An application of centre manifold theory to economics

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Outline

- The 'Minsky moment'
- Steve Keen's model
- What is a centre manifold?
- A centre manifold in Steve Keen's model

H. M. Queen Elizabeth II, visiting the London Scool of Economics, November 2008

- Queen Elizabeth: 'Why had nobody noticed that the credit crunch was on its way?'
- Prof. Tim Besley, FBA, London School of Economics; Bank of England Monetary Policy Committee, and Prof. Peter Hennessy, FBA, Queen Mary, University of London (27 July 2009, after about 6 months' thought and consultation) : 'The failure to foresee the timing, extent and severity of the crisis ... was principally a failure of the collective imagination of many bright people to understand the risks to the system as a whole.'

Hyman Minsky's 'Financial Instability Hypothesis'

Firms that want to borrow money to pay for investment can manage the loan in 3 different ways:

- hedge financing: future profits will be enough to repay both the loan and the interest
- speculative financing: future profits will be enough to pay the interest but not the loan
- Ponzi financing: further borrowing is necessary even to pay the interest. So long as asset values (e.g. house prices) continue to increase, all is well, but eventually the growth of asset prices falters, lenders lose confidence and the edifice collapses.

So long as firms are cautious and stick to hedge financing, the system seems to be stable, but as time progresses and competition intensifies, investors forget the distant past, gain confidence and work their way down the list. Eventually they get into the unstable 'Ponzi' regime and another 'Minsky moment' (i.e. a crash) ensues.

Steve Keen's model[1]

Components of the model

- The Phillips curve
- Goodwin's model of investment
- The role of debt

[1] Keen, S., *A monetary Minsky model of the Great Moderation and the Great Recession* J. Econ. Behav. Organ.(2011), doi:10.1016/j/jebo.2011.01.010

The "Phillips curve"

A. W. H. Phillips (1958)[2] found an empirical association, the so-called Phillips curve, between unemployment and the rate of change of wages and salaries. It is debatable whether the more recent data are consistent with a Phillips-type relationship over any long time interval, but Keen uses such a relationship in his model and for the purpose of this talk I will follow his example.

[2] Phillips, A. W. H. The Relation Between Unemployment and the Rate of Change of Money Wage Rates in the United Kingdom, 1861–1957. Economica, n.s., 25, no. 2 (1958): 283–299.

A "Phillips curve" for the UK, 1990-2000

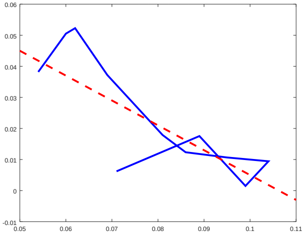

Figure: 1. The abscissa is unemployment; the ordinate is the yearly rate of **real** wage growth. The blue polygon represents the observed data; the red dotted line is a linear approximation to those data.

An interpretation of the negative slope is that a low value of u indicates a high demand for labour, tending to force its 'price', w, to increase. Whereas a high u indicates low demand for labour, so that its price is less likely to increase and may even decrease.

A mathematical statement of the "Phillips" asumption

In general the "Phillips" assumption for real wage growth can be written

$$\frac{1}{w}\frac{dw}{dt} = \Phi(u) \tag{1}$$

where

- w := real wage
- t := time
- u := unemployment fraction, i.e. the fraction of labour force who are available for work but have none
- Φ is a (decreasing) empirical function. The dotted red line in the preceding frame represents the empirical formula

$$\Phi(u) := 0.045 - 0.8(u - 0.05) \tag{2}$$

The predator-prey equation system

Denote the number of foxes by F, the number of rabbits by R. Then the model says

- $\frac{dF}{dt}$ = increasing function of R, e.g. const. $+$R
- $\frac{dR}{dt}$ = decreasing function of F, e.g. const. - F

The model predicts that the two population sizes will oscillate, e.g. $F = \mathrm{const.} + \sin t$, $R = \mathrm{const.} + \cos t$

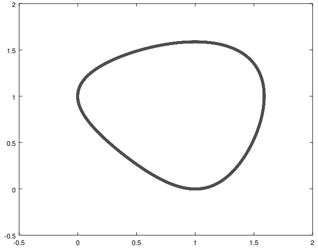

Figure: 2 Schematic phase diagram of a predator-prey system. The abscissa is number of foxes, the ordinate is number of rabbits

Goodwin's ‘predator-prey’ model

- $\frac{dw}{dt}$ = decreasing function of u
- $\frac{du}{dt}$ = increasing function of w

The first equation is a rewriting of the Phillips curve equation. The second equation embodies an assumption that when wages are high, the amount of unemployment tends to increase with time; but if wages are low, the amount of unemployment tends to decrease with time. The rationale for this second equation is explained in the next frame.

Notation:

- u = unemployment
- w = level of "real" wages and salaries

Goodwin's proposed mechanism relating w and du/dt

Assumptions: no government, no foreign trade, AND

- All business profits are re-invested

$$I = \Pi = Y - wL \tag{3}$$

where I = total investment, Π = total business profits, Y = total production (GDP), L = number of employed workers.

- Investment is used to increase K, the existing stock of capital

$$dK/dt = I - \gamma K \tag{4}$$

where γ is the (constant) rate of depreciation

- Output is proportional to capital stock and to employment level

$$Y = K/\nu = aL \tag{5}$$

where ν is the (constant) capital-output ratio and a is labour productivity ($\propto \mathbf{e}^{\alpha t}$ with α constant).

- Available labour force grows as $\mathbf{e}^{\beta t}$

$$L \;\propto\; (1-u)\mathbf{e}^{\beta t} \tag{6}$$

Goodwin's model, continued

The equations from the previous slide are

$$I = \Pi = Y - wL \tag{7}$$
$$dK/dt = I - \gamma K \tag{8}$$
$$Y = K/\nu = aL \propto L\mathrm{e}^{\alpha t} \tag{9}$$
$$L \propto (1-u)\mathbf{e}^{\beta t} \tag{10}$$

whence

$$\begin{aligned}
\frac{1}{1-u}\frac{du}{dt} &= \beta - \frac{1}{L}\frac{dL}{dt} \quad \text{by } (10) \\
&= \alpha + \beta - \frac{1}{K}\frac{dK}{dt} \quad \text{by } (9) \\
&= \alpha + \beta + \gamma - \frac{I}{K} \quad \text{by } (8) \\
&= \alpha + \beta + \gamma - \frac{Y}{K} + \frac{wL}{K} \quad \text{by } (7) \\
&= \alpha + \beta + \gamma - \frac{1}{\nu} + \frac{w}{\nu a} \quad \text{by } (9) \qquad (11)
\end{aligned}$$

Goodwin's model : a predator-prey type system

From the previous slide, the equation giving du/dt in terms of w is

$$\frac{1}{1-u}\frac{du}{dt} = \frac{w}{\nu a} + \alpha + \beta + \gamma - \frac{1}{\nu} \tag{12}$$

while the Phillips eqn $(1/w)dw/dt = \Phi(u)$ is (since $a \propto \mathrm{e}^{\alpha t}$) equivalent to

$$\frac{a}{w}\frac{d}{dt}\left(\frac{w}{a}\right) = \Phi(u) - \alpha \tag{13}$$

Since $\Phi(\cdot)$ is a decreasing function, eqns (12) and (13) constitute a predator-prey type system in the two variables u and w/a. The variable w/a, which is equal to wL/Y, can be interpreted as the share of GDP that is paid out in wages and salaries.
This system of equations has a fixed point u_0, w_0 where

$$\Phi(u_0) = \alpha, \quad w_0/a = 1 - \nu(\alpha + \beta + \gamma) \tag{14}$$

Solutions that start near this fixed point oscillate around it with time period $2\pi\sqrt{\nu/|\Phi'(u_0)|}$
S. Keen takes $\nu = 3$ years, $\alpha = 0.02$/yr, $\beta = \gamma = 0.01$/yr whence (with $\Phi'(u) = -0.08$), $u_0 \approx 0.031, w_0/a \approx 0.88$, period ≈ 38 yr.

Steve Keen's model, which allows for debt

The assumption $I = \Pi$ of the Goodwin model is replaced by

$$I/Y = \Psi(\Pi/Y) \tag{15}$$

where $\Psi(\cdot)$ is an empirical function. Since I is no longer equal to the profit the shortfall is made up by borrowing:

$$I = \Pi + \frac{dD}{dt} \tag{16}$$

where D is the total amount that has already been borrowed by businesses (apparently from banks). The interest on this debt is deducted from the original formula for profit, which is amended to

$$\Pi = Y - wL - rD \tag{17}$$

where r is the rate of interest charged on the loan D.

$$
\begin{aligned}
\frac{dD}{dt} &= I - \Pi && (18)\\
I &= Y\Psi(\Pi/Y) && (19)\\
\Pi &= Y - wL - rD && (20)\\
dK/dt &= I - \gamma K && (21)\\
Y &= K/\nu \;=\; aL \propto L\mathrm{e}^{\alpha t} && (22)\\
L &\propto (1-u)\mathbf{e}^{\beta t} && (23)
\end{aligned}
$$

whence, using the notation $z := D/Y, \pi := \Pi/Y = 1 - w/a - rz$

$$
\begin{aligned}
\frac{1}{1-u}\frac{du}{dt} &= \alpha + \beta + \gamma - \frac{I}{K} \quad \text{as in Goodwin model}\\
&= \alpha + \beta + \gamma - \frac{1}{\nu}\Psi(\pi) \quad \text{by} \;\; (19) && (24)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dz}{dt} = \frac{d}{dt}\left(\frac{D}{Y}\right) &= \frac{I-\Pi}{Y} - \frac{D}{Y}\left(\frac{I-\gamma K}{K}\right)\\
&= \left(1 - \frac{z}{\nu}\right)\Psi(\pi) - \pi + \gamma z && (25)
\end{aligned}
$$

Summary of Keen's model

$$\frac{1}{1-u}\frac{du}{dt} = \alpha + \beta + \gamma - \frac{1}{\nu}\Psi(\pi) \quad (26)$$

$$\frac{1}{w}\frac{dw}{dt} = \Phi(u) - \alpha \quad (27)$$

$$\frac{dz}{dt} = \left(1 - \frac{z}{\nu}\right)\Psi(\pi) - \pi + \gamma z \quad (28)$$

where

$$\pi = 1 - \frac{w}{a} - rz \quad (29)$$

This equation system has a fixed point (u_0, w_0, z_0) where

$$\Phi(u_0) = \alpha \quad (30)$$

$$w_0/a = 1 - \pi_0 - rz_0 \quad (31)$$

$$z_0 = \frac{\Psi(\pi_0) - \pi_0}{\Psi(\pi_0) - \gamma\nu}\nu \quad (32)$$

$$\pi_0 = (\alpha + \beta + \gamma)\nu \quad (33)$$

Behaviour near the fixed point, for small r

To study this model near its fixed point for small values of the interest rate r, we treat r as a new dependent variable, satisfying the equation $dr/dt = 0$. The new system, when linearized, is

$$
\begin{aligned}
\frac{1}{1-u_0}\frac{du}{dt} &= \alpha + \beta + \gamma - \frac{1}{\nu}\Psi(\pi) = -\frac{1}{\nu}\Psi'(\pi_0)(\pi - \pi_0) + \dots && (34)\\
\frac{1}{w_0}\frac{dw}{dt} &= \Phi(u) - \alpha = \Phi'(u_0)(u - u_0) + \dots && (35)\\
\frac{dr}{dt} &= 0 && (36)\\
\frac{dz}{dt} &= \left(1 - \frac{z}{\nu}\right)\Psi(\pi) - \pi + \gamma z + \dots && (37)\\
&= \left(\gamma - \frac{\Psi(\pi_0)}{\nu}\right)(z - z_0) + \left(1 - \frac{z_0}{\nu}\right)\Psi'(\pi_0)(\pi - \pi_0) + \dots
\end{aligned}
$$

where (since $\pi = 1 - w/a - rz$ and r is small, i.e. $r_0 = 0$)

$$\pi - \pi_0 = -\frac{w - w_0}{a} - rz_0 + \dots \qquad (38)$$

Matrix form of the linearized equations

Using the abbreviations $\Psi_0 := \Psi(u_0), \Psi_0' := \Psi'(u_0), \Phi_0 := \Phi(u_0)$, the linearized equation system can be written

$$\begin{bmatrix} \frac{du}{dt} \\ \frac{dw}{dt} \\ \frac{dr}{dt} \\ \frac{dz}{dt} \end{bmatrix} \approx \begin{bmatrix} 0 & \frac{(1-u_0)\Psi_0'}{\nu a} & \frac{(1-u_0)z_0\Psi_0'}{\nu} & 0 \\ w_0\Phi_0' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{(1-z_0/\nu)\Psi_0'}{a} & -(1-\frac{z_0}{\nu})z_0\Psi_0' & \gamma - \frac{\Psi_0}{\nu} \end{bmatrix} \begin{bmatrix} u - u_0 \\ w - w_0 \\ r \\ z - z_0 \end{bmatrix}$$

$$= \left[\begin{array}{c} A|0 \\ \hline C|b \end{array}\right] \begin{bmatrix} x - x_0 \\ z - z_0 \end{bmatrix} \tag{39}$$

in which

$$A := \begin{bmatrix} 0 & (1-u_0)\frac{\Psi_0'}{\nu a} & (1-u_0)z_0\frac{\Psi_0'}{\nu} \\ w_0\Phi_0' & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{40}$$

$$C := \begin{bmatrix} 0 & -\left(1-\frac{z_0}{\nu}\right)\frac{\Psi_0'}{a} & -\left(1-\frac{z_0}{\nu}\right)z_0\Psi_0' \end{bmatrix} \tag{41}$$

$$b := \gamma - \Psi_0/\nu \tag{42}$$

$$x := [u \; w \; r]^T \tag{43}$$

Is there a centre manifold?

To simplify the equations a little, let 1_3 denote the unit 3×3 matrix and define

$$y := z + C(b1_3 - A)^{-1}x \tag{44}$$

Then it follows from the previous equations that

$$dy/dt = b(y - y_0) \tag{45}$$

so that (remembering $x := [u \;\, w \;\, r]$)

$$\frac{d}{dt}\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & b \end{array}\right]\left[\begin{array}{c} x - x_0 \\ y - y_0 \end{array}\right] + \dots \tag{46}$$

If all the eigenvalues of A have zero real parts and b has negative real part, the system has a centre manifold[3]

[3] J. Carr, *Applications of centre manifold theory* Springer 1981.

Applicability of centre manifold theory

The equations are

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & b \end{array}\right]\begin{bmatrix} x \\ y \end{bmatrix} + \dots \tag{47}$$

with

$$A := \begin{bmatrix} 0 & (1-u_0)\frac{\Psi_0'}{\nu a} & (1-u_0)z_0\frac{\Psi_0'}{\nu} \\ w_0\Phi_0' & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{48}$$

$$b := \gamma - \Psi_0/\nu = \frac{\gamma\nu - \Psi(\pi_0)}{\nu} \tag{49}$$

Characteristic polynomial of A is $\lambda(\lambda^2 - (1-u_0)\frac{\Psi_0'}{\nu a}w_0\Phi_0')$.
Eigenvalues are zero and (since $\Phi_0' < 0$) two purely imaginary numbers.
The sign of b is the same as that of $\gamma\nu - \Psi(\pi_0)$; it seems likely to be negative so long as businesses invest more than around 10% of their profits. So the conditions for a centre manifold are satisfied.

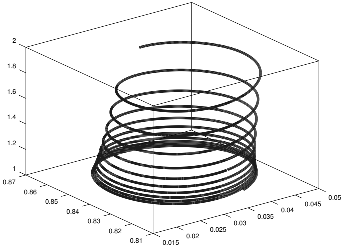

Figure: 3 A trajectory of Keen's model in (u, w, z) space, using $\gamma = 0.03, \Psi(\pi) = 1.2\pi, u(0) = 0.03, w(0) = 0.85, r = 0.01, z(0) = 2$

What centre manifold theory tells us

In the linear approximation, y decays exponentially to its fixed-point value y_0, r stays constant, while the other two variables move on an elliptical orbit. Centre manifold theory enables us to generalize these statements to the non-linear case. By a centre manifold we mean a smooth invariant set in the four-dimensional state space, which at the fixed point is tangential to the linear space containing the elliptical orbits.

- Theorem: There exists a 3-dimensional centre manifold containing the fixed point $(u_0, w_0, 0, y_0)$. The flow on the centre manifold obeys a system of 3 differential equations.
- Theorem: An orbit starting sufficiently close to the centre manifold will approach an orbit that lies in the centre manifold, exponentially fast

In economic terms, these results tell us that the Keen model discussed here has a family of stable oscillatory solutions. To describe the unstable economic behaviour of the real world, one must either look outside this stable family, or use a different model (as Keen himself proposes later on in the paper I have cited.)