cut by 12 £10^9, i.e. $dT = -12$ whence the "fiscal stimulus" theory would predict

$$dY = 3 \times 12 = 36, \quad d(Y/4) = 9$$

The data show that $d(Y/4)$ did not even have the hoped-for sign. This is because, although $T$ decreased, the other variables $I, E, M$ did not stay constant as assumed in the simple theory.
fiscal stimulus and budget deficit $D$

\[ D := G - T \]

\[
\begin{bmatrix}
    dY \\
    dD
\end{bmatrix}
= \begin{bmatrix}
    4 & -3 \\
    1 & -1
\end{bmatrix}
\begin{bmatrix}
    dG \\
    dT
\end{bmatrix}
\]

whence

\[
\begin{bmatrix}
    dG \\
    dT
\end{bmatrix}
= \begin{bmatrix}
    1 & -3 \\
    1 & -4
\end{bmatrix}
\begin{bmatrix}
    dY \\
    dD
\end{bmatrix}
\]

To increase GDP without changing deficit (i.e. $dY > 0$, $dF = 0$) make equal increases in $G$ and $T$.

To reduce deficit without changing GDP (i.e. $dF < 0$, $dG = 0$) increase $T$ and increase $G$ by $3/4$ times as much.
How to reduce the deficit and increase GDP — provided that $I$ and $E - M$ don’t change.
The demand for money

Definition
The quantity of money, $Q$, can be defined as the sum of all cash + readily-available deposits in banks (including Bank of England) and similar institutions.

A simple model for the amount of money needed to support a given GDP is Irving Fisher’s (1911) equation

$$Y = QV$$

where $V$ is the velocity of money (average ratio of people’s spending rate to the amount of money they hold).

The presumption is that $V$ is a constant or at least varies much more slowly than $Y$ and $Q$. So a simple model of the demand for money is

$$Q_{demand} = Y/V$$

In principle the demand depends on the ’price’ of money, which is the interest rate $r$ but for simplicity I’ll ignore this dependence.
Does Irving Fisher’s equation describe the recent UK data?
The supply of money

To meet the demand for money, banks can supply money in the form of loans. Indeed, banks (including but not only the central bank) can create money. They do this by opening new accounts or expanding credit limits on existing ones. In effect they buy financial assets (transferrable promises of future payments) in exchange for newly created money. The amount of money created by the banking system depends on the rate of interest $r$: the more money that is asked from them, the higher price they will charge. Denote by $Q_{supply}(r)$ the amount of money the banking system is willing to provide at interest rate $r$. It is an increasing function of $r$. 
Equality of supply and demand

\[ Q_{\text{supply}}(r) = Q_{\text{demand}}(r) \]

provides an equation from which \( Q \) and \( r \) could be determined — if we knew either of the two functions involved.